LOWER AND UPPER BOUNDS FOR THE SLATER'S GAP OF CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some lower and upper bounds for the *Slater's gap*

$$\Phi(\sigma) - \int_\Omega \Phi \circ f d\mu,$$

where $\int_\Omega \Phi' \circ f d\mu \neq 0$ and the Slater's point is defined by

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu}, \tag{0.1}$$

and assumed to be in $\mathring{I}$, while Φ is a twice differentiable convex function on the interval I. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x - a)\varphi(a) \text{ for any } x,\ a \in I.$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20..

Key words and phrases. Jensen's inequality, Measurable functions, Lebesgue integral, Discrete inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 6, 17 pp. Received 12/01/20

Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\,d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\,d\mu(t)$.

The following result is well known in the literature as *Slater's inequality:*

Theorem 1 (Slater, 1981, [10])**.** *If $\Phi : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $\varphi \in \partial\Phi$ and $f : \Omega \to I$ is μ-measurable and such that $\Phi\circ f$, $\varphi\circ f$, $f\cdot\varphi\circ f \in L(\Omega,\mu)$ then*

$$\int_\Omega \Phi\circ f d\mu \le \Phi\left(\frac{\int_\Omega f\cdot(\varphi\circ f)\,d\mu}{\int_\Omega \varphi\circ f d\mu}\right). \tag{1.1}$$

As pointed out in [9], see also [3, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\int_\Omega f\cdot(\varphi\circ f)\,d\mu}{\int_\Omega \varphi\circ f d\mu} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

For some related results, see [1], [7], [8] and [11]. Extensions for continuous functions of selfadjoint operators may be found in [4]-[6].

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I\subset\mathbb{R}$ be a closed interval, $a\in I$ and let n be a positive integer. If $g : I \longrightarrow \mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x\in I$*

$$g(x) = T_n(g;a,x) + R_n(g;a,x), \tag{2.1}$$

where $T_n(g;c,y)$ is Taylor's polynomial, i.e.,

$$T_n(g;a,x) := \sum_{k=0}^{n}\frac{(x-a)^k}{k!}g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ and $0! := 1$ and the remainder is given by

$$R_n(g;a,x) := \frac{1}{n!}\int_a^x (x-t)^n g^{(n+1)}(t)\,dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s\in[0,1]$ that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c+sd)\,ds.$$

Therefore,

$$\begin{aligned}&\int_a^x g^{(n+1)}(t)(x-t)^n\,dt\\&=(x-a)\int_0^1 g^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^n\,ds\\&=(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds.\end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned}g(x)&=\sum_{k=0}^{n}\frac{1}{k!}g^{(k)}(a)(x-a)^k\\&\quad+\frac{1}{n!}(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds\end{aligned}\tag{2.4}$$

for all $x,\ a\in I$.

We have the following refinement and reverse of Slater's inequality for twice differentiable convex functions:

Theorem 2. *Let $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ (the interior of I) and $f:\Omega\to I$ so that $f,\ f^2,\ \Phi\circ f,\ \Phi''\circ f\in L(\Omega,\mu)$. If $\int_\Omega\Phi'\circ f d\mu\neq 0$ and the Slater's point*

$$\sigma:=\frac{\int_\Omega f\cdot(\Phi'\circ f)\,d\mu}{\int_\Omega\Phi'\circ f d\mu}\in\mathring{I},\tag{2.5}$$

then we have the inequality:

$$\begin{aligned}0&\le\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)\left(\sigma-\int_\Omega f d\mu\right)^2\\&\le\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)\int_\Omega(\sigma-f)^2\,d\mu\\&\le\Phi(\sigma)-\int_\Omega\Phi\circ f d\mu\\&\le\operatorname*{essup}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)\int_\Omega(\sigma-f)^2\,d\mu,\end{aligned}\tag{2.6}$$

provided that $\operatorname{essup}_{t\in\Omega}(\cdot)$ *is finite.*

Proof. We have from (2.4) for $n=2$ that

$$\Phi(x)=\Phi(c)+\Phi'(c)(x-c)+(x-c)^2\int_0^1\Phi''((1-s)c+sx)(1-s)\,ds\tag{2.7}$$

for all $x,\ c\in I$.

If we take in (2.7) $c=f(t)\,,\ t\in\Omega$, then we get

$$\begin{aligned}\Phi(x)&=\Phi(f(t))+\Phi'(f(t))(x-f(t))\\&\quad+(x-f(t))^2\int_0^1\Phi''((1-s)f(t)+sx)(1-s)\,ds\end{aligned}$$

for all $x\in I$.

Now, if we integrate on Ω, then we get

$$\Phi(x) = \int_\Omega \Phi \circ f d\mu + x\int_\Omega \Phi' \circ f d\mu - \int_\Omega f \cdot \Phi' \circ f d\mu + \int_\Omega (x-f)^2 \left(\int_0^1 \Phi''((1-s)f+sx)(1-s)\,ds\right) d\mu \tag{2.8}$$

for all $x \in I$, which is an equality of interest in itself.

Now, if we take in (2.8) $x = \sigma$, then we get the identity

$$\begin{aligned}\Phi(\sigma) &= \int_\Omega \Phi \circ f d\mu + \sigma\int_\Omega \Phi' \circ f d\mu - \int_\Omega f \cdot \Phi' \circ f d\mu \\ &\quad + \int_\Omega (\sigma-f)^2 \left(\int_0^1 \Phi''((1-s)f+s\sigma)(1-s)\,ds\right) d\mu \\ &= \int_\Omega \Phi \circ f d\mu + \int_\Omega (\sigma-f)^2 \left(\int_0^1 \Phi''((1-s)f+s\sigma)(1-s)\,ds\right) d\mu,\end{aligned}$$

namely

$$\Phi(\sigma) - \int_\Omega \Phi \circ f d\mu = \int_\Omega (\sigma-f)^2 \left(\int_0^1 \Phi''((1-s)f+s\sigma)(1-s)\,ds\right) d\mu. \tag{2.9}$$

Now, observe that for all $t \in \Omega$

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right) \\ &\le \int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds \\ &\le \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right),\end{aligned}$$

and if we multiply this inequality by $(\sigma-f)^2 \ge 0$ and take the integral $\int_\Omega$, then we get

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)\int_\Omega(\sigma-f)^2 d\mu \\ &\le \int_\Omega (\sigma-f(t))^2\left(\int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)d\mu(t) \\ &\le \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+s\sigma)(1-s)\,ds\right)\int_\Omega(\sigma-f)^2 d\mu,\end{aligned}$$

and by the identity (2.9) we obtain the third and fourth inequalities (2.6).

The seocnd inequality follows by Jensen's inequality

$$\left(\sigma - \int_\Omega f d\mu\right)^2 \le \int_\Omega (\sigma-f)^2 d\mu.$$

The first inequality is obvious since the function is twice differentiable and convex and hence $\Phi''(u) \ge 0$, $u \in \mathring{I}$. □

Corollary 1. *With the assumptions of Theorem 2 and if there exist* $0 < \gamma < \Gamma < \infty$ *such that*

$$\gamma \le \Phi''(t) \le \Gamma \text{ for almost every } t \in \mathring{I}, \tag{2.10}$$

then

$$0 \le \frac{1}{2}\gamma \left(\sigma - \int_\Omega f d\mu\right)^2 \le \frac{1}{2}\gamma \int_\Omega (\sigma - f)^2 d\mu \tag{2.11}$$
$$\le \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu \le \frac{1}{2}\Gamma \int_\Omega (\sigma - f)^2 d\mu.$$

Proof. We have

$$\int_0^1 \Phi''((1-s) f(t) + s\sigma)(1-s) ds \ge \gamma \int_0^1 (1-s) ds = \frac{1}{2}\gamma$$

and

$$\int_0^1 \Phi''((1-s) f(t) + s\sigma)(1-s) ds \le \Gamma \int_0^1 (1-s) ds = \frac{1}{2}\Gamma$$

and by (2.6) we deduce the desired result (2.11). □

Corollary 2. *With the assumptions of Theorem 2 and if* $\Phi''(\cdot)$ *is monotonic non-decreasing on* $(m, M) \subset \mathring{I}$ *and the Slater's point*

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f) d\mu}{\int_\Omega \Phi' \circ f d\mu} \in (m, M), \tag{2.12}$$

then we have the inequality:

$$0 \le \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\left(\sigma - \int_\Omega f d\mu\right)^2 \tag{2.13}$$
$$\le \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\int_\Omega (\sigma - f)^2 d\mu$$
$$\le \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu$$
$$\le \frac{1}{M - \sigma}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma}\right\}\int_\Omega (\sigma - f)^2 d\mu.$$

If $\Phi''(\cdot)$ *is monotonic nonincreasing on* (m, M), *then*

$$0 \le \frac{1}{M - \sigma}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma}\right\}\left(\sigma - \int_\Omega f d\mu\right)^2 \tag{2.14}$$
$$\le \frac{1}{M - \sigma}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma}\right\}\int_\Omega (\sigma - f)^2 d\mu$$
$$\le \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu$$
$$\le \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\int_\Omega (\sigma - f)^2 d\mu.$$

Proof. Since $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then

$$\Phi''((1-s) m + s\sigma) \le \Phi''((1-s) f(t) + s\sigma) \le \Phi''((1-s) M + s\sigma)$$

for all $s \in (0, 1)$ and μ-a.e. $t \in \Omega$.

This implies that

$$\int_0^1 \Phi''\left((1-s)\,m+s\sigma\right)(1-s)\,ds \le \int_0^1 \Phi''\left((1-s)\,f(t)+s\sigma\right)(1-s)\,ds$$
$$\le \int_0^1 \Phi''\left((1-s)\,M+s\sigma\right)(1-s)\,ds$$

for μ-a.e. $t\in\Omega$.

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)\,m+s\sigma\right)(1-s)\,ds\\
&=\frac{1}{\sigma-m}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\,m+s\sigma\right)\right)\\
&=\frac{1}{\sigma-m}\left\{(1-s)\,\Phi'\left((1-s)\,m+s\sigma\right)\Big|_0^1\right.\\
&\left.+\int_0^1 \Phi'\left((1-s)\,m+s\sigma\right)ds\right\}\\
&=\frac{1}{\sigma-m}\left\{-\Phi'(m)+\frac{\Phi(\sigma)-\Phi(m)}{\sigma-m}\right\}\\
&=\frac{1}{\sigma-m}\left\{\frac{\Phi(\sigma)-\Phi(m)}{\sigma-m}-\Phi'(m)\right\}.
\end{aligned}$$

Similary,

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)\,M+s\sigma\right)(1-s)\,ds\\
&=\frac{1}{\sigma-M}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\,M+s\sigma\right)\right)\\
&=\frac{1}{\sigma-M}\left\{(1-s)\,\Phi'\left((1-s)\,M+s\sigma\right)\Big|_0^1\right.\\
&\left.+\int_0^1 \Phi'\left((1-s)\,M+s\sigma\right)ds\right\}\\
&=\frac{1}{\sigma-M}\left\{-\Phi'(M)+\frac{\Phi(M)-\Phi(\sigma)}{M-\sigma}\right\}\\
&=\frac{1}{M-\sigma}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi(\sigma)}{M-\sigma}\right\}.
\end{aligned}$$

We then have

$$\begin{aligned}
&\frac{1}{\sigma-m}\left\{\frac{\Phi(\sigma)-\Phi(m)}{\sigma-m}-\Phi'(m)\right\}\\
&\le \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\sigma\right)(1-s)\,ds\right)
\end{aligned}$$

and

$$\begin{aligned}&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left(\left(1-s\right)f\left(t\right)+s\sigma\right)\left(1-s\right)ds\right)\\&\leq \frac{1}{M-\sigma}\left\{\Phi'\left(M\right)-\frac{\Phi\left(M\right)-\Phi\left(\sigma\right)}{M-\sigma}\right\}.\end{aligned}$$

By making use of (2.6) we get (2.13).

The case of $\Phi''(\cdot)$ is monotonic nonincreasing on (m,M) follows in a similar way. □

Corollary 3. *With the assumptions of Theorem 2 and if* $\Phi''(\cdot)$ *is convex on* (m,M)*, then*

$$\begin{aligned}0&\leq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\Phi''\left(\frac{2}{3}f\left(t\right)+\frac{1}{3}\sigma\right)\left(\sigma-\int_\Omega f d\mu\right)^2\\&\leq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\Phi''\left(\frac{2}{3}f\left(t\right)+\frac{1}{3}\sigma\right)\int_\Omega\left(\sigma-f\right)^2 d\mu\\&\leq \Phi\left(\sigma\right)-\int_\Omega \Phi\circ f d\mu\\&\leq \frac{1}{2}\left[\frac{2}{3}\operatorname*{essup}_{t\in\Omega}\Phi''\left(f\left(t\right)\right)+\frac{1}{3}\Phi''\left(\sigma\right)\right]\int_\Omega\left(\sigma-f\right)^2 d\mu.\end{aligned}\tag{2.15}$$

If $\Phi''(\cdot)$ *is concave on* (m,M)*, then*

$$\begin{aligned}0&\leq \frac{1}{2}\left[\frac{2}{3}\operatorname*{essup}_{t\in\Omega}\Phi''\left(f\left(t\right)\right)+\frac{1}{3}\Phi''\left(\sigma\right)\right]\left(\sigma-\int_\Omega f d\mu\right)^2\\&\leq \frac{1}{2}\left[\frac{2}{3}\operatorname*{essup}_{t\in\Omega}\Phi''\left(f\left(t\right)\right)+\frac{1}{3}\Phi''\left(\sigma\right)\right]\int_\Omega\left(\sigma-f\right)^2 d\mu\\&\leq \Phi\left(\sigma\right)-\int_\Omega \Phi\circ f d\mu\\&\leq \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\Phi''\left(\frac{2}{3}f\left(t\right)+\frac{1}{3}\sigma\right)\int_\Omega\left(\sigma-f\right)^2 d\mu.\end{aligned}\tag{2.16}$$

Proof. By the convexity of Φ'' and Jensen's integral inequality, we have

$$\begin{aligned}&\int_0^1 \Phi''\left(\left(1-s\right)f\left(t\right)+s\sigma\right)\left(1-s\right)ds\\&\geq\left(\int_0^1\left(1-s\right)ds\right)\Phi''\left(\frac{\int_0^1\left[\left(1-s\right)f\left(t\right)+s\sigma\right]\left(1-s\right)ds}{\int_0^1\left(1-s\right)ds}\right)\\&=\frac{1}{2}\Phi''\left(\frac{\int_0^1\left[\left(1-s\right)f\left(t\right)+s\sigma\right]\left(1-s\right)ds}{\frac{1}{2}}\right)\\&=\frac{1}{2}\Phi''\left(\frac{f\left(t\right)\int_0^1\left(1-s\right)^2ds+\sigma\int_0^1 s\left(1-s\right)ds}{\frac{1}{2}}\right)\\&=\frac{1}{2}\Phi''\left(\frac{\frac{1}{3}f\left(t\right)+\frac{1}{6}\sigma}{\frac{1}{2}}\right)=\frac{1}{2}\Phi''\left(\frac{2}{3}f\left(t\right)+\frac{1}{3}\sigma\right)\end{aligned}$$

for μ-a.e. $t\in\Omega$.

This implies that

$$\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)f(t)+s\sigma\right)(1-s)\,ds\right)\geq\frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\Phi''\left(\frac{2}{3}f(t)+\frac{1}{3}\sigma\right),$$

which proves the first part of (2.15).

By the convexity of Φ'' we also have

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)f(t)+s\sigma\right)(1-s)\,ds\\&\leq\int_0^1\left[(1-s)\Phi''(f(t))+s\Phi''(\sigma)\right](1-s)\,ds\\&=\Phi''(f(t))\int_0^1(1-s)^2\,ds+\Phi''(\sigma)\int_0^1 s(1-s)\,ds\\&=\frac{1}{3}\Phi''(f(t))+\frac{1}{6}\Phi''(\sigma)=\frac{1}{2}\left[\frac{2}{3}\Phi''(f(t))+\frac{1}{3}\Phi''(\sigma)\right]\end{aligned}$$

for μ-a.e. $t\in\Omega$.

This implies that

$$\begin{aligned}&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)f(t)+s\sigma\right)(1-s)\,ds\right)\\&\leq\operatorname*{essup}_{t\in\Omega}\frac{1}{2}\left[\frac{2}{3}\Phi''(f(t))+\frac{1}{3}\Phi''(\sigma)\right]=\frac{1}{2}\left[\frac{2}{3}\operatorname*{essup}_{t\in\Omega}\Phi''(f(t))+\frac{1}{3}\Phi''(\sigma)\right]\end{aligned}$$

and by Theorem 2, we get the second part of (2.15).

In the case when $\Phi''(\cdot)$ is concave on (m,M), the proof goes in a similar way and we omit the details. □

We recall that a function $f:I\to\mathbb{R}$ is called *quasiconvex* on the interval I if

$$f\left((1-s)x+sy\right)\leq\max\{f(x),f(y)\}\tag{2.17}$$

for all x, $y\in I$ and $s\in[0,1]$.

Corollary 4. *With the assumptions of Theorem 2 and if $\Phi''(\cdot)$ is quasiconvex on (m,M), then*

$$\Phi(\sigma)-\int_\Omega\Phi\circ f\,d\mu\leq\frac{1}{2}\max\left\{\Phi''(\sigma),\operatorname*{essup}_{t\in\Omega}\Phi''(f(t))\right\}\int_\Omega(\sigma-f)^2\,d\mu,\tag{2.18}$$

Proof. Since $\Phi''(\cdot)$ is quasiconvex, hence

$$\begin{aligned}&\int_0^1 \Phi''\left((1-s)f(t)+s\sigma\right)(1-s)\,ds\\&\leq\max\{\Phi''(\sigma),\Phi''(f(t))\}\int_0^1(1-s)\,ds=\frac{1}{2}\max\{\Phi''(\sigma),\Phi''(f(t))\}\end{aligned}$$

for μ-a.e. $t\in\Omega$.

Taking the $\operatorname{essup}_{t\in\Omega}$ in this inequality, we get

$$\begin{aligned}&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)f(t)+s\sigma\right)(1-s)\,ds\right)\\&\leq\frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(\max\{\Phi''(\sigma),\Phi''(f(t))\}\right)=\frac{1}{2}\max\left\{\Phi''(\sigma),\operatorname*{essup}_{t\in\Omega}\Phi''(f(t))\right\},\end{aligned}$$

which together with (2.6) gives (2.18). □

3. Exponential Integral Inequalities

We consider the convex function $\Phi_\alpha : \mathbb{R} \to (0, \infty)$ defined by $\Phi_\alpha(t) = \exp(\alpha x)$. We have $\Phi''_\alpha(t) = \alpha^2 \exp(\alpha x)$ which shows that Φ''_α is convex and monotonic decreasing for $\alpha < 0$ and increasing for $\alpha > 0$.

We then have for $t \in [m, M]$ that

$$\begin{aligned} & E_1(\alpha, [m, M]) \\ & := \alpha^2 \begin{cases} \exp(\alpha M), & \alpha < 0 \\ \exp(\alpha m), & \alpha > 0 \end{cases} \leq \Phi''_\alpha(t) \leq \alpha^2 \begin{cases} \exp(\alpha m), & \alpha < 0 \\ \exp(\alpha M), & \alpha > 0 \end{cases} \\ & := E_2(\alpha, [m, M]). \end{aligned}$$

The Slater's point for this function is

$$\sigma_\alpha := \frac{\int_\Omega f \cdot \exp(\alpha f)\, d\mu}{\int_\Omega \exp(\alpha f)\, d\mu}. \tag{3.1}$$

If we assume that $f : \Omega \to [m, M]$ is measurable, then by (2.11) we get

$$\begin{aligned} 0 & \leq \frac{1}{2} E_1(\alpha, [m, M]) \left(\frac{\int_\Omega f \cdot \exp(\alpha f)\, d\mu}{\int_\Omega \exp(\alpha f)\, d\mu} - \int_\Omega f d\mu \right)^2 \\ & \leq \frac{1}{2} E_1(\alpha, [m, M]) \int_\Omega \left(\frac{\int_\Omega f \cdot \exp(\alpha f)\, d\mu}{\int_\Omega \exp(\alpha f)\, d\mu} - f \right)^2 d\mu \\ & \leq \exp\left(\alpha \frac{\int_\Omega f \cdot \exp(\alpha f)\, d\mu}{\int_\Omega \exp(\alpha f)\, d\mu} \right) - \int_\Omega \exp(\alpha f)\, d\mu \\ & \leq \frac{1}{2} E_2(\alpha, [m, M]) \int_\Omega \left(\frac{\int_\Omega f \cdot \exp(\alpha f)\, d\mu}{\int_\Omega \exp(\alpha f)\, d\mu} - f \right)^2 d\mu. \end{aligned} \tag{3.2}$$

If $\alpha > 0$, then Φ''_α is increasing and by (2.13) we get

$$\begin{aligned} 0 & \leq \frac{1}{\sigma_\alpha - m} \left\{ \frac{\exp(\alpha \sigma_\alpha) - \exp(\alpha m)}{\sigma_\alpha - m} - \alpha \exp(\alpha m) \right\} \left(\sigma_\alpha - \int_\Omega f d\mu \right)^2 \\ & \leq \frac{1}{\sigma_\alpha - m} \left\{ \frac{\exp(\alpha \sigma_\alpha) - \exp(\alpha m)}{\sigma_\alpha - m} - \alpha \exp(\alpha m) \right\} \int_\Omega (\sigma_\alpha - f)^2 d\mu \\ & \leq \exp(\alpha \sigma_\alpha) - \int_\Omega \exp(\alpha f)\, d\mu \\ & \leq \frac{1}{M - \sigma_\alpha} \left\{ \alpha \exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha \sigma_\alpha)}{M - \sigma_\alpha} \right\} \int_\Omega (\sigma_\alpha - f)^2 d\mu. \end{aligned} \tag{3.3}$$

If $\alpha < 0$, then by (2.14) we get

$$
\begin{aligned}
(3.4)\qquad 0 &\le \frac{1}{M-\sigma_\alpha}\left\{\alpha \exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha\sigma_\alpha)}{M-\sigma_\alpha}\right\}\left(\sigma_\alpha - \int_\Omega f d\mu\right)^2 \\
&\le \frac{1}{M-\sigma_\alpha}\left\{\alpha \exp(\alpha M) - \frac{\exp(\alpha M) - \exp(\alpha\sigma_\alpha)}{M-\sigma_\alpha}\right\}\int_\Omega (\sigma_\alpha - f)^2 d\mu \\
&\le \exp(\alpha\sigma_\alpha) - \int_\Omega \exp(\alpha f)\, d\mu \\
&\le \frac{1}{\sigma_\alpha - m}\left\{\frac{\exp(\alpha\sigma_\alpha) - \exp(\alpha m)}{\sigma_\alpha - m} - \alpha \exp(\alpha m)\right\}\int_\Omega (\sigma_\alpha - f)^2 d\mu.
\end{aligned}
$$

Since the function Φ_α'' is convex, then by (2.15) we have for all $\alpha \neq 0$ that

$$
\begin{aligned}
(3.5)\qquad 0 &\le \frac{1}{2}\alpha^2 \operatorname*{essinf}_{t\in\Omega} \exp\left[\alpha\left(\frac{2}{3}f(t) + \frac{1}{3}\sigma_\alpha\right)\right]\left(\sigma_\alpha - \int_\Omega f d\mu\right)^2 \\
&\le \frac{1}{2}\alpha^2 \operatorname*{essinf}_{t\in\Omega} \exp\left[\alpha\left(\frac{2}{3}f(t) + \frac{1}{3}\sigma_\alpha\right)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu \\
&\le \exp(\alpha\sigma_\alpha) - \int_\Omega \exp(\alpha f)\, d\mu \\
&\le \frac{1}{2}\alpha^2\left[\frac{2}{3}\operatorname*{essup}_{t\in\Omega} \exp(\alpha f(t)) + \frac{1}{3}\exp(\alpha\sigma_\alpha)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu.
\end{aligned}
$$

If $\alpha > 0$, then by (3.5) we get

$$
\begin{aligned}
(3.6)\qquad 0 &\le \frac{1}{2}\alpha^2 \exp\left[\alpha\left(\frac{2}{3}m + \frac{1}{3}\sigma_\alpha\right)\right]\left(\sigma_\alpha - \int_\Omega f d\mu\right)^2 \\
&\le \frac{1}{2}\alpha^2 \exp\left[\alpha\left(\frac{2}{3}m + \frac{1}{3}\sigma_\alpha\right)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu \\
&\le \exp(\alpha\sigma_\alpha) - \int_\Omega \exp(\alpha f)\, d\mu \\
&\le \frac{1}{2}\alpha^2\left[\frac{2}{3}\exp(\alpha M) + \frac{1}{3}\exp(\alpha\sigma_\alpha)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu,
\end{aligned}
$$

while for $\alpha < 0$ we get

$$
\begin{aligned}
(3.7)\qquad 0 &\le \frac{1}{2}\alpha^2 \exp\left[\alpha\left(\frac{2}{3}M + \frac{1}{3}\sigma_\alpha\right)\right]\left(\sigma_\alpha - \int_\Omega f d\mu\right)^2 \\
&\le \frac{1}{2}\alpha^2 \exp\left[\alpha\left(\frac{2}{3}M + \frac{1}{3}\sigma_\alpha\right)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu \\
&\le \exp(\alpha\sigma_\alpha) - \int_\Omega \exp(\alpha f)\, d\mu \\
&\le \frac{1}{2}\alpha^2\left[\frac{2}{3}\exp(\alpha m) + \frac{1}{3}\exp(\alpha\sigma_\alpha)\right]\int_\Omega (\sigma_\alpha - f)^2 d\mu.
\end{aligned}
$$

4. Logarithmic Integral Inequalities

We consider the convex function $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$. Then $\Phi''(x) = \frac{1}{x^2}$, which is convex and decreasing on $(0,\infty)$. If $x \in [m, M] \subset (0,\infty)$, then we also have

$$\frac{1}{M^2} \leq \Phi''(x) \leq \frac{1}{m^2}.$$

The Slater's point for this function is

$$\sigma_\ell := \frac{1}{\int_\Omega \frac{1}{f} d\mu}. \tag{4.1}$$

If $f : \Omega \to [m, M]$, then from (2.11) we get

$$\begin{aligned} 0 &\leq \frac{1}{2M^2}\left(\frac{1}{\int_\Omega \frac{1}{f} d\mu} - \int_\Omega f d\mu\right)^2 \leq \frac{1}{2M^2}\int_\Omega \left(\frac{1}{\int_\Omega \frac{1}{f} d\mu} - f\right)^2 d\mu \\ &\leq \int_\Omega \ln f d\mu - \ln\left(\frac{1}{\int_\Omega \frac{1}{f} d\mu}\right) \leq \frac{1}{2m^2}\int_\Omega \left(\frac{1}{\int_\Omega \frac{1}{f} d\mu} - f\right)^2 d\mu. \end{aligned} \tag{4.2}$$

Since Φ'' is decreasing, hence by (2.14) we get

$$\begin{aligned} 0 &\leq \frac{1}{M - \sigma_\ell}\left\{\frac{\ln(M) - \ln(\sigma_\ell)}{M - \sigma_\ell} - \frac{1}{M}\right\}\left(\sigma_\ell - \int_\Omega f d\mu\right)^2 \\ &\leq \frac{1}{M - \sigma_\ell}\left\{\frac{\ln(M) - \ln(\sigma_\ell)}{M - \sigma_\ell} - \frac{1}{M}\right\}\int_\Omega (\sigma_\ell - f)^2 d\mu \\ &\leq \int_\Omega \ln f d\mu - \ln\left(\frac{1}{\int_\Omega \frac{1}{f} d\mu}\right) \\ &\leq \frac{1}{\sigma_\ell - m}\left\{\frac{1}{m} - \frac{\ln(\sigma_\ell) - \ln(m)}{\sigma_\ell - m}\right\}\int_\Omega (\sigma_\ell - f)^2 d\mu. \end{aligned} \tag{4.3}$$

Since Φ'' is convex, then by (2.15) we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(\frac{2}{3}M + \frac{1}{3}\sigma_\ell\right)^{-2}\left(\sigma_\ell - \int_\Omega f d\mu\right)^2 \\ &\leq \frac{1}{2}\left(\frac{2}{3}M + \frac{1}{3}\sigma_\ell\right)^{-2}\int_\Omega (\sigma_\ell - f)^2 d\mu \\ &\leq \int_\Omega \ln f d\mu - \ln\left(\frac{1}{\int_\Omega \frac{1}{f} d\mu}\right) \\ &\leq \frac{1}{2}\left(\frac{2}{3}m^{-2} + \frac{1}{3}\sigma_\ell^{-2}\right)\int_\Omega (\sigma_\ell - f)^2 d\mu. \end{aligned} \tag{4.4}$$

5. Power Inequalities

The power function $\Phi_p : (0, \infty) \to (0, \infty)$, $\Phi_p(x) = x^p$ is convex for $p \in (-\infty, 0) \cup [1, \infty)$. We have $\Phi_p''(x) = p(p-1)x^{p-2}$, $x \in (0, \infty)$. If $p \in (-\infty, 0) \cup [1, 2)$ then Φ_p'' is decreasing and if $p \in [2, \infty)$ then Φ_p'' is increasing.

For $p \in (-\infty, 0) \cup [1, \infty)$ and $[m, M] \subset (0, \infty)$ we define the bounds

$$B_1(p, [m, M]) := p(p-1)\begin{cases} M^{p-2} \text{ if } (-\infty, 0) \cup [1, 2), \\ \\ m^{p-2} \text{ if } p \in [2, \infty) \end{cases} \tag{5.1}$$

and

$$B_2(p,[m,M]) := p(p-1)\begin{cases} m^{p-2} \text{ if } (-\infty,0)\cup[1,2), \\ M^{p-2} \text{ if } p\in[2,\infty). \end{cases} \tag{5.2}$$

Then we have

$$B_1(p,[m,M]) \le \Phi_p''(x) \le B_2(p,[m,M]) \text{ for } x\in[m,M].$$

The Slater's point for this function is

$$\sigma_p := \frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}. \tag{5.3}$$

If $f:\Omega\to[m,M]$ is so that $f,\ f^2,\ f^p\in L(\Omega,\mu)$, then by (2.11) we get

$$\begin{aligned} 0 &\le \frac{1}{2}B_1(p,[m,M])\left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}-\int_\Omega f d\mu\right)^2 \\ &\le \frac{1}{2}B_1(p,[m,M])\int_\Omega\left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}-f\right)^2 d\mu \\ &\le \left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}\right)^p-\int_\Omega f^p d\mu \\ &\le \frac{1}{2}B_2(p,[m,M])\int_\Omega\left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}-f\right)^2 d\mu. \end{aligned} \tag{5.4}$$

If $p\in(-\infty,0)\cup[1,2)$ then Φ_p'' is decreasing and by (2.14) we have

$$\begin{aligned} 0 &\le \frac{1}{M-\sigma_p}\left(pM^{p-1}-\frac{M^p-\sigma_p^p}{M-\sigma_p}\right)\left(\sigma_p-\int_\Omega f d\mu\right)^2 \\ &\le \frac{1}{M-\sigma_p}\left(pM^{p-1}-\frac{M^p-\sigma_p^p}{M-\sigma_p}\right)\int_\Omega(\sigma_p-f)^2 d\mu \\ &\le \left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}\right)^p-\int_\Omega f^p d\mu \\ &\le \frac{1}{\sigma_p-m}\left(\frac{\sigma_p^p-m^p}{\sigma_p-m}-pm^{p-1}\right)\int_\Omega(\sigma_p-f)^2 d\mu. \end{aligned} \tag{5.5}$$

If $p\in[2,\infty)$ then Φ_p'' is increasing and by (2.13) we get

$$\begin{aligned} 0 &\le \frac{1}{\sigma-m}\left(\frac{\sigma_p^p-m^p}{\sigma_p-m}-pm^{p-1}\right)\left(\sigma-\int_\Omega f d\mu\right)^2 \\ &\le \frac{1}{\sigma-m}\left(\frac{\sigma_p^p-m^p}{\sigma_p-m}-pm^{p-1}\right)\int_\Omega(\sigma-f)^2 d\mu \\ &\le \left(\frac{\int_\Omega f^p d\mu}{\int_\Omega f^{p-1} d\mu}\right)^p-\int_\Omega f^p d\mu \\ &\le \frac{1}{M-\sigma}\left(pM^{p-1}-\frac{M^p-\sigma_p^p}{M-\sigma_p}\right)\int_\Omega(\sigma-f)^2 d\mu. \end{aligned} \tag{5.6}$$

6. Discrete Case

The discrete case is useful for applications and we state some examples here.

In this section the function $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable convex function on (m, M) and $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

If $\sum_{k=1}^{n} p_k \Phi'(x_k) \neq 0$ and the Slater's point

$$\sigma := \frac{\sum_{k=1}^{n} p_k x_k \Phi'(x_k)}{\sum_{k=1}^{n} p_k \Phi'(x_k)} \in (m, M), \tag{6.1}$$

then we have the inequality:

$$\begin{aligned} 0 &\leq \min_{k\in\{1,\dots,n\}} \left(\int_0^1 \Phi''((1-s)x_k + s\sigma)(1-s)\,ds \right) \left(\sigma - \sum_{k=1}^{n} p_k x_k \right)^2 \\ &\leq \min_{k\in\{1,\dots,n\}} \left(\int_0^1 \Phi''((1-s)x_k + s\sigma)(1-s)\,ds \right) \sum_{k=1}^{n} p_k (\sigma - x_k)^2 \\ &\leq \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\ &\leq \max_{k\in\{1,\dots,n\}} \left(\int_0^1 \Phi''((1-s)x_k + s\sigma)(1-s)\,ds \right) \sum_{k=1}^{n} p_k (\sigma - x_k)^2 . \end{aligned} \tag{6.2}$$

If there exist $0 < \gamma < \Gamma < \infty$ such that

$$\gamma \leq \Phi''(t) \leq \Gamma \text{ for almost every } t \in [m, M],$$

then,

$$\begin{aligned} 0 &\leq \frac{1}{2}\gamma \left(\sigma - \sum_{k=1}^{n} p_k x_k \right)^2 \leq \frac{1}{2}\gamma \sum_{k=1}^{n} p_k (\sigma - x_k)^2 \\ &\leq \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \leq \frac{1}{2}\Gamma \sum_{k=1}^{n} p_k (\sigma - x_k)^2 . \end{aligned} \tag{6.3}$$

If $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then we have the inequality:

$$\begin{aligned} 0 &\leq \frac{1}{\sigma - m} \left\{ \frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m) \right\} \left(\sigma - \sum_{k=1}^{n} p_k x_k \right)^2 \\ &\leq \frac{1}{\sigma - m} \left\{ \frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m) \right\} \sum_{k=1}^{n} p_k (\sigma - x_k)^2 \\ &\leq \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\ &\leq \frac{1}{M - \sigma} \left\{ \Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma} \right\} \sum_{k=1}^{n} p_k (\sigma - x_k)^2 . \end{aligned} \tag{6.4}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m,M), then

$$0 \le \frac{1}{M-\sigma}\left\{\Phi'(M) - \frac{\Phi(M)-\Phi(\sigma)}{M-\sigma}\right\}\left(\sigma - \sum_{k=1}^{n} p_k x_k\right)^2 \tag{6.5}$$
$$\le \frac{1}{M-\sigma}\left\{\Phi'(M) - \frac{\Phi(M)-\Phi(\sigma)}{M-\sigma}\right\}\sum_{k=1}^{n} p_k (\sigma - x_k)^2$$
$$\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k)$$
$$\le \frac{1}{\sigma-m}\left\{\frac{\Phi(\sigma)-\Phi(m)}{\sigma-m} - \Phi'(m)\right\}\sum_{k=1}^{n} p_k (\sigma - x_k)^2 .$$

If $\Phi''(\cdot)$ is convex on (m,M), then

$$0 \le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \Phi''\left(\frac{2}{3}x_k + \frac{1}{3}\sigma\right)\left(\sigma - \sum_{k=1}^{n} p_k x_k\right)^2 \tag{6.6}$$
$$\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \Phi''\left(\frac{2}{3}x_k + \frac{1}{3}\sigma\right)\sum_{k=1}^{n} p_k (\sigma - x_k)^2$$
$$\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k)$$
$$\le \frac{1}{2}\left[\frac{2}{3}\max_{k\in\{1,\dots,n\}} \Phi''(x_k) + \frac{1}{3}\Phi''(\sigma)\right]\sum_{k=1}^{n} p_k (\sigma - x_k)^2 .$$

If $\Phi''(\cdot)$ is concave on (m,M), then

$$0 \le \frac{1}{2}\left[\frac{2}{3}\max_{k\in\{1,\dots,n\}} \Phi''(x_k) + \frac{1}{3}\Phi''(\sigma)\right]\left(\sigma - \sum_{k=1}^{n} p_k x_k\right)^2 \tag{6.7}$$
$$\le \frac{1}{2}\left[\frac{2}{3}\max_{k\in\{1,\dots,n\}} \Phi''(x_k) + \frac{1}{3}\Phi''(\sigma)\right]\sum_{k=1}^{n} p_k (\sigma - x_k)^2$$
$$\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k)$$
$$\le \frac{1}{2}\min_{k\in\{1,\dots,n\}} \Phi''\left(\frac{2}{3}x_k + \frac{1}{3}\sigma\right)\sum_{k=1}^{n} p_k (\sigma - x_k)^2 .$$

If $\Phi''(\cdot)$ is quasiconvex on (m,M), then

$$\Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \le \frac{1}{2}\max\left\{\Phi''(\sigma), \max_{k\in\{1,\dots,n\}} \Phi''(x_k)\right\}\sum_{k=1}^{n} p_k (\sigma - x_k)^2 . \tag{6.8}$$

If $\Phi(x) = \exp x$, then Φ'' is convex and increasing on any interval $[m,M]$. If $x_k \in [m,M]$, $p_k \ge 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$, then the Slater point is

$$\sigma_e := \frac{\sum_{k=1}^{n} p_k x_k \exp(x_k)}{\sum_{k=1}^{n} p_k \exp(x_k)} \in (m,M), \tag{6.9}$$

and by (6.3) we get

$$0 \leq \frac{1}{2}\exp(m)\left(\sigma_e - \sum_{k=1}^{n} p_k x_k\right)^2 \leq \frac{1}{2}\exp(m)\sum_{k=1}^{n} p_k (\sigma_e - x_k)^2 \leq \exp(\sigma_e) - \sum_{k=1}^{n} p_k \exp(x_k) \leq \frac{1}{2}\exp(M)\sum_{k=1}^{n} p_k(\sigma_e - x_k)^2, \tag{6.10}$$

while by (6.4) we get

$$\begin{aligned} 0 &\leq \frac{1}{\sigma_e - m}\left\{\frac{\Phi(\sigma_e) - \Phi(m)}{\sigma_e - m} - \Phi'(m)\right\}\left(\sigma_e - \sum_{k=1}^{n} p_k x_k\right)^2 \\ &\leq \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma_e) - \Phi(m)}{\sigma_e - m} - \Phi'(m)\right\}\sum_{k=1}^{n} p_k(\sigma_e - x_k)^2 \\ &\leq \Phi(\sigma_e) - \sum_{k=1}^{n} p_k \Phi(x_k) \\ &\leq \frac{1}{M - \sigma_e}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma_e)}{M - \sigma_e}\right\}\sum_{k=1}^{n} p_k(\sigma_e - x_k)^2 . \end{aligned} \tag{6.11}$$

From (6.6) we also have

$$\begin{aligned} 0 &\leq \frac{1}{2}\exp\left(\frac{2}{3}\min_{k\in\{1,\dots,n\}} x_k + \frac{1}{3}\sigma_e\right)\left(\sigma_e - \sum_{k=1}^{n} p_k x_k\right)^2 \\ &\leq \frac{1}{2}\exp\left(\frac{2}{3}\min_{k\in\{1,\dots,n\}} x_k + \frac{1}{3}\sigma_e\right)\sum_{k=1}^{n} p_k(\sigma_e - x_k)^2 \\ &\leq \exp(\sigma_e) - \sum_{k=1}^{n} p_k \exp(x_k) \\ &\leq \frac{1}{2}\left[\frac{2}{3}\exp\left(\max_{k\in\{1,\dots,n\}} x_k\right) + \frac{1}{3}\exp(\sigma_e)\right]\sum_{k=1}^{n} p_k(\sigma_e - x_k)^2 . \end{aligned} \tag{6.12}$$

Now, let $y_k \in [a,b] \subset (0,\infty)$, $k \in \{1,...,n\}$. If we take $x_k = \ln y_k$, $m = \ln a$ and $M = \ln b$, then the Slater point becomes

$$\sigma_l := \frac{\sum_{k=1}^{n} p_k y_k \ln y_k}{\sum_{k=1}^{n} p_k y_k} = \frac{\ln\left(\prod_{k=1}^{n} y_k^{p_k y_k}\right)}{\sum_{k=1}^{n} p_k y_k} \in (\ln a, \ln b), \tag{6.13}$$

and by (6.10) we obtain

$$\begin{aligned} 0 &\leq \frac{1}{2}a\left(\sigma_l - \ln\left(\prod_{k=1}^{n} y_k^{p_k}\right)\right)^2 \leq \frac{1}{2}a\sum_{k=1}^{n} p_k(\sigma_l - \ln y_k)^2 \\ &\leq \left(\prod_{k=1}^{n} y_k^{p_k y_k}\right)^{\sum_{k=1}^{n} p_k y_k} - \sum_{k=1}^{n} p_k y_k \leq \frac{1}{2}b\sum_{k=1}^{n} p_k(\sigma_l - \ln y_k)^2 . \end{aligned} \tag{6.14}$$

We consider the convex function $\Phi : (0,\infty) \to \mathbb{R}$, $\Phi(x) = -\ln x$. Then $\Phi''(x) = \frac{1}{x^2}$ which is convex and decreasing on $(0,\infty)$. If $x \in [m,M] \subset (0,\infty)$ then we also have

$$\frac{1}{M^2} \le \Phi''(x) \le \frac{1}{m^2}$$

and the Slater point is

$$\sigma_\ell := \frac{1}{\sum_{k=1}^n \frac{p_k}{x_k}}. \tag{6.15}$$

If $x_k \in [m, M]$, $p_k \ge 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$, then by (6.3) we get

$$\begin{aligned} 0 &\le \frac{1}{2M^2}\left(\frac{1}{\sum_{k=1}^n \frac{p_k}{x_k}} - \sum_{k=1}^n p_k x_k\right)^2 \\ &\le \frac{1}{2M^2}\sum_{k=1}^n p_k\left(\frac{1}{\sum_{i=1}^n \frac{p_i}{x_i}} - x_k\right)^2 \le \sum_{k=1}^n p_k \ln x_k - \ln\left(\frac{1}{\sum_{k=1}^n \frac{p_k}{x_k}}\right) \\ &\le \frac{1}{2m^2}\sum_{k=1}^n p_k\left(\frac{1}{\sum_{i=1}^n \frac{p_i}{x_i}} - x_k\right)^2. \end{aligned} \tag{6.16}$$

The power function $\Phi_p : (0, \infty) \to (0, \infty)$, $\Phi_p(x) = x^p$ is convex for $p \in (-\infty, 0) \cup [1, \infty)$. We have $\Phi_p''(x) = p(p-1)x^{p-2}$, $x \in (0, \infty)$.

The Slater's point for this function is

$$\sigma_p := \frac{\sum_{k=1}^n p_k x_k^p}{\sum_{k=1}^n p_k x_k^{p-1}}. \tag{6.17}$$

From (2.11) we have

$$\begin{aligned} 0 &\le \frac{1}{2}B_1(p, [m, M])\left(\frac{\sum_{k=1}^n p_k x_k^p}{\sum_{k=1}^n p_k x_k^{p-1}} - \sum_{k=1}^n p_k x_k\right)^2 \\ &\le \frac{1}{2}B_1(p, [m, M])\sum_{i=1}^n p_i\left(\frac{\sum_{k=1}^n p_k x_k^p}{\sum_{k=1}^n p_k x_k^{p-1}} - x_i\right)^2 \\ &\le \left(\frac{\sum_{k=1}^n p_k x_k^p}{\sum_{k=1}^n p_k x_k^{p-1}}\right)^p - \sum_{k=1}^n p_k x_k^p \\ &\le \frac{1}{2}B_2(p, [m, M])\sum_{i=1}^n p_i\left(\frac{\sum_{k=1}^n p_k x_k^p}{\sum_{k=1}^n p_k x_k^{p-1}} - x_i\right)^2, \end{aligned} \tag{6.18}$$

where $B_1(p, [m, M])$ and $B_2(p, [m, M])$ are defined by (5.1) and (5.2).

References

[1] S. Abramovich, Jensen, Hölder, Minkowski, Jensen-Steffensen and Slater-Pečarić inequalities derived through N-quasiconvexity. *Math. Inequal. Appl.* **19** (2016), no. 4, 1203–1226.

[2] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), No. 3, 551–562. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suplement, Art. 12. [Online http://rgmia.org/papers/v5e/GTIILFApp.pdf].

[3] S. S. Dragomir, *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers, NY, 2004.

[4] S. S. Dragomir, Some Slater type inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Rev. Un. Mat. Argentina* **52** (2011), no. 1, 109–120.

[5] S. S. Dragomir, Some Slater's type inequalities for convex functions defined on linear spaces and applications. *Abstr. Appl. Anal.* **2012**, Art. ID 168405, 16 pp.

[6] S. S. Dragomir, Some Slater's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Toyama Math. J.* **38** (2016), 75–99.
[7] S. S. Dragomir and J. E. Pečarić, A generalization of Slater's inequality. *Zb. Rad.* (Kragujevac) No. **13** (1992), 5–9.
[8] A. M. Khan and J. Pečarić, On Slater's integral inequality. *J. Math. Inequal.* **5** (2011), no. 2, 231–241.
[9] J. Pečarić, A multidimensional generalization of Slater's inequality. *J. Approx. Theory* **44** (1985), no. 3, 292–294.
[10] M. S. Slater, A companion inequality to Jensen's inequality, *J. Approx. Theory*, **32**(1981), 160-166.
[11] G.-S. Yang, and K.-L. Tseng, Some refinements of Slater, Pečarić and Wang-Wang's inequalities. *Math. Inequal. Appl.* **6** (2003), no. 3, 437–443.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.