NEW LOWER AND UPPER BOUNDS FOR THE SLATER'S GAP OF CONVEX FUNCTIONS

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some new lower and upper bounds for the *Slater's gap*

$$\Phi(\sigma) - \int_\Omega \Phi \circ f d\mu,$$

where $\int_\Omega \Phi' \circ f d\mu \neq 0$ and the Slater's point is defined by

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu},$$

and assumed to be in $\mathring{I}$, while Φ is a twice differentiable convex function on the interval I. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x - a)\varphi(a) \text{ for any } x,\ a \in I.$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

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Assume, for simplicity, that $\int_{\Omega} d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w d\mu$ instead of $\int_{\Omega} w(t)\, d\mu(t)$.

The following result is well known in the literature as *Slater's inequality:*

Theorem 1 (Slater, 1981, [11])**.** *If $\Phi : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $\varphi \in \partial\Phi$ and $f : \Omega \to I$ is μ-measurable and such that $\Phi \circ f$, $\varphi \circ f$, $f \cdot \varphi \circ f \in L(\Omega, \mu)$ then*

$$\int_{\Omega} \Phi \circ f d\mu \le \Phi\left(\frac{\int_{\Omega} f \cdot (\varphi \circ f)\, d\mu}{\int_{\Omega} \varphi \circ f d\mu}\right). \tag{1.1}$$

As pointed out in [10], see also [3, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\int_{\Omega} f \cdot (\varphi \circ f)\, d\mu}{\int_{\Omega} \varphi \circ f d\mu} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

In the recent paper [7] we obtained between others the following result:

Theorem 2. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ (the interior of I) and $f : \Omega \to I$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. If $\int_{\Omega} \Phi' \circ f d\mu \neq 0$, the Slater's point*

$$\sigma := \frac{\int_{\Omega} f \cdot (\Phi' \circ f)\, d\mu}{\int_{\Omega} \Phi' \circ f d\mu} \in \mathring{I},$$

and if there exist $0 < \gamma < \Gamma < \infty$ such that

$$\gamma \le \Phi''(t) \le \Gamma \text{ for almost every } t \in \mathring{I}, \tag{1.3}$$

then

$$\begin{aligned} 0 &\le \frac{1}{2}\gamma\left(\sigma - \int_{\Omega} f d\mu\right)^2 \le \frac{1}{2}\gamma \int_{\Omega} (\sigma - f)^2\, d\mu \\ &\le \Phi(\sigma) - \int_{\Omega} \Phi \circ f d\mu \le \frac{1}{2}\Gamma \int_{\Omega} (\sigma - f)^2\, d\mu. \end{aligned} \tag{1.4}$$

In this paper we establish some new lower and upper bounds for the *Slater's gap*

$$\Phi(\sigma) - \int_{\Omega} \Phi \circ f d\mu,$$

in the case that Φ is a twice differentiable convex function on the interval I. Applications for exponential, logarithm and power functions are also given.

For some related results, see [1], [8], [9] and [12]. Extensions for continuous functions of selfadjoint operators may be found in [4]-[6].

2. Main Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $g : I \longrightarrow \mathbb{C}$ is such that the n-derivative $g^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$g(x) = T_n(g; a, x) + R_n(g; a, x), \tag{2.1}$$

where $T_n(g; c, y)$ is Taylor's polynomial, i.e.,

$$T_n(g; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ and $0! := 1$ and the remainder is given by

$$R_n(g; a, x) := \frac{1}{n!} \int_a^x (x-t)^n g^{(n+1)}(t)\, dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\, dt = (d-c) \int_0^1 h((1-s)c + sd)\, ds.$$

Therefore,

$$\begin{aligned} &\int_a^x g^{(n+1)}(t)(x-t)^n\, dt \\ &= (x-a) \int_0^1 g^{(n+1)}((1-s)a + sx)(x - (1-s)a - sx)^n\, ds \\ &= (x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds. \end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned} g(x) = &\sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(a)(x-a)^k \\ &+ \frac{1}{n!}(x-a)^{n+1} \int_0^1 g^{(n+1)}((1-s)a + sx)(1-s)^n\, ds \end{aligned} \tag{2.4}$$

for all x, $a \in I$.

We have the following refinement and reverse of Slater's inequality for twice differentiable convex functions:

Theorem 3. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ (the interior of I) and $f : \Omega \to I$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. If $\int_\Omega \Phi' \circ f d\mu \neq 0$ and the Slater's point*

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in \mathring{I}, \tag{2.5}$$

then we have the inequality:

$$0 \le \operatorname*{essinf}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu \tag{2.6}$$
$$\le \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu$$
$$\le \operatorname*{essup}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu.$$

Proof. We have from (2.4) for $n = 2$ that

$$\Phi(x) = \Phi(c) + \Phi'(c)(x-c) + (x-c)^2 \int_0^1 \Phi''((1-s)c + sx)(1-s)\, ds \tag{2.7}$$

for all $x, c \in I$.

If we take in (2.7) $c = f(t)$, $t \in \Omega$, then we get

$$\Phi(x) = \Phi(f(t)) + \Phi'(f(t))(x - f(t))$$
$$+ (x - f(t))^2 \int_0^1 \Phi''((1-s) f(t) + sx)(1-s)\, ds$$

for all $x \in I$.

Now, if we integrate on Ω, then we get

$$\Phi(x) = \int_\Omega \Phi \circ f d\mu + x \int_\Omega \Phi' \circ f d\mu - \int_\Omega f \cdot \Phi' \circ f d\mu \tag{2.8}$$
$$+ \int_\Omega (x-f)^2 \left(\int_0^1 \Phi''((1-s) f + sx)(1-s)\, ds \right) d\mu$$

for all $x \in I$, which is an equality of interest in itself.

Now, if we take in (2.8) $x = \sigma$, then we get the identity

$$\Phi(\sigma) = \int_\Omega \Phi \circ f d\mu + \sigma \int_\Omega \Phi' \circ f d\mu - \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$+ \int_\Omega (\sigma - f)^2 \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu$$
$$= \int_\Omega \Phi \circ f d\mu + \int_\Omega (\sigma - f)^2 \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu,$$

namely

$$\Phi(\sigma) - \int_\Omega \Phi \circ f d\mu = \int_\Omega (\sigma - f)^2 \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu. \tag{2.9}$$

Since

$$\operatorname*{essinf}_{t\in\Omega} (\sigma - f(t))^2 \le (\sigma - f(t))^2 \le \operatorname*{essup}_{t\in\Omega} (\sigma - f(t))^2$$

for μ-a.e. $t \in \Omega$, hence

$$\begin{aligned}
&\operatorname*{essinf}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu \\
&\leq \int_\Omega (\sigma - f)^2 \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu \\
&\leq \operatorname*{essup}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \left(\int_0^1 \Phi''((1-s) f + s\sigma)(1-s)\, ds \right) d\mu
\end{aligned}$$

and by the identity (2.9) we deduce the desired result (2.6). □

Corollary 1. *With the assumptions of Theorem 3 and if* $\Phi''(\cdot)$ *is monotonic nondecreasing on* $(m, M) \subset \mathring{I}$ *and the Slater's point*

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in (m, M), \tag{2.10}$$

then we have the inequality:

$$\begin{aligned}
0 &\leq \operatorname*{essinf}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \frac{1}{f-m} \left\{ \Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f-m} \right\} d\mu \\
&\leq \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu \\
&\leq \operatorname*{essup}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \frac{1}{M-f} \left\{ \frac{\Phi(M) - \Phi \circ f}{M-f} - \Phi' \circ f \right\} d\mu.
\end{aligned} \tag{2.11}$$

If $\Phi''(\cdot)$ *is monotonic nonincreasing on* (m, M)*, then*

$$\begin{aligned}
0 &\leq \operatorname*{essinf}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \frac{1}{M-f} \left\{ \frac{\Phi(M) - \Phi \circ f}{M-f} - \Phi' \circ f \right\} d\mu \\
&\leq \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu \\
&\leq \operatorname*{essup}_{t\in\Omega} (\sigma - f(t))^2 \int_\Omega \frac{1}{f-m} \left\{ \Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f-m} \right\} d\mu.
\end{aligned} \tag{2.12}$$

Proof. Since $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then

$$\Phi''((1-s) f(t) + sm) \leq \Phi''((1-s) f(t) + s\sigma) \leq \Phi''((1-s) f(t) + sM)$$

for all $s \in (0,1)$ and μ-a.e. $t \in \Omega$.

This implies that

$$\begin{aligned}
\int_0^1 \Phi''((1-s) f(t) + sm)\, ds &\leq \int_0^1 \Phi''((1-s) f(t) + s\sigma)\, ds \\
&\leq \int_0^1 \Phi''((1-s) f(t) + sM)\, ds
\end{aligned}$$

for μ-a.e. $t \in \Omega$.

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)f(t)+sm\right)(1-s)\,ds \\
&= \frac{1}{m-f(t)}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)f(t)+sm\right)\right) \\
&= \frac{1}{m-f(t)}\left\{(1-s)\,\Phi'\left((1-s)f(t)+sm\right)\Big|_0^1\right. \\
&\left.+\int_0^1 \Phi'\left((1-s)f(t)+sm\right)ds\right\} \\
&= \frac{1}{m-f(t)}\left\{-\Phi'(f(t))+\frac{\Phi(m)-\Phi(f(t))}{m-f(t)}\right\} \\
&= \frac{1}{f(t)-m}\left\{\Phi'(f(t))-\frac{\Phi(f(t))-\Phi(m)}{f(t)-m}\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \Phi''\left((1-s)f(t)+sM\right)(1-s)\,ds \\
&= \frac{1}{M-f(t)}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)f(t)+sM\right)\right) \\
&= \frac{1}{M-f(t)}\left\{(1-s)\,\Phi'\left((1-s)f(t)+sM\right)\Big|_0^1\right. \\
&\left.+\int_0^1 \Phi'\left((1-s)f(t)+sM\right)ds\right\} \\
&= \frac{1}{M-f(t)}\left\{-\Phi'(f(t))+\frac{\Phi(f(t))-\Phi(M)}{f(t)-M}\right\} \\
&= \frac{1}{M-f(t)}\left\{\frac{\Phi(M)-\Phi(f(t))}{M-f(t)}-\Phi'(f(t))\right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega \frac{1}{f(t)-m}\left\{\Phi'(f(t))-\frac{\Phi(f(t))-\Phi(m)}{f(t)-m}\right\}d\mu(t) \\
&\leq \int_\Omega\left(\int_0^1 \Phi''\left((1-s)f+s\sigma\right)(1-s)\,ds\right)d\mu \\
&\leq \int_\Omega \frac{1}{M-f(t)}\left\{\frac{\Phi(M)-\Phi(f(t))}{M-f(t)}-\Phi'(f(t))\right\}d\mu(t)
\end{aligned}$$

and by (2.6) we get (2.11).

The inequality (2.12) follows in a similar way. □

Corollary 2. *With the assumptions of Theorem 3 and if* $\Phi''(\cdot)$ *is convex on* (m,M)*, then*

$$\begin{aligned}
0 &\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}(\sigma-f(t))^2\,\Phi''\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}\sigma\right) \\
&\le \operatorname*{essinf}_{t\in\Omega}(\sigma-f(t))^2\int_0^1\Phi''\left((1-s)\int_\Omega f d\mu+s\sigma\right)(1-s)\,ds \\
&\le \Phi(\sigma)-\int_\Omega\Phi\circ f d\mu \\
&\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}(\sigma-f(t))^2\left[\frac{2}{3}\int_\Omega\Phi''\circ f d\mu+\frac{1}{3}\Phi''(\sigma)\right].
\end{aligned}\tag{2.13}$$

If $\Phi''(\cdot)$ *is concave on* (m,M)*, then*

$$\begin{aligned}
0 &\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}(\sigma-f(t))^2\left[\frac{2}{3}\int_\Omega\Phi''\circ f d\mu+\frac{1}{3}\Phi''(\sigma)\right] \\
&\le \Phi(\sigma)-\int_\Omega\Phi\circ f d\mu \\
&\le \operatorname*{essup}_{t\in\Omega}(\sigma-f(t))^2\int_0^1\Phi''\left((1-s)\int_\Omega f d\mu+s\sigma\right)(1-s)\,ds \\
&\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}(\sigma-f(t))^2\,\Phi''\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}\sigma\right).
\end{aligned}\tag{2.14}$$

Proof. By the convexity of $\Phi''(\cdot)$ and Jensen's inequality applied twice we have

$$\begin{aligned}
&\int_\Omega\left(\int_0^1\Phi''((1-s)f+s\sigma)(1-s)\,ds\right)d\mu \\
&=\int_0^1\left(\int_\Omega\Phi''((1-s)f+s\sigma)\,d\mu\right)(1-s)\,ds \\
&\ge\int_0^1\Phi''\left((1-s)\int_\Omega f d\mu+s\sigma\right)(1-s)\,ds \\
&\ge\int_0^1(1-s)\,ds\Phi''\left(\frac{\int_0^1\left[(1-s)\int_\Omega f d\mu+s\sigma\right](1-s)\,ds}{\int_0^1(1-s)\,ds}\right) \\
&=\frac{1}{2}\Phi''\left(\frac{\int_\Omega f d\mu\int_0^1(1-s)^2\,ds+\sigma\int_0^1 s(1-s)\,ds}{\frac{1}{2}}\right) \\
&=\frac{1}{2}\Phi''\left(\frac{2}{3}\int_\Omega f d\mu+\frac{1}{3}\sigma\right),
\end{aligned}$$

which proves the first part of (2.13).

By the convexity of Φ'' we also have

$$\begin{aligned}
&\int_\Omega\left(\int_0^1\Phi''((1-s)f+s\sigma)(1-s)\,ds\right)d\mu \\
&\le\int_\Omega\left(\int_0^1(1-s)\Phi''\circ f+s\Phi''(\sigma)(1-s)\,ds\right)d\mu \\
&=\frac{1}{3}\int_\Omega\Phi''\circ f d\mu+\frac{1}{6}\Phi''(\sigma)=\frac{1}{2}\left[\frac{2}{3}\int_\Omega\Phi''\circ f d\mu+\frac{1}{3}\Phi''(\sigma)\right],
\end{aligned}$$

which proves the second part of (2.13).

In the case when $\Phi''(\cdot)$ is concave on (m,M), the proof goes in a similar way and we omit the details. □

We recall that a function $f:I\to\mathbb{R}$ is called *quasiconvex* on the interval I if

$$f((1-s)x+sy)\le\max\{f(x),f(y)\}$$

for all x, $y\in I$ and $s\in[0,1]$.

Corollary 3. *With the assumptions of Theorem 3 and if $\Phi''(\cdot)$ is quasiconvex on (m,M), then*

$$\begin{aligned}(2.15)\qquad &\Phi(\sigma)-\int_\Omega\Phi\circ f d\mu\\ &\le\frac{1}{4}\operatorname*{essup}_{t\in\Omega}(f(t)-\sigma)^2\\ &\times\left[\Phi''(\sigma)+\int_\Omega\Phi''\circ f d\mu+\int_\Omega|\Phi''(f(t))-\Phi''(\sigma)|\,d\mu(t)\right].\end{aligned}$$

Proof. Since $\Phi''(\cdot)$ is quasiconvex, hence

$$\begin{aligned}&\int_0^1\Phi''((1-s)f+s\sigma)(1-s)\,ds\\ &\le\max\{\Phi''(\sigma),\Phi''(f(t))\}\int_0^1(1-s)\,ds=\frac{1}{2}\max\{\Phi''(\sigma),\Phi''(f(t))\}\\ &=\frac{1}{4}[\Phi''(\sigma)+\Phi''(f(t))+|\Phi''(f(t))-\Phi''(\sigma)|]\end{aligned}$$

for μ-a.e. $t\in\Omega$.

Taking the integral $\int_\Omega$ in this inequality, we get

$$\begin{aligned}&\int_\Omega\left(\int_0^1\Phi''((1-s)\sigma+sf(t))(1-s)\,ds\right)d\mu(t)\\ &\le\frac{1}{4}\int_\Omega[\Phi''(\sigma)+\Phi''(f(t))+|\Phi''(f(t))-\Phi''(\sigma)|]\,d\mu(t)\\ &=\frac{1}{4}\left[\Phi''(\sigma)+\int_\Omega\Phi''(f(t))\,d\mu(t)\right]+\frac{1}{4}\int_\Omega|\Phi''(f(t))-\Phi''(\sigma)|\,d\mu(t),\end{aligned}$$

which together with (2.6), produce the desired result (2.15). □

3. Discrete Case

The discrete case is useful for applications and we state some examples here.

In this section the function $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable convex function on (m,M) and $x_k\in[m,M]$, $p_k\ge0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^n p_k=1$.

If $\sum_{k=1}^n p_k\Phi'(x_k)\ne0$ and the Slater's point

$$(3.1)\qquad \sigma:=\frac{\sum_{k=1}^n p_k x_k\Phi'(x_k)}{\sum_{k=1}^n p_k\Phi'(x_k)}\in(m,M).$$

By (2.6) we get

$$\begin{aligned}
0 &\le \min_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} p_k \left(\int_0^1 \Phi''\left((1-s)\, x_k + s\sigma\right)(1-s)\, ds \right) \\
&\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\
&\le \max_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} p_k \left(\int_0^1 \Phi''\left((1-s)\, x_k + s\sigma\right)(1-s)\, ds \right).
\end{aligned} \tag{3.2}$$

If $\Phi''(\cdot)$ is monotonic nondecreasing on (m, M), then we have the inequality:

$$\begin{aligned}
0 &\le \min_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{x_k - m} \left\{ \Phi'(x_k) - \frac{\Phi(x_k) - \Phi(m)}{x_k - m} \right\} \\
&\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\
&\le \max_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{M - x_k} \left\{ \frac{\Phi(M) - \Phi(x_k)}{M - x_k} - \Phi'(x_k) \right\}.
\end{aligned} \tag{3.3}$$

If $\Phi''(\cdot)$ is monotonic nonincreasing on (m, M), then

$$\begin{aligned}
0 &\le \min_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{M - x_k} \left\{ \frac{\Phi(M) - \Phi(x_k)}{M - x_k} - \Phi'(x_k) \right\} \\
&\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\
&\le \max_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{x_k - m} \left\{ \Phi'(x_k) - \frac{\Phi(x_k) - \Phi(m)}{x_k - m} \right\}.
\end{aligned} \tag{3.4}$$

If $\Phi''(\cdot)$ is convex on (m, M), then

$$\begin{aligned}
0 &\le \frac{1}{2} \min_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \Phi'' \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3}\sigma \right) \\
&\le \min_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \int_0^1 \Phi'' \left((1-s) \sum_{i=1}^{n} p_i x_i + s\sigma \right) (1-s)\, ds \\
&\le \Phi(\sigma) - \sum_{k=1}^{n} p_k \Phi(x_k) \\
&\le \frac{1}{2} \max_{k\in\{1,\ldots,n\}} (\sigma - x_k)^2 \left[\frac{2}{3} \sum_{k=1}^{n} p_k \Phi''(x_k) + \frac{1}{3} \Phi''(\sigma) \right].
\end{aligned} \tag{3.5}$$

If $\Phi''(\cdot)$ is concave on (m,M), then

$$0 \le \frac{1}{2}\min_{k\in\{1,\dots,n\}}(\sigma - x_k)^2\left(\frac{2}{3}\sum_{k=1}^n p_k\Phi''(x_k)+\frac{1}{3}\Phi''(\sigma)\right) \tag{3.6}$$
$$\le \Phi(\sigma)-\sum_{k=1}^n p_k\Phi(x_k)$$
$$\le \max_{k\in\{1,\dots,n\}}(\sigma - x_k)^2\int_0^1\Phi''\left((1-s)\sum_{i=1}^n p_ix_i + s\sigma\right)(1-s)\,ds$$
$$\le \frac{1}{2}\max_{k\in\{1,\dots,n\}}(\sigma - x_k)^2\,\Phi''\left(\frac{2}{3}\sum_{i=1}^n p_ix_i+\frac{1}{3}\sigma\right).$$

4. Some Examples

We consider the exponential function, $\Phi(x)=\exp x$, $x\in[m,M]$. If $x_k\in[m,M]$, $p_k\ge 0$ for $k\in\{1,\dots,n\}$ with $\sum_{k=1}^n p_k = 1$. Put

$$\sigma_e := \frac{\sum_{k=1}^n p_kx_k\exp(x_k)}{\sum_{k=1}^n p_k\exp(x_k)}.$$

Then

$$0\le \min_{k\in\{1,\dots,n\}}(\sigma_e - x_k)^2\sum_{k=1}^n\frac{p_k}{x_k-m}\left\{\exp(x_k)-\frac{\exp(x_k)-\exp(m)}{x_k-m}\right\} \tag{4.1}$$
$$\le \exp(\sigma_e)-\sum_{k=1}^n p_k\exp(x_k)$$
$$\le \max_{k\in\{1,\dots,n\}}(\sigma_e - x_k)^2\sum_{k=1}^n\frac{p_k}{M-x_k}\left\{\frac{\exp(M)-\exp(x_k)}{M-x_k}-\exp(x_k)\right\}$$

and

$$0\le\frac{1}{2}\min_{k\in\{1,\dots,n\}}(\sigma_e - x_k)^2\exp\left(\frac{2}{3}\sum_{i=1}^n p_ix_i+\frac{1}{3}\sigma\right) \tag{4.2}$$
$$\le \exp\left(\frac{\sum_{k=1}^n p_kx_k\exp(x_k)}{\sum_{k=1}^n p_k\exp(x_k)}\right)-\sum_{k=1}^n p_k\exp(x_k)$$
$$\le \frac{1}{2}\max_{k\in\{1,\dots,n\}}(\sigma_e - x_k)^2\left[\frac{2}{3}\sum_{k=1}^n p_k\exp(x_k)+\frac{1}{3}\exp(\sigma_e)\right].$$

If we consider the logarithmic function $\Phi(x)=-\ln x$, $x\in[m,M]\subset(0,\infty)$, and $x_k\in[m,M]$, $p_k\ge 0$ for $k\in\{1,\dots,n\}$ with $\sum_{k=1}^n p_k=1$, and if we consider

$$\sigma_\ell := \frac{1}{\sum_{k=1}^n\frac{p_k}{x_k}},$$

then

$$0 \leq \min_{k\in\{1,\dots,n\}} (\sigma_\ell - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{M - x_k} \left\{ \frac{1}{x_k} - \frac{\ln(M) - \ln(x_k)}{M - x_k} \right\} \tag{4.3}$$
$$\leq \sum_{k=1}^{n} p_k \ln(x_k) - \ln\left(\frac{1}{\sum_{k=1}^{n} \frac{p_k}{x_k}} \right)$$
$$\leq \max_{k\in\{1,\dots,n\}} (\sigma_\ell - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{x_k - m} \left\{ \frac{\ln(x_k) - \ln(m)}{x_k - m} - \frac{1}{x_k} \right\}$$

and

$$0 \leq \frac{1}{2} \min_{k\in\{1,\dots,n\}} (\sigma_\ell - x_k)^2 \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3}\sigma \right)^{-2} \tag{4.4}$$
$$\leq \sum_{k=1}^{n} p_k \ln(x_k) - \ln\left(\frac{1}{\sum_{k=1}^{n} \frac{p_k}{x_k}} \right)$$
$$\leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} (\sigma_\ell - x_k)^2 \left[\frac{2}{3} \sum_{k=1}^{n} p_k x_k^{-2} + \frac{1}{3}\sigma^{-2} \right].$$

The power function $\Phi_p : (0,\infty) \to (0,\infty)$, $\Phi_p(x) = x^p$ is convex for $p \in (-\infty, 0) \cup [1,\infty)$. We have $\Phi_p''(x) = p(p-1)x^{p-2}$, $x \in (0,\infty)$.

The Slater's point for this function is

$$\sigma_p := \frac{\sum_{k=1}^{n} p_k x_k^p}{\sum_{k=1}^{n} p_k x_k^{p-1}}. \tag{4.5}$$

For $p \geq 2$, $\Phi_p''(\cdot)$ is increasing and by (3.3) we get

$$0 \leq \min_{k\in\{1,\dots,n\}} (\sigma_p - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{x_k - m} \left(p x_k^{p-1} - \frac{x_k^p - m^p}{x_k - m} \right) \tag{4.6}$$
$$\leq \left(\frac{\sum_{k=1}^{n} p_k x_k^p}{\sum_{k=1}^{n} p_k x_k^{p-1}} \right)^p - \sum_{k=1}^{n} p_k x_k^p$$
$$\leq \max_{k\in\{1,\dots,n\}} (\sigma_p - x_k)^2 \sum_{k=1}^{n} \frac{p_k}{M - x_k} \left(\frac{M^p - x_k^p}{M - x_k} - p x_k^{p-1} \right).$$

For $p \geq 3$, $\Phi_p''(\cdot)$ is convex and by (3.5) we get

$$0 \leq \frac{1}{2} p(p-1) \min_{k\in\{1,\dots,n\}} (\sigma_p - x_k)^2 \left(\frac{2}{3} \sum_{i=1}^{n} p_i x_i + \frac{1}{3}\sigma_p \right)^{p-2} \tag{4.7}$$
$$\leq \left(\frac{\sum_{k=1}^{n} p_k x_k^p}{\sum_{k=1}^{n} p_k x_k^{p-1}} \right)^p - \sum_{k=1}^{n} p_k x_k^p$$
$$\leq \frac{1}{2} p(p-1) \max_{k\in\{1,\dots,n\}} (\sigma_p - x_k)^2 \left(\frac{2}{3} \sum_{k=1}^{n} p_k x_k^{p-2} + \frac{1}{3}\sigma_p^{p-2} \right).$$

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