SOME UPPER BOUNDS FOR THE PERTURBED SLATER'S GAP OF CONVEX FUNCTIONS

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some upper bounds for the perturbed Slater's gap

$$\Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f) d\mu$$

for some classes of differentiable convex functions Φ defined on an interval I and $u \in I$. Applications for exponential, logarithm and power functions are also given.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \leq \Phi'_+(x) \leq \Phi'_-(y) \leq \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \geq \Phi(a) + (x - a)\varphi(a) \text{ for any } x, \ a \in I.$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \leq \varphi(x) \leq \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R}, \ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) d\mu(t)$.

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The following result is well known in the literature as *Slater's inequality:*

Theorem 1 (Slater, 1981, [10]). *If $\Phi : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $\varphi \in \partial\Phi$ and $f : \Omega \to I$ is μ-measurable and such that $\Phi \circ f$, $\varphi \circ f$, $f \cdot \varphi \circ f \in L(\Omega, \mu)$ then*

$$\int_\Omega \Phi \circ f d\mu \le \Phi\left(\frac{\int_\Omega f \cdot (\varphi \circ f)\, d\mu}{\int_\Omega \varphi \circ f d\mu}\right). \tag{1.1}$$

As pointed out in [9], see also [3, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\int_\Omega f \cdot (\varphi \circ f)\, d\mu}{\int_\Omega \varphi \circ f d\mu} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

For some related results, see [1], [7], [8] and [11]. Extensions for continuous functions of selfadjoint operators may be found in [4]-[6].

In this paper we establish some upper bounds for the perturbed Slater's gap

$$\Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f)\, d\mu$$

for some classes of differentiable convex functions Φ defined on an interval I and $u \in I$. Applications for exponential, logarithm and power functions are also given.

2. Inequalities Related to Slater's Result

We have:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior of I denoted by $\mathring{I}$. Assume that $f : \Omega \to I$ is μ-measurable and such that f, $\Phi \circ f$, $\Phi' \circ f$, $f \cdot \Phi' \circ f \in L(\Omega, \mu)$. Then for all $u \in I$,*

$$\begin{aligned} 0 &\le \Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f)\, d\mu \\ &\le \int_\Omega [\Phi' \circ f - \Phi'(u)](f - u)\, d\mu \\ &\le \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(u)| \int_\Omega |f - u|\, d\mu, \\[2mm] \left(\int_\Omega |\Phi' \circ f - \Phi'(u)|^p d\mu\right)^{1/p} \left(\int_\Omega |f - u|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\[2mm] \operatorname{essup}_\Omega |f - u| \int_\Omega |\Phi' \circ f - \Phi'(u)|\, d\mu, \end{cases} \end{aligned} \tag{2.1}$$

provided the integrals in the last term are finite.

Proof. By the gradient inequality we have

$$\Phi'(u)(u - v) \ge \Phi(u) - \Phi(v) \ge \Phi'(v)(u - v)$$

for all u, $v \in I$. This can be written as

$$\Phi'(u)(u - v) - \Phi'(v)(u - v) \ge \Phi(u) - \Phi(v) - \Phi'(v)(u - v) \ge 0,$$

which implies that

$$[\Phi'(u) - \Phi'(f(x))](u - f(x)) \geq \Phi(u) - \Phi(f(x)) - \Phi'(f(x))(u - f(x)) \geq 0, \tag{2.2}$$

for all $u \in I$ and $x \in \Omega$.

Taking the integral over $x \in \Omega$ in (2.2) we get

$$\int_\Omega [\Phi'(u) - \Phi' \circ f](u - f)\, d\mu \geq \Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f)\, d\mu \geq 0,$$

for all $u \in I$, which proves the first two inequalities in (2.1).

Observe that

$$\begin{aligned} 0 &\leq \int_\Omega [\Phi' \circ f - \Phi'(u)](f - u)\, d\mu = \left| \int_\Omega [\Phi' \circ f - \Phi'(u)](f - u)\, d\mu \right| \\ &\leq \int_\Omega |\Phi' \circ f - \Phi'(u)|\, |f - u|\, d\mu \\ &\leq \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(u)| \int_\Omega |f - u|\, d\mu, \\ \left(\int_\Omega |\Phi' \circ f - \Phi'(u)|^p d\mu\right)^{1/p} \left(\int_\Omega |f - u|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \operatorname{essup}_\Omega |f - u| \int_\Omega |\Phi' \circ f - \Phi'(u)|\, d\mu, \end{cases} \end{aligned}$$

which proves the last part of (2.1). □

We have the following reverse of perturbed Jensen's inequality, see also [2] for the first inequality below:

Corollary 1. *With the assumptions of Theorem 2, we have*

$$\begin{aligned} 0 &\leq \Phi\left(\int_\Omega f d\mu\right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f)\, d\mu \\ &\leq \int_\Omega \left[\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right]\left(f - \int_\Omega f d\mu\right) d\mu \\ &\leq \begin{cases} \operatorname{essup}_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right| \int_\Omega \left|f - \int_\Omega f d\mu\right| d\mu, \\ \left(\int_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right|^p d\mu\right)^{1/p} \left(\int_\Omega \left|f - \int_\Omega f d\mu\right|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \textit{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \operatorname{essup}_\Omega \left|f - \int_\Omega f d\mu\right| \int_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right| d\mu. \end{cases} \end{aligned} \tag{2.3}$$

We have the following reverse of perturbed Slater's inequality:

Corollary 2. *With the assumptions of Theorem 2 and if* $\int_\Omega \Phi' \circ f d\mu \neq 0$ *and the Slater's point*

$$s := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in I, \tag{2.4}$$

then

$$0 \leq \Phi(s) - \int_\Omega \Phi \circ f d\mu \leq \int_\Omega [\Phi' \circ f - \Phi'(s)](f - s) d\mu \tag{2.5}$$
$$\leq \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(s)| \int_\Omega |f - s| d\mu, \\ \\ \left(\int_\Omega |\Phi' \circ f - \Phi'(s)|^p d\mu\right)^{1/p} \left(\int_\Omega |f - s|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega |f - s| \int_\Omega |\Phi' \circ f - \Phi'(s)| d\mu. \end{cases}$$

Remark 1. *Since for the Slater point s we have*

$$\begin{aligned} &\int_\Omega [\Phi' \circ f - \Phi'(s)](f - s) d\mu \\ &= \int_\Omega f\Phi' \circ f - \Phi'(s) \int_\Omega f d\mu - s \int_\Omega \Phi' \circ f + \Phi'(s) s \\ &= \Phi'(s)\left(s - \int_\Omega f d\mu\right), \end{aligned}$$

then (2.5) can also be written as

$$0 \leq \Phi(s) - \int_\Omega \Phi \circ f d\mu \leq \Phi'(s)\left(s - \int_\Omega f d\mu\right) \tag{2.6}$$
$$\leq \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(s)| \int_\Omega |f - s| d\mu, \\ \\ \left(\int_\Omega |\Phi' \circ f - \Phi'(s)|^p d\mu\right)^{1/p} \left(\int_\Omega |f - s|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega |f - s| \int_\Omega |\Phi' \circ f - \Phi'(s)| d\mu. \end{cases}$$

Corollary 3. *With the assumptions of Theorem 2 and if there exists an interval $[m, M] \subset I$ such that $m \leq f \leq M$ μ-a.e. on Ω, then*

$$0 \leq \Phi\left(\frac{m+M}{2}\right) - \int_\Omega \Phi \circ f d\mu - \frac{m+M}{2} \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f) d\mu \tag{2.7}$$
$$\leq \int_\Omega \left[\Phi' \circ f - \Phi'\left(\frac{m+M}{2}\right)\right]\left(f - \frac{m+M}{2}\right) d\mu$$
$$\leq \begin{cases} \operatorname{essup}_\Omega \left|\Phi' \circ f - \Phi'\left(\frac{m+M}{2}\right)\right| \int_\Omega \left|f - \frac{m+M}{2}\right| d\mu, \\ \\ \left(\int_\Omega \left|\Phi' \circ f - \Phi'\left(\frac{m+M}{2}\right)\right|^p d\mu\right)^{1/p} \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega \left|\Phi' \circ f - \Phi'\left(\frac{m+M}{2}\right)\right| d\mu. \end{cases}$$

Remark 2. *Since $m \leq f \leq M$ μ-a.e. on Ω is equivalent to*

$$\left|f - \frac{m+M}{2}\right| \leq \frac{1}{2}(M - m) \ \ \mu\text{-a.e.on } \Omega,$$

then from the third branch of (2.7) we obtain the following inequality of interest

$$\begin{aligned} (2.8)\quad 0 &\leq \Phi\left(\frac{m+M}{2}\right)-\int_{\Omega}\Phi\circ f d\mu-\frac{m+M}{2}\int_{\Omega}\Phi'\circ f d\mu+\int_{\Omega}f\cdot(\Phi'\circ f)\,d\mu \\ &\leq \frac{1}{2}(M-m)\int_{\Omega}\left|\Phi'\circ f-\Phi'\left(\frac{m+M}{2}\right)\right|d\mu. \end{aligned}$$

Corollary 4. *With the assumptions of Corollary 4 we have*

$$\begin{aligned} (2.9)\quad 0 &\leq \frac{1}{M-m}\int_m^M\Phi(u)\,du-\int_{\Omega}\Phi\circ f d\mu \\ &-\frac{m+M}{2}\int_{\Omega}\Phi'\circ f d\mu+\int_{\Omega}f\cdot(\Phi'\circ f)\,d\mu \\ &\leq \frac{1}{M-m}\int_m^M\int_{\Omega}\left[\Phi'\circ f-\Phi'(u)\right](f-u)\,d\mu \\ &\leq \begin{cases} \frac{1}{M-m}\int_m^M\left(\operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|\int_{\Omega}|f-u|\,d\mu\right)du, \\ \\ \frac{1}{M-m}\int_m^M\left[\left(\int_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|^p d\mu\right)^{1/p}\left(\int_{\Omega}|f-u|^q d\mu\right)^{1/q}\right]du, \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{M-m}\int_m^M\left[\operatorname{essup}_{\Omega}|f-u|\int_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|d\mu\right]du, \end{cases} \\ &\leq \begin{cases} \sup_{u\in[m,M]}\left(\operatorname{essup}_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|\right) \\ \times\frac{1}{M-m}\int_m^M\int_{\Omega}|f-u|\,d\mu du, \\ \\ \left(\frac{1}{M-m}\int_m^M\int_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|^p d\mu du\right)^{1/p} \\ \times\left(\frac{1}{M-m}\int_m^M\int_{\Omega}|f-u|^q d\mu du\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \sup_{u\in[m,M]}\left(\operatorname{essup}_{\Omega}|f-u|\right) \\ \times\frac{1}{M-m}\int_m^M\int_{\Omega}\left|\Phi'\circ f-\Phi'(u)\right|d\mu du. \end{cases} \end{aligned}$$

The proof follows by taking the integral mean $\frac{1}{M-m}\int_m^M$ in (2.1) and making use of the Hölder's integral inequality.

Corollary 5. *With the assumptions of Corollary 4 and if there exists $t\in I$, a trapezoid point, such that*

$$(2.10)\qquad \Phi(t)=\frac{\Phi(m)+\Phi(M)}{2},$$

then

$$0 \le \frac{\Phi(m)+\Phi(M)}{2} - \int_\Omega \Phi\circ f d\mu - t\int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot(\Phi'\circ f)\,d\mu \tag{2.11}$$
$$\le \int_\Omega [\Phi'\circ f - \Phi'(t)](f-t)\,d\mu$$
$$\le \begin{cases} \operatorname{essup}_\Omega |\Phi'\circ f - \Phi'(t)| \int_\Omega |f-t|\,d\mu, \\ \\ \left(\int_\Omega |\Phi'\circ f - \Phi'(t)|^p d\mu\right)^{1/p} \left(\int_\Omega |f-t|^q d\mu\right)^{1/q}, \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega |f-t| \int_\Omega |\Phi'\circ f - \Phi'(t)|\,d\mu. \end{cases}$$

Remark 3. *If the function Φ is strictly monotonic on $[m,M]$, then we can take*

$$t = \Phi^{-1}\left(\frac{\Phi(m)+\Phi(M)}{2}\right).$$

When more conditions are imposed on the derivative, we have:

Corollary 6. *With the assumptions of Theorem 2 and if Φ' is Lipschitzian with the constant $L>0$ on $\mathring{I}$, namely*

$$|\Phi'(u)-\Phi'(v)| \le L|u-v| \text{ for all } u,\ v\in \mathring{I},$$

then

$$0 \le \Phi(u) - \int_\Omega \Phi\circ f d\mu - u\int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot(\Phi'\circ f)\,d\mu \tag{2.12}$$
$$\le L\int_\Omega (f-u)^2 d\mu = L\left(\int_\Omega f^2 d\mu - 2u\int_\Omega f d\mu + u^2\right).$$

In particular, we have the following upper bound in terms of variance

$$0 \le \Phi\left(\int_\Omega f d\mu\right) - \int_\Omega \Phi\circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot(\Phi'\circ f)\,d\mu \tag{2.13}$$
$$\le L\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].$$

If, in addition, the condition (2.4) is satisfied, then

$$0 \le \Phi(s) - \int_\Omega \Phi\circ f d\mu \le L\left(\int_\Omega f^2 d\mu - 2s\int_\Omega f d\mu + s^2\right). \tag{2.14}$$

If there exists an interval $[m,M]\subset I$ such that $m\le f\le M$ μ-a.e. on Ω, then

$$0 \le \Phi\left(\frac{m+M}{2}\right) - \int_\Omega \Phi\circ f d\mu \tag{2.15}$$
$$- \frac{m+M}{2}\int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot(\Phi'\circ f)\,d\mu$$
$$\le L\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu \le \frac{1}{4}(M-m)^2.$$

We also have

$$0\le \frac{1}{M-m}\int_m^M \Phi(u)\,du-\int_\Omega \Phi\circ f d\mu -\frac{m+M}{2}\int_\Omega \Phi'\circ f d\mu+\int_\Omega f\cdot(\Phi'\circ f)\,d\mu \le \frac{1}{M-m}\int_m^M\left(\int_\Omega (f-u)^2\,d\mu\right)du = \int_\Omega f^2 d\mu-2\frac{m+M}{2}\int_\Omega f d\mu+\frac{m^2+mM+M^2}{3}. \tag{2.16}$$

If the condition (2.10) is satisfied, then

$$0\le \frac{\Phi(m)+\Phi(M)}{2}-\int_\Omega \Phi\circ f d\mu - t\int_\Omega \Phi'\circ f d\mu+\int_\Omega f\cdot(\Phi'\circ f)\,d\mu \le L\int_\Omega (f-t)^2\,d\mu. \tag{2.17}$$

The discrete case is important for applications.

Let $\Phi : I\to\mathbb{R}$ be a differentiable convex function on the interior of I denoted by $\mathring{I}$. Assume that $x_i\in I$ and $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$. Then for all $u\in I$ we have by (2.1) that

$$0\le \Phi(u)-\sum_{i=1}^n p_i\Phi(x_i)-u\sum_{i=1}^n p_i\Phi'(x_i)+\sum_{i=1}^n p_i x_i\Phi'(x_i) \le \sum_{i=1}^n p_i\left[\Phi'(x_i)-\Phi'(u)\right](x_i-u) \le \begin{cases} \max_{i\in\{1,\dots,n\}}\left|\Phi'(x_i)-\Phi'(u)\right|\sum_{i=1}^n p_i|x_i-u|, \\ \left(\sum_{i=1}^n p_i\left|\Phi'(x_i)-\Phi'(u)\right|^p d\mu\right)^{1/p}\left(\sum_{i=1}^n p_i|x_i-u|^q\right)^{1/q} \\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \max_{i\in\{1,\dots,n\}}|x_i-u|\sum_{i=1}^n p_i\left|\Phi'(x_i)-\Phi'(u)\right|. \end{cases} \tag{2.18}$$

From (2.3) we have

$$0\le \Phi\left(\sum_{i=1}^n p_i x_k\right)-\sum_{i=1}^n p_i\Phi(x_i)-\sum_{i=1}^n p_i x_k\sum_{i=1}^n p_i\Phi'(x_i)+\sum_{i=1}^n p_i x_i\Phi'(x_i) \le \sum_{i=1}^n p_i\left[\Phi'(x_i)-\Phi'\left(\sum_{i=1}^n p_i x_k\right)\right]\left(x_i-\sum_{i=1}^n p_i x_k\right) \le \begin{cases} \max_{i\in\{1,\dots,n\}}\left|\Phi'(x_i)-\Phi'\left(\sum_{i=1}^n p_i x_k\right)\right| \\ \times\sum_{i=1}^n p_i\left|x_i-\sum_{i=1}^n p_i x_k\right|, \\ \left(\sum_{i=1}^n p_i\left|\Phi'(x_i)-\Phi'\left(\sum_{i=1}^n p_i x_k\right)\right|^p\right)^{1/p} \\ \times\left(\sum_{i=1}^n p_i\left|x_i-\sum_{i=1}^n p_i x_k\right|^q\right)^{1/q}, \ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \max_{i\in\{1,\dots,n\}}\left|x_i-\sum_{i=1}^n p_i x_k\right|\sum_{i=1}^n p_i\left|\Phi'(x_i)-\Phi'\left(\sum_{i=1}^n p_i x_k\right)\right|. \end{cases} \tag{2.19}$$

Now, assume that $\sum_{i=1}^{n} p_i \Phi'(x_i) \neq 0$ and

$$s := \frac{\sum_{i=1}^{n} p_i x_i \Phi'(x_i)}{\sum_{i=1}^{n} p_i \Phi'(x_i)} \in I, \tag{2.20}$$

then by (2.5) we get

$$0 \le \Phi(s) - \sum_{i=1}^{n} p_i \Phi(x_i) \le \sum_{i=1}^{n} p_i [\Phi'(x_i) - \Phi'(s)](x_i - s) \tag{2.21}$$
$$\le \begin{cases} \max_{i\in\{1,\dots,n\}} |\Phi'(x_i) - \Phi'(s)| \sum_{i=1}^{n} p_i |x_i - s|, \\ \\ \left(\sum_{i=1}^{n} p_i |\Phi'(x_i) - \Phi'(s)|^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i |x_i - s|^q\right)^{1/q} \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max_{i\in\{1,\dots,n\}} |x_i - s| \sum_{i=1}^{n} p_i |\Phi'(x_i) - \Phi'(s)|. \end{cases}$$

If there exists an interval $[m, M] \subset I$ such that $m \le x_i \le M$ for $i \in \{1, ..., n\}$, then

$$0 \le \Phi\left(\frac{m+M}{2}\right) - \sum_{i=1}^{n} p_i \Phi(x_i) - \frac{m+M}{2} \sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i) \tag{2.22}$$
$$\le \sum_{i=1}^{n} p_i \left[\Phi'(x_i) - \Phi'\left(\frac{m+M}{2}\right)\right]\left(x_i - \frac{m+M}{2}\right)$$
$$\le \begin{cases} \max_{i\in\{1,\dots,n\}} \left|\Phi'(x_i) - \Phi'\left(\frac{m+M}{2}\right)\right| \sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right|, \\ \\ \left(\sum_{i=1}^{n} p_i \left|\Phi'(x_i) - \Phi'\left(\frac{m+M}{2}\right)\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right|^q\right)^{1/q} \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \max_{i\in\{1,\dots,n\}} \left|x_i - \frac{m+M}{2}\right| \sum_{i=1}^{n} p_i \left|\Phi'(x_i) - \Phi'\left(\frac{m+M}{2}\right)\right|. \end{cases}$$

Since $m \le x_i \le M$ for $i \in \{1, ..., n\}$ is equivalent to

$$\left|x_i - \frac{m+M}{2}\right| \le \frac{1}{2}(M - m),\ i \in \{1, ..., n\},$$

then from the third branch of (2.7) we obtain the following inequality of interest

$$0 \le \Phi\left(\frac{m+M}{2}\right) - \sum_{i=1}^{n} p_i \Phi(x_i) - \frac{m+M}{2} \sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i) \tag{2.23}$$
$$\le \frac{1}{2}(M - m) \sum_{i=1}^{n} p_i \left|\Phi'(x_i) - \Phi'\left(\frac{m+M}{2}\right)\right|.$$

If there exists $t \in I$ such that

$$\Phi(t) = \frac{\Phi(m) + \Phi(M)}{2}, \tag{2.24}$$

then

$$0 \leq \frac{\Phi(m)+\Phi(M)}{2} - \sum_{i=1}^{n} p_i \Phi(x_i) - t\sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i) \tag{2.25}$$
$$\leq \sum_{i=1}^{n} p_i \left[\Phi'(x_i) - \Phi'(t)\right](x_i - t)$$
$$\leq \begin{cases} \max_{i\in\{1,...,n\}} \left|\Phi'(x_i) - \Phi'(t)\right| \sum_{i=1}^{n} p_i |x_i - t|, \\ \left(\sum_{i=1}^{n} p_i \left|\Phi'(x_i) - \Phi'(t)\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i |x_i - t|^q\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \max_{i\in\{1,...,n\}} |x_i - t| \sum_{i=1}^{n} p_i \left|\Phi'(x_i) - \Phi'(t)\right|. \end{cases}$$

If Φ' is Lipschitzian with the constant $L > 0$ on $\mathring{I}$, namely

$$\left|\Phi'(u) - \Phi'(v)\right| \leq L|u - v| \text{ for all } u,\ v \in \mathring{I},$$

then

$$0 \leq \Phi(u) - \sum_{i=1}^{n} p_i \Phi(x_i) - u\sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i) \tag{2.26}$$
$$\leq L\sum_{i=1}^{n} p_i (x_i - u)^2 = L\left(\sum_{i=1}^{n} p_i x_i^2 - 2u\sum_{i=1}^{n} p_i x_i + u^2\right).$$

In particular, we have the following upper bound in terms of variance

$$0 \leq \Phi\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \Phi(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i) \tag{2.27}$$
$$\leq L\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right].$$

If, in addition, the condition (2.4) is satisfied, then

$$0 \leq \Phi(s) - \sum_{i=1}^{n} p_i \Phi(x_i) \leq L\left(\sum_{i=1}^{n} p_i x_i^2 - 2s\sum_{i=1}^{n} p_i x_i + s^2\right). \tag{2.28}$$

If there exists an interval $[m, M] \subset I$ such that $m \leq x_i \leq M$ for $i \in \{1, ..., n\}$, then

$$0 \leq \Phi\left(\frac{m+M}{2}\right) - \sum_{i=1}^{n} p_i \Phi(x_i) \tag{2.29}$$
$$- \frac{m+M}{2}\sum_{i=1}^{n} p_i \Phi'(x_i) + \sum_{i=1}^{n} p_i x_i \Phi'(x_i)$$
$$\leq L\sum_{i=1}^{n} p_i \left(x_i - \frac{m+M}{2}\right)^2 d\mu \leq \frac{1}{4}(M - m)^2.$$

We also have

$$0 \leq \frac{1}{M-m} \int_m^M \Phi(u)\,du - \sum_{i=1}^n p_i \Phi(x_i) - \frac{m+M}{2} \sum_{i=1}^n p_i \Phi'(x_i) + \sum_{i=1}^n p_i x_i \Phi'(x_i) \leq \sum_{i=1}^n p_i x_i^2 - 2\frac{m+M}{2} \sum_{i=1}^n p_i x_i + \frac{m^2+mM+M^2}{3}. \tag{2.30}$$

If the condition (2.10) is satisfied, then

$$0 \leq \frac{\Phi(m)+\Phi(M)}{2} - \sum_{i=1}^n p_i \Phi(x_i) - t \sum_{i=1}^n p_i \Phi'(x_i) + \sum_{i=1}^n p_i x_i \Phi'(x_i) \leq L \sum_{i=1}^n p_i (x_i - t)^2. \tag{2.31}$$

3. Some Inequalities for Exponential

If we consider the exponential function $\Phi_\alpha : \mathbb{R} \to (0,\infty)$, $\Phi_\alpha(x) = \exp(\alpha x)$, $\alpha \neq 0$, then by (2.1) we get

$$0 \leq \exp(\alpha u) - \int_\Omega (1 + \alpha u - \alpha f) \exp(\alpha f)\,d\mu \leq \alpha \int_\Omega [\exp(\alpha f) - \exp(\alpha u)](f-u)\,d\mu \leq \begin{cases} |\alpha| \operatorname{essup}_\Omega |\exp(\alpha f) - \exp(\alpha u)| \int_\Omega |f-u|\,d\mu, \\ |\alpha| \left(\int_\Omega |\exp(\alpha f) - \exp(\alpha u)|^p d\mu\right)^{1/p} \left(\int_\Omega |f-u|^q d\mu\right)^{1/q}, \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ |\alpha| \operatorname{essup}_\Omega |f-u| \int_\Omega |\exp(\alpha f) - \exp(\alpha u)|\,d\mu. \end{cases} \tag{3.1}$$

If we take in (3.1) $u = \int_\Omega f d\mu$, then we get

$$0 \leq \exp\left(\alpha \int_\Omega f d\mu\right) - \int_\Omega \left(1 + \alpha \int_\Omega f d\mu - \alpha f\right) \exp(\alpha f)\,d\mu \leq \alpha \int_\Omega \left[\exp(\alpha f) - \exp\left(\alpha \int_\Omega f d\mu\right)\right] \left(f - \int_\Omega f d\mu\right) d\mu \leq \begin{cases} |\alpha| \operatorname{essup}_\Omega \left|\exp(\alpha f) - \exp\left(\alpha \int_\Omega f d\mu\right)\right| \int_\Omega \left|f - \int_\Omega f d\mu\right| d\mu, \\ |\alpha| \left(\int_\Omega \left|\exp(\alpha f) - \exp\left(\alpha \int_\Omega f d\mu\right)\right|^p d\mu\right)^{1/p} \\ \times \left(\int_\Omega \left|f - \int_\Omega f d\mu\right|^q d\mu\right)^{1/q},\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ |\alpha| \operatorname{essup}_\Omega \left|f - \int_\Omega f d\mu\right| \\ \times \int_\Omega \left|\exp(\alpha f) - \exp\left(\alpha \int_\Omega f d\mu\right)\right| d\mu. \end{cases} \tag{3.2}$$

Also, if we take in (3.1)

$$s=\frac{\int_{\Omega}f\cdot\exp(\alpha f)\,d\mu}{\int_{\Omega}\exp(\alpha f)\,d\mu},$$

then we get

$$\begin{aligned}
(3.3)\qquad 0&\leq\exp\left(\alpha\frac{\int_{\Omega}f\cdot\exp(\alpha f)\,d\mu}{\int_{\Omega}\exp(\alpha f)\,d\mu}\right)-\int_{\Omega}\exp(\alpha f)\,d\mu\\
&\leq\alpha\int_{\Omega}\left[\exp(\alpha f)-\exp\left(\alpha\frac{\int_{\Omega}f\cdot\exp(\alpha f)\,d\mu}{\int_{\Omega}\exp(\alpha f)\,d\mu}\right)\right]\\
&\times\left(f-\frac{\int_{\Omega}f\cdot\exp(\alpha f)\,d\mu}{\int_{\Omega}\exp(\alpha f)\,d\mu}\right)d\mu\\
&\leq\begin{cases}
|\alpha|\operatorname{essup}_{\Omega}\left|\exp(\alpha f)-\exp\left(\alpha\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right)\right|\\
\times\int_{\Omega}\left|f-\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right|d\mu,\\
\\
|\alpha|\left(\int_{\Omega}\left|\exp(\alpha f)-\exp\left(\alpha\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right)\right|^{p}d\mu\right)^{1/p}\\
\times\left(\int_{\Omega}\left|f-\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right|^{q}d\mu\right)^{1/q},\ p,\ q>1\text{ with }\frac{1}{p}+\frac{1}{q}=1,\\
\\
\operatorname{essup}_{\Omega}\left|f-\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right|\\
\times\int_{\Omega}\left|\exp(\alpha f)-\exp\left(\alpha\frac{\int_{\Omega}f\cdot\exp(\alpha f)d\mu}{\int_{\Omega}\exp(\alpha f)d\mu}\right)\right|d\mu.
\end{cases}
\end{aligned}$$

If $m\leq f\leq M$ μ-a.e. on Ω, then from the third branch of (2.7) we obtain the following inequality of interest

$$\begin{aligned}
(3.4)\qquad 0&\leq\exp\left(\alpha\frac{m+M}{2}\right)-\int_{\Omega}\exp(\alpha f)\,d\mu\\
&-\alpha\frac{m+M}{2}\int_{\Omega}\exp(\alpha f)\,d\mu+\alpha\int_{\Omega}f\cdot\exp(\alpha f)\,d\mu\\
&\leq\frac{1}{2}|\alpha|(M-m)\int_{\Omega}\left|\exp(\alpha f)-\exp\left(\alpha\frac{m+M}{2}\right)\right|d\mu.
\end{aligned}$$

If

$$\exp(\alpha t)=\frac{\exp(\alpha m)+\exp(\alpha M)}{2},$$

namely

$$t=\frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right),$$

then by (2.11) we get

$$
\begin{aligned}
(3.5)\quad 0 &\leq \frac{\exp(\alpha m)+\exp(\alpha M)}{2} - \int_\Omega \exp(\alpha f)\,d\mu \\
&- \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\int_\Omega \exp\alpha(f)\,d\mu + \alpha\int_\Omega f\cdot\exp(\alpha f)\,d\mu \\
&\leq \alpha\int_\Omega \left[\exp(\alpha f) - \frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right] \\
&\times \left[f - \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\right] d\mu \\
&\leq \begin{cases}
|\alpha| \operatorname{essup}_\Omega \left|\exp(\alpha f) - \frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right| \\
\times \int_\Omega \left|f - \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\right| d\mu, \\
\\
|\alpha| \left(\int_\Omega \left|\exp(\alpha f) - \frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right|^p d\mu\right)^{1/p} \\
\times \left(\int_\Omega \left|f - \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\right|^q d\mu\right)^{1/q} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\
\\
|\alpha| \operatorname{essup}_\Omega \left|f - \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\right| \\
\times \int_\Omega \left|f - \frac{1}{\alpha}\ln\left(\frac{\exp(\alpha m)+\exp(\alpha M)}{2}\right)\right| d\mu.
\end{cases}
\end{aligned}
$$

If we write the inequality (2.19) for the exponential function, then we get

$$
\begin{aligned}
(3.6)\quad 0 &\leq \exp\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i \exp(x_i) \\
&- \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i \exp(x_i) + \sum_{i=1}^n p_i x_i \exp(x_i) \\
&\leq \sum_{i=1}^n p_i \left[\exp(x_i) - \exp\left(\sum_{i=1}^n p_i x_i\right)\right]\left(x_i - \sum_{i=1}^n p_i x_i\right) \\
&\leq \begin{cases}
\max_{i\in\{1,\ldots,n\}} \left|\exp(x_i) - \exp\left(\sum_{i=1}^n p_i x_i\right)\right| \\
\times \sum_{i=1}^n p_i \left|x_i - \sum_{i=1}^n p_i x_i\right|, \\
\\
\left(\sum_{i=1}^n p_i \left|\exp(x_i) - \exp\left(\sum_{i=1}^n p_i x_i\right)\right|^p\right)^{1/p} \\
\times \left(\sum_{i=1}^n p_i \left|x_i - \sum_{i=1}^n p_i x_i\right|^q\right)^{1/q},\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\
\\
\max_{i\in\{1,\ldots,n\}} \left|x_i - \sum_{i=1}^n p_i x_i\right| \sum_{i=1}^n p_i \left|\exp(x_i) - \exp\left(\sum_{i=1}^n p_i x_i\right)\right|
\end{cases}
\end{aligned}
$$

for all $x_i \in \mathbb{R}$, $i \in \{1,...,n\}$ and $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$.

If $x_i = \ln y_i$, $y_i > 0$, $i \in \{1, ..., n\}$, then by (3.6) we get

$$
\begin{aligned}
(3.7)\quad 0 &\le \prod_{i=1}^{n} y_i^{p_i} - \sum_{i=1}^{n} p_i y_i - \ln\left(\prod_{i=1}^{n} y_i^{p_i}\right) \sum_{i=1}^{n} p_i y_i + \ln\left(\prod_{i=1}^{n} y_i^{p_i y_i}\right) \\
&\le \sum_{i=1}^{n} p_i \left(y_i - \prod_{i=1}^{n} y_i^{p_i}\right)\left[\ln y_i - \ln\left(\prod_{i=1}^{n} y_i^{p_i}\right)\right] \\
&\le \begin{cases}
\max_{i\in\{1,\ldots,n\}} \left|y_i - \prod_{i=1}^{n} y_i^{p_i}\right| \sum_{i=1}^{n} p_i \left|\ln y_i - \ln\left(\prod_{i=1}^{n} y_i^{p_i}\right)\right|, \\
\\
\left(\sum_{i=1}^{n} p_i \left|y_i - \prod_{i=1}^{n} y_i^{p_i}\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i \left|\ln y_i - \ln\left(\prod_{i=1}^{n} y_i^{p_i}\right)\right|^q\right)^{1/q}, \\
p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\
\\
\max_{i\in\{1,\ldots,n\}} \left|\ln y_i - \ln\left(\prod_{i=1}^{n} y_i^{p_i}\right)\right| \sum_{i=1}^{n} p_i \left|y_i - \prod_{i=1}^{n} y_i^{p_i}\right|.
\end{cases}
\end{aligned}
$$

For $\Phi(x) = \exp x$, $x \in \mathbb{R}$, consider

$$
s = \frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}, \tag{3.8}
$$

then by (2.5) we get

$$
\begin{aligned}
(3.9)\quad 0 &\le \exp\left(\frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right) - \sum_{i=1}^{n} p_i \exp(x_i) \\
&\le \sum_{i=1}^{n} p_i \left[\exp(x_i) - \exp\left(\frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right)\right] \\
&\quad \times \left(x_i - \frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right) \\
&\le \begin{cases}
\max_{i\in\{1,\ldots,n\}} \left|\exp(x_i) - \exp\left(\frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right)\right| \\
\times \sum_{i=1}^{n} p_i \left|x_i - \frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right|, \\
\\
\left(\sum_{i=1}^{n} p_i \left|\exp(x_i) - \exp\left(\frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right)\right|^p\right)^{1/p} \\
\times \left(\sum_{i=1}^{n} p_i \left|x_i - \frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right|^q\right)^{1/q}, \\
p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\
\\
\max_{i\in\{1,\ldots,n\}} \left|x_i - \frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right| \\
\times \sum_{i=1}^{n} p_i \left|\exp(x_i) - \exp\left(\frac{\sum_{i=1}^{n} p_i x_i \exp(x_i)}{\sum_{i=1}^{n} p_i \exp(x_i)}\right)\right|.
\end{cases}
\end{aligned}
$$

If $x_i = \ln y_i$, $y_i > 0$, $i \in \{1, ..., n\}$, then

$$
s := \frac{\sum_{i=1}^{n} p_i y_i \ln y_i}{\sum_{i=1}^{n} p_i y_i} = \frac{\ln\left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)}{\sum_{i=1}^{n} p_i y_i}, \tag{3.10}
$$

then by (3.9) we get

$$
\begin{aligned}
0 &\le \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} - \sum_{i=1}^{n} p_i y_i \\
&\le \sum_{i=1}^{n} p_i \left[y_i - \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right] \\
&\quad \times \left(\ln y_i - \ln \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right) \\
&\le \begin{cases}
\max_{i\in\{1,\dots,n\}} \left| y_i - \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right| \\
\times \sum_{i=1}^{n} p_i \left| \ln y_i - \ln \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right|, \\
\\
\left(\sum_{i=1}^{n} p_i \left| y_i - \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right|^p \right)^{1/p} \\
\times \left(\sum_{i=1}^{n} p_i \left| \ln y_i - \ln \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right|^q \right)^{1/q}, \\
p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\
\\
\max_{i\in\{1,\dots,n\}} \left| \ln y_i - \ln \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right| \\
\times \sum_{i=1}^{n} p_i \left| y_i - \left(\prod_{i=1}^{n} y_i^{p_i y_i}\right)^{\sum_{i=1}^{n} p_i y_i} \right|.
\end{cases}
\end{aligned} \tag{3.11}
$$

4. Some Inequalities for Logarithm

If we consider the convex function $\Phi(x) = -\ln x$, $x > 0$, $u > 0$, then by (2.1) we get

$$
\begin{aligned}
0 &\le \int_\Omega \ln f d\mu + u \int_\Omega \frac{1}{f} d\mu - \ln u - 1 \le \frac{1}{u} \int_\Omega \frac{(f-u)^2}{f} d\mu \\
&\le \begin{cases}
\frac{1}{u} \operatorname{essup}_\Omega \left| \frac{f-u}{f} \right| \int_\Omega |f-u| \, d\mu, \\
\\
\frac{1}{u} \left(\int_\Omega \left| \frac{f-u}{f} \right|^p d\mu \right)^{1/p} \left(\int_\Omega |f-u|^q \, d\mu \right)^{1/q}, \\
p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\
\\
\frac{1}{u} \operatorname{essup}_\Omega |f-u| \int_\Omega \left| \frac{f-u}{f} \right| d\mu.
\end{cases}
\end{aligned} \tag{4.1}
$$

If we take $u = \int_\Omega f d\mu > 0$ in (4.1), then we get

$$
\begin{aligned}
0 &\le \int_\Omega \ln f d\mu + \int_\Omega f d\mu \int_\Omega \frac{1}{f} d\mu - \ln \left(\int_\Omega f d\mu \right) - 1 \\
&\le \frac{1}{\int_\Omega f d\mu} \int_\Omega \frac{(f-u)^2}{f} d\mu
\end{aligned} \tag{4.2}
$$

$$\leq \begin{cases} \frac{1}{\int_\Omega f d\mu} \operatorname{essup}_\Omega \left| \frac{f - \int_\Omega f d\mu}{f} \right| \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu, \\ \\ \frac{1}{\int_\Omega f d\mu} \left(\int_\Omega \left| \frac{f - \int_\Omega f d\mu}{f} \right|^p d\mu \right)^{1/p} \left(\int_\Omega \left| f - \int_\Omega f d\mu \right|^q d\mu \right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{\int_\Omega f d\mu} \operatorname{essup}_\Omega \left| f - \int_\Omega f d\mu \right| \int_\Omega \left| \frac{f - \int_\Omega f d\mu}{f} \right| d\mu. \end{cases}$$

Observe that for the logarithmic function the Slater point is

$$s = \frac{1}{\int_\Omega \frac{d\mu}{f}}, \tag{4.3}$$

then by (2.5)

$$0 \leq \int_\Omega \ln f d\mu - \ln \left(\frac{1}{\int_\Omega \frac{d\mu}{f}} \right) \leq \int_\Omega \left(\int_\Omega \frac{d\mu}{f} - \frac{1}{f} \right) \left(f - \frac{1}{\int_\Omega \frac{d\mu}{f}} \right) d\mu \tag{4.4}$$

$$\leq \begin{cases} \operatorname{essup}_\Omega \left| \int_\Omega \frac{d\mu}{f} - \frac{1}{f} \right| \int_\Omega \left| f - \frac{1}{\int_\Omega \frac{d\mu}{f}} \right| d\mu, \\ \\ \left(\int_\Omega \left| \int_\Omega \frac{d\mu}{f} - \frac{1}{f} \right|^p d\mu \right)^{1/p} \left(\int_\Omega \left| f - \frac{1}{\int_\Omega \frac{d\mu}{f}} \right|^q d\mu \right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega \left| f - \frac{1}{\int_\Omega \frac{d\mu}{f}} \right| \int_\Omega \left| \int_\Omega \frac{d\mu}{f} - \frac{1}{f} \right| d\mu. \end{cases}$$

5. Some Inequalities for Power Function

The power function $\Phi_r(x) = x^r$, $r \in (-\infty, 0) \cup [1, \infty)$ is convex on $(0, \infty)$ and by (2.1) we get

$$\begin{aligned} 0 &\leq u^r - ru \int_\Omega f^{r-1} d\mu + (r-1) \int_\Omega f^r d\mu \\ &\leq r \int_\Omega \left(f^{r-1} - u^{r-1} \right) (f - u)\, d\mu \\ &\leq \begin{cases} |r| \operatorname{essup}_\Omega \left| f^{r-1} - u^{r-1} \right| \int_\Omega |f - u|\, d\mu, \\ \\ |r| \left(\int_\Omega \left| f^{r-1} - u^{r-1} \right|^p d\mu \right)^{1/p} \left(\int_\Omega |f - u|^q d\mu \right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ |r| \operatorname{essup}_\Omega |f - u| \int_\Omega \left| f^{r-1} - u^{r-1} \right| d\mu. \end{cases} \end{aligned} \tag{5.1}$$

If in this inequality, we take $u = \int_\Omega f d\mu$, then

$$\begin{aligned} 0 &\leq \left(\int_\Omega f d\mu \right)^r - r \int_\Omega f d\mu \int_\Omega f^{r-1} d\mu + (r-1) \int_\Omega f^r d\mu \\ &\leq r \int_\Omega \left(f^{r-1} - \left(\int_\Omega f d\mu \right)^{r-1} \right) \left(f - \int_\Omega f d\mu \right) d\mu \end{aligned} \tag{5.2}$$

$$\leq \begin{cases} |r| \operatorname{essup}_{\Omega} \left| f^{r-1} - \left(\int_{\Omega} f d\mu\right)^{r-1} \right| \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu, \\ \\ |r| \left(\int_{\Omega} \left| f^{r-1} - \left(\int_{\Omega} f d\mu\right)^{r-1} \right|^{p} d\mu\right)^{1/p} \left(\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right|^{q} d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ |r| \operatorname{essup}_{\Omega} \left| f - \int_{\Omega} f d\mu \right| \int_{\Omega} \left| f^{r-1} - \left(\int_{\Omega} f d\mu\right)^{r-1} \right| d\mu. \end{cases}$$

For the power function the Slater point is

$$s = \frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}, \tag{5.3}$$

then by (2.6) we get

$$\begin{aligned} 0 &\leq \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r} - \int_{\Omega} f^{r} d\mu \\ &\leq r \int_{\Omega} \left[f^{r-1} - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right] \left[f - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right] d\mu \end{aligned} \tag{5.4}$$

$$\leq \begin{cases} |r| \operatorname{essup}_{\Omega} \left| f^{r-1} - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right| \\ \times \int_{\Omega} \left| f - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right| d\mu, \\ \\ |r| \left(\int_{\Omega} \left| f^{r-1} - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right|^{p} d\mu\right)^{1/p} \\ \times \left(\int_{\Omega} \left| f - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right|^{q} d\mu\right)^{1/q},\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ |r| \operatorname{essup}_{\Omega} \left| f - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right| \\ \times \int_{\Omega} \left| f^{r-1} - \left(\frac{\int_{\Omega} f^{r} d\mu}{\int_{\Omega} f^{r-1} d\mu}\right)^{r-1} \right| d\mu. \end{cases}$$

If we take $r = -1$ in (5.2), then we get

$$\begin{aligned} 0 &\leq \left(\int_{\Omega} f d\mu\right)^{-1} + \int_{\Omega} f d\mu \int_{\Omega} f^{-2} d\mu - 2 \int_{\Omega} f^{-1} d\mu \\ &\leq \int_{\Omega} \left(\left(\int_{\Omega} f d\mu\right)^{-2} - f^{-2} \right) \left(f - \int_{\Omega} f d\mu \right) d\mu \end{aligned} \tag{5.5}$$

$$\leq \begin{cases} \operatorname{essup}_{\Omega} \left| f^{r-2} - \left(\int_{\Omega} f d\mu\right)^{-2} \right| \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu, \\ \\ \left(\int_{\Omega} \left| f^{r-2} - \left(\int_{\Omega} f d\mu\right)^{-2} \right|^{p} d\mu\right)^{1/p} \left(\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right|^{q} d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_{\Omega} \left| f - \int_{\Omega} f d\mu \right| \int_{\Omega} \left| f^{r-2} - \left(\int_{\Omega} f d\mu\right)^{-2} \right| d\mu. \end{cases}$$

If we take in (5.4) $r = 2$, then we get

$$
\begin{aligned}
(5.6)\qquad 0 &\le \left(\frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right)^2 - \int_\Omega f^2 d\mu \\
&\le 2\int_\Omega \left(f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right)\left(f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right) d\mu \\
&\le \begin{cases} 2\,\mathrm{essup}_\Omega \left|f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right| \int_\Omega \left|f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right| d\mu, \\ \\ 2\left(\int_\Omega \left|f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}\right|^p d\mu\right)^{1/p} \times \left(\int_\Omega \left|f - \frac{\int_\Omega f^2 d\mu}{\int_\Omega f d\mu}^{r-1}\right|^q d\mu\right)^{1/q}, \\ p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \end{cases}
\end{aligned}
$$

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