

REFINING MINKOWSKI INTEGRAL INEQUALITY FOR PARTITIONS OF WEIGHTS

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ABSTRACT. In this paper we establish a refinement and some reverses for Minkowski inequality for the general Lebesgue integral on measurable spaces and partitions of weights. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. INTRODUCTION

The Minkowski inequality plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_w^r(\Omega, \mu)$, $r \geq 1$ the Banach space of all \mathbb{C} -valued functions f defined on Ω that are r - w -integrable on Ω , i.e., $\int_{\Omega} w(x) |f(x)|^r d\mu(x) < \infty$, where $w : \Omega \rightarrow [0, \infty)$ is a given μ -measurable function on Ω . We write for simplicity $\int_{\Omega} w |f|^r d\mu$ instead of $\int_{\Omega} w(x) |f(x)|^r d\mu(x)$.

The following inequality is well known in the literature as the *integral Minkowski inequality*:

$$(1.1) \quad \left(\int_{\Omega} w |f + g|^p d\mu \right)^{1/p} \leq \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p}$$

provided that $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$.

For some recent results related to Minkowski inequality, see [1] and [4]-[4].

For the μ -integrable positive μ -a.e. weight w and a given $n \geq 2$ we consider the set $\mathfrak{P}_n(w)$ all possible n -tuples of μ -integrable positive weights $\bar{w} = (w_1, \dots, w_n)$ with the property that

$$\sum_{i=1}^n w_i = w.$$

This is called a partition of w .

It is clear that $\sum_{i=1}^n \int_{\Omega} w_i d\mu = \int_{\Omega} w d\mu$ for any $\bar{w} = (w_1, \dots, w_n) \in \mathfrak{P}_n(w)$ and $\int_{\Omega} w_i d\mu > 0$.

1991 *Mathematics Subject Classification*. Primary 26D15; Secondary 26D10.

Key words and phrases. Minkowski-Inequality, Measurable spaces, Lebesgue integral, Weighted means.

For $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$ we consider the functional $\kappa_p(|f|, |g|, \cdot) : \mathfrak{P}_n(w) \rightarrow [0, \infty)$ defined by

$$(1.2) \quad \kappa_p(|f|, |g|, \bar{w}) := \left(\sum_{i=1}^n \left[\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right]^p \right)^{1/p}.$$

In this paper we establish some inequalities concerning the functional $\kappa_p(|f|, |g|, \cdot)$ that provide refinements and reverses for the Minkowski integral inequality (1.1). Applications for discrete inequalities and weighted means of positive numbers are also given.

2. SOME GENERAL FACTS

The following refinement of the Minkowski inequality holds:

Theorem 1. For $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$ and $\bar{w} = (w_1, \dots, w_n) \in \mathfrak{P}_n(w)$ we have

$$(2.1) \quad \begin{aligned} \left(\int_{\Omega} w |f + g|^p d\mu \right)^{1/p} &\leq \kappa_p(|f|, |g|, \bar{w}) \\ &\leq \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p}. \end{aligned}$$

Proof. By Minkowski weighted integral inequality for $w_i, i \in \{1, \dots, n\}$ we have

$$\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \geq \left(\int_{\Omega} w_i |f + g|^p d\mu \right)^{1/p}.$$

This implies, by summation over $i \in \{1, \dots, n\}$, that

$$\begin{aligned} \kappa_p^p(|f|, |g|, \bar{w}) &= \sum_{i=1}^n \left[\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right]^p \\ &\geq \sum_{i=1}^n \int_{\Omega} w_i |f + g|^p d\mu = \int_{\Omega} \sum_{i=1}^n w_i |f + g|^p d\mu \\ &= \int_{\Omega} w |f + g|^p d\mu. \end{aligned}$$

Therefore, by taking the power $1/p$ we get

$$\kappa_p(|f|, |g|, \bar{w}) \geq \left(\int_{\Omega} w |f + g|^p d\mu \right)^{1/p}$$

and the first inequality in (2.1) is proved.

Consider now the Minkowski discrete inequality for the nonnegative numbers $a_i, b_i, i \in \{1, \dots, n\}$

$$\left[\sum_{i=1}^n (a_i + b_i)^p \right]^{1/p} \leq \left(\sum_{i=1}^n a_i^p \right)^{1/p} + \left(\sum_{i=1}^n b_i^p \right)^{1/p}.$$

If we take in this inequality $a_i = \left(\int_{\Omega} w_i |f|^p d\mu\right)^{1/p}$, $b_i = \left(\int_{\Omega} w_i |g|^p d\mu\right)^{1/p}$, $i \in \{1, \dots, n\}$, we have

$$\begin{aligned}
& \kappa_p(|f|, |g|, \bar{w}) \\
&= \left(\sum_{i=1}^n \left[\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right]^p \right)^{1/p} \\
&\leq \left(\sum_{i=1}^n \left[\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} \right]^p \right)^{1/p} + \left(\sum_{i=1}^n \left[\left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right]^p \right)^{1/p} \\
&= \left(\sum_{i=1}^n \left(\int_{\Omega} w_i |f|^p d\mu \right) \right)^{1/p} + \left(\sum_{i=1}^n \left(\int_{\Omega} w_i |g|^p d\mu \right) \right)^{1/p} \\
&= \left(\int_{\Omega} \sum_{i=1}^n w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} \sum_{i=1}^n w_i |g|^p d\mu \right)^{1/p} \\
&= \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p},
\end{aligned}$$

which proves the second part (2.1). \square

We consider the n -tuple of integrable functions on Ω , $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i(x) \geq 0$, $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n \alpha_i(x) = 1$ for μ -almost every $x \in \Omega$. We define

$$\begin{aligned}
(2.2) \quad & \kappa_p(|f|, |g|, \bar{w}, \boldsymbol{\alpha}) \\
& := \left(\sum_{i=1}^n \left[\left(\int_{\Omega} \alpha_i w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} \alpha_i w |g|^p d\mu \right)^{1/p} \right]^p \right)^{1/p}.
\end{aligned}$$

Observe that

$$\sum_{i=1}^n \alpha_i w = w$$

and by (2.1) we get

$$\begin{aligned}
(2.3) \quad & \left(\int_{\Omega} w |f + g|^p d\mu \right)^{1/p} \leq \kappa_p(|f|, |g|, \bar{w}, \boldsymbol{\alpha}) \\
& \leq \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p}.
\end{aligned}$$

This inequality obviously hold if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a probability, namely α_i are real numbers with $\alpha_i \geq 0$, $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n \alpha_i = 1$.

For $t \in (0, 1)$ we define

$$\begin{aligned}
(2.4) \quad & \kappa_p(|f|, |g|, \bar{w}, \mathbf{t}) \\
& := \left\{ \left[\left(\int_{\Omega} t w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} t w |g|^p d\mu \right)^{1/p} \right]^p \right. \\
& \quad \left. + \left[\left(\int_{\Omega} (1-t) w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} (1-t) w |g|^p d\mu \right)^{1/p} \right]^p \right\}^{1/p}
\end{aligned}$$

that is derived from (2.2) for $\alpha_1 = t$ and $\alpha_2 = t$, and by (2.3) we get

$$(2.5) \quad \left(\int_{\Omega} w |f + g|^p d\mu \right)^{1/p} \leq \kappa_p(|f|, |g|, \bar{w}, \mathbf{t}) \\ \leq \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p}.$$

If $\Omega = [a, b] \subset \mathbb{R}$ and if we take $\alpha_1(x) = \frac{x-a}{b-a}$ and $\alpha_2(x) = \frac{b-x}{b-a}$, $x \in [a, b]$, then we get the refinement of Minkowski's univariate integral inequality

$$(2.6) \quad \left(\int_a^b w(x) |f(x) + g(x)|^p dx \right)^{1/p} \\ \leq \frac{1}{b-a} \left\{ \left[\left(\int_a^b (x-a) w(x) |f(x)|^p dx \right)^{1/p} + \left(\int_a^b (x-a) w(x) |g(x)|^p dx \right)^{1/p} \right]^p \right. \\ \left. + \frac{1}{b-a} \left[\left(\int_a^b (b-x) w(x) |f(x)|^p dx \right)^{1/p} + \left(\int_a^b (b-x) w(x) |g(x)|^p dx \right)^{1/p} \right]^p \right\}^{1/p} \\ \leq \left(\int_a^b w(x) |f(x)|^p dx \right)^{1/p} + \left(\int_a^b w(x) |g(x)|^p dx \right)^{1/p}.$$

In the case of uniform weight $w \equiv 1$, we get

$$(2.7) \quad \left(\int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \\ \leq \frac{1}{b-a} \left\{ \left[\left(\int_a^b (x-a) |f(x)|^p dx \right)^{1/p} + \left(\int_a^b (x-a) |g(x)|^p dx \right)^{1/p} \right]^p \right. \\ \left. + \frac{1}{b-a} \left[\left(\int_a^b (b-x) |f(x)|^p dx \right)^{1/p} + \left(\int_a^b (b-x) |g(x)|^p dx \right)^{1/p} \right]^p \right\}^{1/p} \\ \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)|^p dx \right)^{1/p}.$$

We observe that $\mathfrak{P}_n(w)$ is a convex set. Indeed if $\bar{w} = (w_1, \dots, w_n)$, $\bar{p} = (p_1, \dots, p_n) \in \mathfrak{P}_n(w)$ then for $t \in [0, 1]$ we have

$$\sum_{i=1}^n [(1-t)w_i + tp_i] = (1-t) \sum_{i=1}^n w_i + t \sum_{i=1}^n p_i = (1-t)w + tw = w,$$

which shows that $(1-t)\bar{w} + t\bar{p} \in \mathfrak{P}_n(w)$.

Theorem 2. For $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$ we have that $\kappa_p^p(|f|, |g|, \cdot)$ is a concave mapping on $\mathfrak{P}_n(w)$.

Proof. Let $\bar{w} = (w_1, \dots, w_n)$, $\bar{p} = (p_1, \dots, p_n) \in \mathfrak{P}_n(w)$ and $t \in [0, 1]$. Then

$$\begin{aligned}
(2.8) \quad & \kappa_p^p(|f|, |g|, (1-t)\bar{w} + t\bar{p}) \\
&= \sum_{i=1}^n \left\{ \left(\int_{\Omega} ((1-t)w_i + tp_i) |f|^p d\mu \right)^{1/p} \right. \\
&\quad \left. + \left(\int_{\Omega} ((1-t)w_i + tp_i) |g|^p d\mu \right)^{1/p} \right\}^p \\
&= \sum_{i=1}^n \left\{ \left((1-t) \int_{\Omega} w_i |f|^p d\mu + t \int_{\Omega} p_i |f|^p d\mu \right)^{1/p} \right. \\
&\quad \left. + \left((1-t) \int_{\Omega} w_i |g|^p d\mu + t \int_{\Omega} p_i |g|^p d\mu \right)^{1/p} \right\}^p.
\end{aligned}$$

We use the following elementary inequality

$$\begin{aligned}
& ((1-t)a^p + tb^p)^{1/p} + ((1-t)c^p + td^p)^{1/p} \\
& \geq [(1-t)(a+c)^p + t(b+d)^p]^{1/p}
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
(2.9) \quad & \left[((1-t)a^p + tb^p)^{1/p} + ((1-t)c^p + td^p)^{1/p} \right]^p \\
& \geq [(1-t)(a+c)^p + t(b+d)^p]
\end{aligned}$$

for $a, b, c, d > 0$ and $t \in (0, 1)$.

By taking $a = \left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p}$, $b = \left(\int_{\Omega} p_i |f|^p d\mu \right)^{1/p}$, $c = \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p}$ and $d = \left(\int_{\Omega} p_i |g|^p d\mu \right)^{1/p}$ in (2.9) and summing the obtained inequality, we have

$$\begin{aligned}
& \kappa_p^p(|f|, |g|, (1-t)\bar{w} + t\bar{p}) \\
&= \sum_{i=1}^n \left\{ \left((1-t) \int_{\Omega} w_i |f|^p d\mu + t \int_{\Omega} p_i |f|^p d\mu \right)^{1/p} \right. \\
&\quad \left. + \left((1-t) \int_{\Omega} w_i |g|^p d\mu + t \int_{\Omega} p_i |g|^p d\mu \right)^{1/p} \right\}^p \\
&\geq \sum_{i=1}^n \left[(1-t) \left(\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right)^p \right. \\
&\quad \left. + t \left(\left(\int_{\Omega} p_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} p_i |g|^p d\mu \right)^{1/p} \right)^p \right] \\
&= (1-t) \sum_{i=1}^n \left(\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right)^p \\
&\quad + t \sum_{i=1}^n \left(\left(\int_{\Omega} p_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} p_i |g|^p d\mu \right)^{1/p} \right)^p \\
&= (1-t) \kappa_p^p(|f|, |g|, \bar{w}) + t \kappa_p^p(|f|, |g|, \bar{p}).
\end{aligned}$$

This proves the concavity of $\kappa_p^p(|f|, |g|, \cdot)$. \square

We have:

Corollary 1. *If $\bar{w}_k \in \mathfrak{F}_n(w)$ for $k \in \{1, \dots, m\}$ and $p_k \geq 0$ with $\sum_{k=1}^n p_k = 1$, then by Jensen's inequality we have*

$$(2.10) \quad \sum_{k=1}^n p_k \kappa_p^p(|f|, |g|, \bar{w}_k) \leq \kappa_p^p \left(|f|, |g|, \sum_{k=1}^n p_k \bar{w}_k \right)$$

for $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$.

3. REVERSE INEQUALITIES

In [3] the authors obtained the following reverse of Minkowski's inequality:

Lemma 1. *Let $p > 1$, and $f, g \in L_w^p(\Omega, \mu)$. If*

$$(3.1) \quad 0 < m \leq \frac{(f+g)^{p-1}}{f}, \quad \frac{(f+g)^{p-1}}{g} \leq M < \infty$$

μ -almost everywhere on Ω , then

$$(3.2) \quad \left(\int_{\Omega} w f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w g^p d\mu \right)^{1/p} \leq \left(\frac{M}{m} \right)^{\frac{p-1}{p^2}} \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p}.$$

We observe that, if the following simpler conditions are valid

$$(3.3) \quad 0 < m_1 \leq f \leq M_1 < \infty, \quad 0 < m_2 \leq g \leq M_2 < \infty \quad \mu\text{-a.e. on } \Omega,$$

then

$$\frac{(m_1 + m_2)^{p-1}}{M_1} \leq \frac{(f+g)^{p-1}}{f} \leq \frac{(M_1 + M_2)^{p-1}}{m_1}$$

and

$$\frac{(m_1 + m_2)^{p-1}}{M_2} \leq \frac{(f+g)^{p-1}}{g} \leq \frac{(M_1 + M_2)^{p-1}}{m_2}.$$

If we take

$$M = \max \left\{ \frac{(M_1 + M_2)^{p-1}}{m_1}, \frac{(M_1 + M_2)^{p-1}}{m_2} \right\} = \frac{(M_1 + M_2)^{p-1}}{\min \{m_1, m_2\}}$$

and

$$m = \min \left\{ \frac{(m_1 + m_2)^{p-1}}{M_1}, \frac{(m_1 + m_2)^{p-1}}{M_2} \right\} = \frac{(m_1 + m_2)^{p-1}}{\max \{M_1, M_2\}}$$

then by (3.2) we get

$$(3.4) \quad \left(\int_{\Omega} w f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w g^p d\mu \right)^{1/p} \leq \left(\frac{\max \{M_1, M_2\} (M_1 + M_2)^{p-1}}{\min \{m_1, m_2\} (m_1 + m_2)^{p-1}} \right)^{\frac{p-1}{p^2}} \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p}.$$

In the same paper [3] the authors obtained the inequality:

Lemma 2. Let $p > 1$ and $f, g \in L_w^p(\Omega, \mu)$. If

$$(3.5) \quad 0 < m \leq \frac{f}{g} \leq M < \infty$$

μ -almost everywhere on Ω , then

$$(3.6) \quad \left(\int_{\Omega} w f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w g^p d\mu \right)^{1/p} \\ \leq \frac{M(m+1) + M+1}{(m+1)(M+1)} \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p}.$$

The case of functions of a real variable and Lebesgue integral was obtained by Bougoffa in [2].

Now, if the conditions (3.3) are valid, then

$$\frac{m_1}{M_2} \leq \frac{f}{g} \leq \frac{M_1}{m_2} \quad \mu\text{-a.e. on } \Omega$$

and by taking $M = \frac{M_1}{m_2}$ and $m = \frac{m_1}{M_2}$ in (3.6), we get

$$(3.7) \quad \left(\int_{\Omega} w f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w g^p d\mu \right)^{1/p} \\ \leq \frac{M_1(m_1 + M_2) + M_2(M_1 + m_2)}{(m_1 + M_2)(M_1 + m_2)} \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p}.$$

For other reverses of Minkowski's inequality, see [1] and [5]-[6].

The discrete case is as follows.

Assume that

$$(3.8) \quad 0 < m_1 \leq a_i \leq M_1 < \infty, \quad 0 < m_2 \leq b_i \leq M_2 < \infty \quad \text{for } i \in \{1, \dots, n\},$$

then

$$(3.9) \quad \left(\sum_{i=1}^n p_i a_i^p \right)^{1/p} + \left(\sum_{i=1}^n p_i b_i^p \right)^{1/p} \\ \leq \left(\frac{\max\{M_1, M_2\} (M_1 + M_2)^{p-1}}{\min\{m_1, m_2\} (m_1 + m_2)^{p-1}} \right)^{\frac{p-1}{p^2}} \left(\sum_{i=1}^n p_i (a_i + b_i)^p \right)^{1/p}$$

and

$$(3.10) \quad \left(\sum_{i=1}^n p_i a_i^p \right)^{1/p} + \left(\sum_{i=1}^n p_i b_i^p \right)^{1/p} \\ \leq \frac{M_1(m_1 + M_2) + M_2(M_1 + m_2)}{(m_1 + M_2)(M_1 + m_2)} \left(\sum_{i=1}^n p_i (a_i + b_i)^p \right)^{1/p},$$

where $p_i \geq 0$ for $i \in \{1, \dots, n\}$.

We assume that, in general, we have the following reverse of Minkowski's inequality

$$(3.11) \quad \left(\sum_{i=1}^n p_i a_i^p \right)^{1/p} + \left(\sum_{i=1}^n p_i b_i^p \right)^{1/p} \\ \leq B_p(m_1, M_1, m_2, M_2) \left(\sum_{i=1}^n p_i (a_i + b_i)^p \right)^{1/p},$$

provided that $a_i, b_i, i \in \{1, \dots, n\}$ satisfy the condition (3.1), where $p \geq 1$. Here $B_p(m_1, M_1, m_2, M_2)$ is a function depending on the parameters m_1, M_1, m_2, M_2 with the property (3.8) and $p \geq 1$.

The following result holds:

Theorem 3. *Let $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$ and $\bar{w} = (w_1, \dots, w_n) \in \mathfrak{F}_n(w)$ be such that there exists $k, K, l, L > 0$ with the property*

$$(3.12) \quad 0 < k \leq \left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} \leq K < \infty$$

and

$$(3.13) \quad 0 < l \leq \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \leq L < \infty$$

for each $i \in \{1, \dots, n\}$. Then

$$(3.14) \quad \left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p} \leq B_p(k, K, l, L) \kappa_p(|f|, |g|, \bar{w}).$$

Proof. Now, if we take $a_i = \left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p}$, $b_i = \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p}$, $p_i = 1$, $i \in \{1, \dots, n\}$, $m_1 = k$, $M_1 = K$, $m_2 = l$ and $M_2 = L$, then by (3.11) we get

$$\left(\sum_{i=1}^n \int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\sum_{i=1}^n \int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \\ \leq B_p(k, K, l, L) \left(\sum_{i=1}^n \left(\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right)^p \right)^{1/p},$$

namely

$$\left(\int_{\Omega} w |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w |g|^p d\mu \right)^{1/p} \\ \leq B_p(k, K, l, L) \left(\sum_{i=1}^n \left(\left(\int_{\Omega} w_i |f|^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i |g|^p d\mu \right)^{1/p} \right)^p \right)^{1/p},$$

and the inequality (3.14) is proved. \square

Corollary 2. *With the assumptions of Theorem 3 we have the inequality (3.7) with*

$$B_p(k, K, l, L) = \left(\frac{\max\{K, L\} (K + L)^{p-1}}{\min\{k, l\} (k + l)^{p-1}} \right)^{\frac{p-1}{p^2}}$$

or

$$B_p(k, K, l, L) = \frac{K(k+L) + L(K+l)}{(k+L)(K+l)}.$$

Further, we assume that, in general, we have the following reverse of Minkowski's inequality

$$(3.15) \quad \left(\int_{\Omega} w f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w g^p d\mu \right)^{1/p} \\ \leq M_p(m_1, M_1, m_2, M_2) \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p},$$

provided that f, g be μ -measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that (3.3) is valid.

Theorem 4. *Let $f, g \in L_w^p(\Omega, \mu)$ with $p \geq 1$ and $\bar{w} = (w_1, \dots, w_n) \in \mathfrak{P}_n(w)$ be such that there exists the constants m_1, M_1, m_2, M_2 for which the condition (3.3) is valid. Then*

$$(3.16) \quad \kappa_p(f, g, \bar{w}) \leq M_p(m_1, M_1, m_2, M_2) \left(\int_{\Omega} w (f+g)^p d\mu \right)^{1/p}.$$

Proof. By the inequality (3.15) we have

$$\left(\int_{\Omega} w_i f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i g^p d\mu \right)^{1/p} \\ \leq M_p(m_1, M_1, m_2, M_2) \left(\int_{\Omega} w_i (f+g)^p d\mu \right)^{1/p},$$

for all $i \in \{1, \dots, n\}$.

If we take the power $p \geq 1$ and sum over i from 1 to n , then we get

$$\sum_{i=1}^n \left[\left(\int_{\Omega} w_i f^p d\mu \right)^{1/p} + \left(\int_{\Omega} w_i g^p d\mu \right)^{1/p} \right]^p \\ \leq M_p^p(m_1, M_1, m_2, M_2) \sum_{i=1}^n \int_{\Omega} w_i (f+g)^p d\mu \\ = M_p^p(m_1, M_1, m_2, M_2) \int_{\Omega} w (f+g)^p d\mu$$

namely

$$\kappa_p^p(f, g, \bar{w}) \leq M_p^p(m_1, M_1, m_2, M_2) \int_{\Omega} w (f+g)^p d\mu,$$

which is equivalent to (3.16). \square

Corollary 3. *With the assumptions of Theorem 4 we have the inequality (3.16) with*

$$M_p(m_1, M_1, m_2, M_2) = \left(\frac{\max\{M_1, M_2\} (M_1 + M_2)^{p-1}}{\min\{m_1, m_2\} (m_1 + m_2)^{p-1}} \right)^{\frac{p-1}{p^2}}$$

or

$$M_p(m_1, M_1, m_2, M_2) = \frac{M_1(m_1 + M_2) + M_2(M_1 + m_2)}{(m_1 + M_2)(M_1 + m_2)}.$$

4. DISCRETE INEQUALITIES

When μ is the discrete measure on $\Omega = \{1, \dots, n\}$, then the corresponding discrete inequalities for complex (real) numbers can be stated as well. We give here some examples.

Consider the sequences of complex numbers $\bar{x} = (x_1, \dots, x_m)$, $\bar{y} = (y_1, \dots, y_m) \in \mathbb{C}^m$, $\bar{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$, with $w_k > 0$, $k \in \{1, \dots, m\}$. Let $w_{ki} > 0$ for $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$ with $m, n \geq 2$ and

$$(4.1) \quad \sum_{i=1}^n w_{ki} = w_k \text{ for any } k \in \{1, \dots, m\}.$$

We consider the functional associated with the matrix $W := \{w_{ki}\}_{k \in \{1, \dots, m\}, i \in \{1, \dots, n\}}$ that satisfy (4.1),

$$(4.2) \quad \kappa_p(|\bar{x}|, |\bar{y}|, W) := \left(\sum_{i=1}^n \left[\left(\sum_{k=1}^m w_{ki} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^m w_{ki} |y_k|^p \right)^{1/p} \right]^p \right)^{1/p},$$

where $p \geq 1$.

Using (2.1) we have the following refinement of Minkowski's discrete inequality

$$(4.3) \quad \begin{aligned} & \left(\sum_{k=1}^m w_k |x_k + y_k|^p \right)^{1/p} \\ & \leq \left(\sum_{i=1}^n \left[\left(\sum_{k=1}^m w_{ki} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^m w_{ki} |y_k|^p \right)^{1/p} \right]^p \right)^{1/p} \\ & \leq \left(\sum_{k=1}^m w_k |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^m w_k |y_k|^p \right)^{1/p}, \end{aligned}$$

where $p \geq 1$.

If there exists $k, K, l, L > 0$ with the property that

$$0 < k \leq \left(\sum_{j=1}^m w_{ji} |x_j|^p \right)^{1/p} \leq K < \infty$$

and

$$0 < l \leq \left(\sum_{j=1}^m w_{ji} w_i |y_j|^q d\mu \right)^{1/q} \leq L < \infty$$

for each $i \in \{1, \dots, n\}$, then

$$(4.4) \quad \left(\sum_{j=1}^m w_j |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^m w_j |y_j|^p \right)^{1/p} \leq B_p(k, K, l, L) \kappa_p(|\bar{x}|, |\bar{y}|, W),$$

where $p \geq 1$.

In the inequality (4.4) we can choose

$$B_p(k, K, l, L) = \left(\frac{\max\{K, L\} (K + L)^{p-1}}{\min\{k, l\} (k + l)^{p-1}} \right)^{\frac{p-1}{p^2}}$$

or

$$B_p(k, K, l, L) = \frac{K(k + L) + L(K + l)}{(k + L)(K + l)}.$$

If there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq x_j \leq M_1 < \infty, \quad 0 < m_2 \leq y_j \leq M_2 < \infty$$

for all $j \in \{1, \dots, m\}$, then

$$(4.5) \quad \kappa_p(\bar{x}, \bar{y}, W) \leq M_p(m_1, M_1, m_2, M_2) \left(\sum_{j=1}^m w_j (x_j + y_j)^p \right)^{1/p},$$

where $p \geq 1$.

In the inequality (4.5) we can choose

$$M_p(m_1, M_1, m_2, M_2) = \left(\frac{\max\{M_1, M_2\} (M_1 + M_2)^{p-1}}{\min\{m_1, m_2\} (m_1 + m_2)^{p-1}} \right)^{\frac{p-1}{p^2}}$$

or

$$M_p(m_1, M_1, m_2, M_2) = \frac{M_1(m_1 + M_2) + M_2(M_1 + m_2)}{(m_1 + M_2)(M_1 + m_2)}.$$

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