REFINING MINKOWSKI INTEGRAL INEQUALITY FOR DIVISIONS OF MEASURABLE SPACE

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Abstract. In this paper we establish a refinement and some reverses for Minkowski inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given.

1. Introduction

The Minkowski inequality plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L_w^r(\Omega,\mu)$, $r\geq 1$ the Banach space of all $\mathbb{C}$-valued functions f defined on Ω that are r-w-integrable on Ω, i.e., $\int_\Omega w(x)|f(x)|^r d\mu(x)<\infty$, where $w:\Omega\to[0,\infty)$ is a given μ-measurable function on Ω. We write for simplicity $\int_\Omega w|f|^r d\mu$ instead of $\int_\Omega w(x)|f(x)|^r d\mu(x)$.

The following inequality is well known in the literature as the *integral Minkowski inequality*:

$$\left(\int_\Omega w|f+g|^p d\mu\right)^{1/p}\leq\left(\int_\Omega w|f|^p d\mu\right)^{1/p}+\left(\int_\Omega w|g|^p d\mu\right)^{1/p} \tag{1.1}$$

provided that $f,\ g\in L_w^p(\Omega,\mu)$ with $p\geq 1$.

For some recent results related to Minkowski inequality, see [1] and [4]-[4].

We say that the family of measurable sets $F_n(\Omega)=\{\Omega_i\}_{i\in\{1,\dots,n\}}$ is a *n-division* for Ω if $\Omega=\bigcup_{i=1}^n\Omega_i$, $\Omega_i\cap\Omega_j=\emptyset$ for any $i,\ j\in\{1,...,n\}$ with $i\neq j$ and $\mu(\Omega_i)>0$ for any $i\in\{1,...,n\}$. In this situation, if $f\in L_w(\Omega,\mu)$ then $f\in L_w(\Omega_i,\mu)$ for any $i\in\{1,...,n\}$ and $\int_\Omega fwd\mu=\sum_{i=1}^n\int_{\Omega_i}fwd\mu$. Also, $\int_\Omega wd\mu=\sum_{i=1}^n\int_{\Omega_i}wd\mu$ with $\int_{\Omega_i}wd\mu>0$ for any $i\in\{1,...,n\}$.

For $f,\ g\in L_w^p(\Omega,\mu)$ with $p\geq 1$, we consider, for a given $n\geq 2$, $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and define the functional $\kappa_p(|f|,|g|,\cdot):\mathfrak{D}_n(\Omega)\to\mathbb{R}$ given

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by

$$\kappa_p(|f|,|g|,F_n(\Omega)) := \left(\sum_{i=1}^{n}\left[\left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p} + \left(\int_{\Omega_i} w|g|^p d\mu\right)^{1/p}\right]^p\right)^{1/p}. \tag{1.2}$$

In this paper we establish some inequalities concerning the functional $\kappa_p(|f|,|g|,\cdot)$ that provide refinements and reverses for the Minkowski integral inequality (1.1). Applications for discrete inequalities and weighted means of positive numbers are also given.

2. The Main Results

The following refinement of the Minkowski inequality holds:

Theorem 1. *For f, $g\in L_w^p(\Omega,\mu)$ with $p\geq 1$ and $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$ we have*

$$\left(\int_{\Omega} w|f+g|^p d\mu\right)^{1/p} \leq \kappa_p(|f|,|g|,F_n(\Omega)) \leq \left(\int_{\Omega} w|f|^p d\mu\right)^{1/p} + \left(\int_{\Omega} w|g|^p d\mu\right)^{1/p}. \tag{2.1}$$

Proof. By Minkowski weighted integral inequality for w and the set Ω_i, $i\in\{1,...,n\}$, we have

$$\left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p} + \left(\int_{\Omega_i} w|g|^p d\mu\right)^{1/p} \geq \left(\int_{\Omega_i} w|f+g|^p d\mu\right)^{1/p}.$$

This implies, by sumation over $i\in\{1,...,n\}$, that

$$\kappa_p^p(|f|,|g|,F_n(\Omega)) = \sum_{i=1}^{n}\left[\left(\int_{\Omega_i} w|f|^p d\mu\right)^{1/p} + \left(\int_{\Omega_i} w|g|^p d\mu\right)^{1/p}\right]^p \geq \sum_{i=1}^{n}\int_{\Omega_i} w|f+g|^p d\mu = \int_{\Omega} w|f+g|^p d\mu.$$

Therefore, by taking the power $1/p$ we get

$$\kappa_p(|f|,|g|,F_n(\Omega)) \geq \left(\int_{\Omega} w|f+g|^p d\mu\right)^{1/p}$$

and the first inequality in (2.1) is proved.

Consider now the Minkowski discrete inequality for the nonnegative numbers a_i, b_i, $i\in\{1,...,n\}$

$$\left[\sum_{i=1}^{n}(a_i+b_i)^p\right]^{1/p} \leq \left(\sum_{i=1}^{n}a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n}b_i^p\right)^{1/p}.$$

If we take in this inequality $a_i = \left(\int_{\Omega_i} w\,|f|^p\,d\mu\right)^{1/p}$, $b_i = \left(\int_{\Omega_i} w\,|g|^p\,d\mu\right)^{1/p}$, then we have

$$\begin{aligned}
&\kappa_p\left(|f|, |g|, F_n(\Omega)\right) \\
&= \left(\sum_{i=1}^{n}\left[\left(\int_{\Omega_i} w\,|f|^p\,d\mu\right)^{1/p} + \left(\int_{\Omega_i} w\,|g|^p\,d\mu\right)^{1/p}\right]^p\right)^{1/p} \\
&\le \left(\sum_{i=1}^{n}\left[\left(\int_{\Omega_i} w\,|f|^p\,d\mu\right)^{1/p}\right]^p\right)^{1/p} + \left(\sum_{i=1}^{n}\left[\left(\int_{\Omega_i} w\,|g|^p\,d\mu\right)^{1/p}\right]^p\right)^{1/p} \\
&= \left(\sum_{i=1}^{n}\left(\int_{\Omega_i} w\,|f|^p\,d\mu\right)\right)^{1/p} + \left(\sum_{i=1}^{n}\left(\int_{\Omega_i} w\,|g|^p\,d\mu\right)\right)^{1/p} \\
&= \left(\int_{\Omega} w\,|f|^p\,d\mu\right)^{1/p} + \left(\int_{\Omega} w\,|g|^p\,d\mu\right)^{1/p},
\end{aligned}$$

which proves the second part (2.1). □

Remark 1. *If $\Omega = [a, b] \subset \mathbb{R}$ and if we take $\Omega_1 = [a, y]$ and $\Omega_1 = [y, b]$, $y \in (a, b)$, then we get the refinement of Minkowski's univariate integral inequality*

$$\begin{aligned}
&\left(\int_a^b w(x)\,|f(x) + g(x)|^p\,dx\right)^{1/p} \\
&\le \left\{\left[\left(\int_a^y w(x)\,|f(x)|^p\,dx\right)^{1/p} + \left(\int_a^y w(x)\,|g(x)|^p\,dx\right)^{1/p}\right]^p\right. \\
&\left. + \left[\left(\int_y^b w(x)\,|f(x)|^p\,dx\right)^{1/p} + \left(\int_y^b w(x)\,|g(x)|^p\,dx\right)^{1/p}\right]^p\right\}^{1/p} \\
&\le \left(\int_{\Omega} w(x)\,|f(x)|^p\,dx\right)^{1/p} + \left(\int_{\Omega} w(x)\,|g(x)|^p\,dx\right)^{1/p}.
\end{aligned} \tag{2.2}$$

For the uniform weight $w \equiv 1$ we have

$$\begin{aligned}
&\left(\int_a^b |f(x) + g(x)|^p\,dx\right)^{1/p} \\
&\le \left\{\left[\left(\int_a^y |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^y |g(x)|^p\,dx\right)^{1/p}\right]^p\right. \\
&\left. + \left[\left(\int_y^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_y^b |g(x)|^p\,dx\right)^{1/p}\right]^p\right\}^{1/p} \\
&\le \left(\int_{\Omega} |f(x)|^p\,dx\right)^{1/p} + \left(\int_{\Omega} w\,|g(x)|^p\,dx\right)^{1/p}.
\end{aligned} \tag{2.3}$$

In [3] the authors obtained the following reverses of Minkowski's inequality:

Lemma 1. *Let $p>1$ and $f,\, g\in L_{w}^{p}(\Omega,\mu)$. If*

$$0<m\leq\frac{(f+g)^{p-1}}{f},\ \frac{(f+g)^{p-1}}{g}\leq M<\infty \tag{2.4}$$

μ-almost everywhere on Ω, then

$$\left(\int_{\Omega}wf^{p}d\mu\right)^{1/p}+\left(\int_{\Omega}wg^{p}d\mu\right)^{1/p}\leq\left(\frac{M}{m}\right)^{\frac{p-1}{p^{2}}}\left(\int_{\Omega}w(f+g)^{p}d\mu\right)^{1/p}. \tag{2.5}$$

We observe that, if the following simpler conditions are valid

$$0<m_{1}\leq f\leq M_{1}<\infty,\ 0<m_{2}\leq g\leq M_{2}<\infty\ \mu\text{-a.e. on }\Omega, \tag{2.6}$$

then

$$\frac{(m_{1}+m_{2})^{p-1}}{M_{1}}\leq\frac{(f+g)^{p-1}}{f}\leq\frac{(M_{1}+M_{2})^{p-1}}{m_{1}}$$

and

$$\frac{(m_{1}+m_{2})^{p-1}}{M_{2}}\leq\frac{(f+g)^{p-1}}{g}\leq\frac{(M_{1}+M_{2})^{p-1}}{m_{2}}.$$

If we take

$$M=\max\left\{\frac{(M_{1}+M_{2})^{p-1}}{m_{1}},\frac{(M_{1}+M_{2})^{p-1}}{m_{2}}\right\}=\frac{(M_{1}+M_{2})^{p-1}}{\min\{m_{1},m_{2}\}}$$

and

$$m=\min\left\{\frac{(m_{1}+m_{2})^{p-1}}{M_{1}},\frac{(m_{1}+m_{2})^{p-1}}{M_{2}}\right\}=\frac{(m_{1}+m_{2})^{p-1}}{\max\{M_{1},M_{2}\}}$$

then by (2.5) we get

$$\begin{aligned}&\left(\int_{\Omega}wf^{p}d\mu\right)^{1/p}+\left(\int_{\Omega}wg^{p}d\mu\right)^{1/p}\\&\leq\left(\frac{\max\{M_{1},M_{2}\}}{\min\{m_{1},m_{2}\}}\frac{(M_{1}+M_{2})^{p-1}}{(m_{1}+m_{2})^{p-1}}\right)^{\frac{p-1}{p^{2}}}\left(\int_{\Omega}w(f+g)^{p}d\mu\right)^{1/p}.\end{aligned} \tag{2.7}$$

In the same paper [3] the authors obtained the inequality:

Lemma 2. *Let $p>1$ and $f,\, g\in L_{w}^{p}(\Omega,\mu)$. If*

$$0<m\leq\frac{f}{g}\leq M<\infty \tag{2.8}$$

μ-almost everywhere on Ω, then

$$\begin{aligned}&\left(\int_{\Omega}wf^{p}d\mu\right)^{1/p}+\left(\int_{\Omega}wg^{p}d\mu\right)^{1/p}\\&\leq\frac{M(m+1)+M+1}{(m+1)(M+1)}\left(\int_{\Omega}w(f+g)^{p}d\mu\right)^{1/p}.\end{aligned} \tag{2.9}$$

The case of functions of a real variable and Lebesgue integral was obtained by Bougoffa in [2].

Now, if the conditions (2.6) are valid, then

$$\frac{m_{1}}{M_{2}}\leq\frac{f}{g}\leq\frac{M_{1}}{m_{2}}\ \mu\text{-a.e. on }\Omega$$

and by taking $M=\frac{M_1}{m_2}$ and $m=\frac{m_1}{M_2}$ in (2.9), we get

$$\left(\int_\Omega wf^p d\mu\right)^{1/p}+\left(\int_\Omega wg^p d\mu\right)^{1/p} \le \frac{M_1\left(m_1+M_2\right)+M_2\left(M_1+m_2\right)}{\left(m_1+M_2\right)\left(M_1+m_2\right)}\left(\int_\Omega w\left(f+g\right)^p d\mu\right)^{1/p}. \tag{2.10}$$

For other reverses of Minkowski's inequality, see [1] and [5]-[6].

The discrete case is as follows.

Assume that

$$0<m_1\le a_i\le M_1<\infty,\ 0<m_2\le b_i\le M_2<\infty \text{ for } i\in\{1,...,n\}, \tag{2.11}$$

then

$$\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}+\left(\sum_{i=1}^n p_i b_i^p\right)^{1/p} \le \left(\frac{\max\{M_1,M_2\}}{\min\{m_1,m_2\}}\frac{\left(M_1+M_2\right)^{p-1}}{\left(m_1+m_2\right)^{p-1}}\right)^{\frac{p-1}{p^2}}\left(\sum_{i=1}^n p_i\left(a_i+b_i\right)^p\right)^{1/p} \tag{2.12}$$

and

$$\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}+\left(\sum_{i=1}^n p_i b_i^p\right)^{1/p} \le \frac{M_1\left(m_1+M_2\right)+M_2\left(M_1+m_2\right)}{\left(m_1+M_2\right)\left(M_1+m_2\right)}\left(\sum_{i=1}^n p_i\left(a_i+b_i\right)^p\right)^{1/p}, \tag{2.13}$$

where $p_i\ge 0$ for $i\in\{1,...,n\}$.

We assume that, in general, we have the following reverse of Minkowski's inequality

$$\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}+\left(\sum_{i=1}^n p_i b_i^p\right)^{1/p} \le B_p\left(m_1,M_1,m_2,M_2\right)\left(\sum_{i=1}^n p_i\left(a_i+b_i\right)^p\right)^{1/p}, \tag{2.14}$$

provided that a_i, b_i, $i\in\{1,...,n\}$ satisfy the condition (2.4), where $p\ge 1$. Here $B_p\left(m_1,M_1,m_2,M_2\right)$ is a function depending on the parameters m_1, M_1, m_2, M_2 with the property (2.11) and $p\ge 1$.

The following result holds:

Theorem 2. *Let f, $g\in L_w^p\left(\Omega,\mu\right)$ with $p\ge 1$ and $F_n\left(\Omega\right)\in\mathfrak{D}_n\left(\Omega\right)$ be such that there exist h, H, u, $U>0$ with the property*

$$0<h\le\left(\int_{\Omega_i}\left|f\right|^p d\mu\right)^{1/p}\le H<\infty \tag{2.15}$$

and

$$0<u\le\left(\int_{\Omega_i}w\left|g\right|^p d\mu\right)^{1/p}\le U<\infty \tag{2.16}$$

for each $i \in \{1, ..., n\}$. *Then*

$$\left(\int_{\Omega} w|f|^p \, d\mu\right)^{1/p} + \left(\int_{\Omega} w|g|^p \, d\mu\right)^{1/p} \leq B_p(h, H, u, U) \, \kappa_p(|f|, |g|, F_n(\Omega)). \tag{2.17}$$

Proof. Now, if we take $a_i = \left(\int_{\Omega_i} w|f|^p \, d\mu\right)^{1/p}$, $b_i = \left(\int_{\Omega_i} w|g|^p \, d\mu\right)^{1/p}$, $p_i = 1$, $i \in \{1, ..., n\}$, $m_1 = h$, $M_1 = H$, $m_2 = u$ and $M_2 = U$, then by (2.14) we get

$$\left(\sum_{i=1}^{n} \int_{\Omega_i} w|f|^p \, d\mu\right)^{1/p} + \left(\sum_{i=1}^{n} \int_{\Omega_i} w|g|^p \, d\mu\right)^{1/p} \leq B_p(h, H, u, U) \left(\sum_{i=1}^{n} \left(\left(\int_{\Omega_i} w|f|^p \, d\mu\right)^{1/p} + \left(\int_{\Omega_i} w|g|^p \, d\mu\right)^{1/p}\right)^p\right)^{1/p},$$

namely

$$\left(\int_{\Omega} w|f|^p \, d\mu\right)^{1/p} + \left(\int_{\Omega} w|g|^p \, d\mu\right)^{1/p} \leq B_p(k, K, l, L) \left(\sum_{i=1}^{n} \left(\left(\int_{\Omega_i} w|f|^p \, d\mu\right)^{1/p} + \left(\int_{\Omega_i} w|g|^p \, d\mu\right)^{1/p}\right)^p\right)^{1/p},$$

and the inequality (2.17) is proved. □

Corollary 1. *With the assumptions of Theorem 2 we have the inequality (2.10) with*

$$B_p(h, H, u, U) = \left(\frac{\max\{H, U\}}{\min\{h, u\}} \frac{(H+U)^{p-1}}{(h+u)^{p-1}}\right)^{\frac{p-1}{p^2}}$$

or

$$B_p(h, K, l, L) = \frac{H(h+U) + U(H+u)}{(h+U)(H+u)}.$$

Further, we assume that, in general, we have the following reverse of Minkowski's inequality

$$\left(\int_{\Omega} wf^p d\mu\right)^{1/p} + \left(\int_{\Omega} wg^p d\mu\right)^{1/p} \leq B_p(m_1, M_1, m_2, M_2) \left(\int_{\Omega} w(f+g)^p \, d\mu\right)^{1/p}, \tag{2.18}$$

provided that f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that (2.6) is valid.

Theorem 3. *Let* $f, g \in L_w^p(\Omega, \mu)$ *with* $p \geq 1$ *be such that there exists the constants* m_1, M_1, m_2, M_2 *for which the condition (2.6) is valid. If* $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, *then*

$$\kappa_p(|f|, |g|, F_n(\Omega)) \leq B_p(m_1, M_1, m_2, M_2) \left(\int_{\Omega} w(f+g)^p \, d\mu\right)^{1/p}. \tag{2.19}$$

Proof. By the inequality (2.18) we have

$$\left(\int_{\Omega_i} wf^p d\mu\right)^{1/p} + \left(\int_{\Omega_i} wg^p d\mu\right)^{1/p}$$
$$\leq B_p\left(m_1, M_1, m_2, M_2\right)\left(\int_{\Omega_i} w\left(f+g\right)^p d\mu\right)^{1/p},$$

for all $i \in \{1, ..., n\}$.

If we take the power $p \geq 1$ and sum over i from 1 to n, then we get

$$\sum_{i=1}^{n}\left[\left(\int_{\Omega_i} wf^p d\mu\right)^{1/p} + \left(\int_{\Omega_i} wg^p d\mu\right)^{1/p}\right]^p$$
$$\leq B_p^p\left(m_1, M_1, m_2, M_2\right)\sum_{i=1}^{n}\int_{\Omega_i} w\left(f+g\right)^p d\mu$$
$$= B_p^p\left(m_1, M_1, m_2, M_2\right)\int_{\Omega} w\left(f+g\right)^p d\mu$$

namely

$$\kappa_p^p\left(|f|, |g|, F_n(\Omega)\right) \leq B_p^p\left(m_1, M_1, m_2, M_2\right)\int_{\Omega} w\left(f+g\right)^p d\mu,$$

which is equivalent to (2.19). □

Corollary 2. *With the assumptions of Theorem 3 we have the inequality (2.19) with*

$$B_p\left(m_1, M_1, m_2, M_2\right) = \left(\frac{\max\{M_1, M_2\}}{\min\{m_1, m_2\}}\frac{\left(M_1+M_2\right)^{p-1}}{\left(m_1+m_2\right)^{p-1}}\right)^{\frac{p-1}{p^2}}$$

or

$$B_p\left(m_1, M_1, m_2, M_2\right) = \frac{M_1\left(m_1+M_2\right) + M_2\left(M_1+m_2\right)}{\left(m_1+M_2\right)\left(M_1+m_2\right)}.$$

3. Discrete Inequalities

When μ is the discrete measure, then the corresponding discrete inequalities for complex (real) numbers can be stated as well. We give here some examples.

Assume that, for $n \geq 2$, the family J of indices contains more than n elements and $F_n(J) = \{J_i\}_{i\in\{1,...,n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^{n} J_i$ and $J_i \cap J_j = \emptyset$ for any i, $j \in \{1, ..., n\}$ with $i \neq j$.

For a given $n \geq 2$ we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\kappa_p(|x|, |y|, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ with $p \geq 1$ defined by

$$\kappa_p\left(|x|, |y|, F_n(J)\right) := \left(\sum_{i=1}^{n}\left[\left(\sum_{j\in J_i} w_j |x_j|^p\right)^{1/p} + \left(\sum_{j\in J_i} w_j |y_j|^p\right)^{1/p}\right]^p\right)^{1/p}, \tag{3.1}$$

where $x = \{x_j\}_{j\in J}$, $y = \{y_j\}_{j\in J} \subset \mathbb{C}$ and $w_j > 0$ for $j \in J$.

From the inequality (2.1) we have

$$\left(\sum_{j\in J} w_j |x_j+y_j|^p\right)^{1/p} \leq \kappa_p(|x|,|y|,F_n(J)) \leq \left(\sum_{j\in J} w_j |x_j|^p\right)^{1/p} + \left(\sum_{j\in J} w_j |y_j|^p\right)^{1/p}. \tag{3.2}$$

If there exists h, H, u, $U>0$ with the property

$$0<h\leq \left(\sum_{j\in J_i} w_j |x_j|^p\right)^{1/p} \leq H<\infty \tag{3.3}$$

and

$$0<u\leq \left(\sum_{j\in J_i} w_j |y_j|^p\right)^{1/p} \leq U<\infty \tag{3.4}$$

for each $i\in\{1,...,n\}$. Then

$$\left(\sum_{j\in J} w_j |x_j|^p\right)^{1/p} + \left(\sum_{j\in J} w_j |y_j|^p\right)^{1/p} \leq B_p(h,H,u,U)\,\kappa_p(|x|,|y|,F_n(J)). \tag{3.5}$$

Observe that we have the inequality (3.5) with

$$B_p(h,H,u,U) = \left(\frac{\max\{H,U\}}{\min\{h,u\}}\frac{(H+U)^{p-1}}{(h+u)^{p-1}}\right)^{\frac{p-1}{p^2}}$$

or

$$B_p(h,K,l,L) = \frac{H(h+U)+U(H+u)}{(h+U)(H+u)}.$$

Assume that there exist the constants m_1, M_1, m_2, M_2 such that

$$0<m_1\leq x_j\leq M_1<\infty,\ 0<m_2\leq y_j\leq M_2<\infty \tag{3.6}$$

for $j\in J$. Then by (2.19) we get

$$\kappa_p(|x|,|y|,F_n(J)) \leq B_p(m_1,M_1,m_2,M_2)\left(\sum_{j\in J} w_j (x_j+y_j)^p\right)^{1/p}. \tag{3.7}$$

The inequality (3.7) holds for

$$B_p(m_1,M_1,m_2,M_2) = \left(\frac{\max\{M_1,M_2\}}{\min\{m_1,m_2\}}\frac{(M_1+M_2)^{p-1}}{(m_1+m_2)^{p-1}}\right)^{\frac{p-1}{p^2}}$$

or

$$B_p(m_1,M_1,m_2,M_2) = \frac{M_1(m_1+M_2)+M_2(M_1+m_2)}{(m_1+M_2)(M_1+m_2)}.$$

References

[1] S. Abramovich, Jensen, Hölder, Minkowski, Jensen-Steffensen and Slater-Pečarić inequalities derived through N-quasiconvexity. *Math. Inequal. Appl.* **19** (2016), no. 4, 1203–1226.
[2] L. Bougoffa, On Minkowski and Hardy integral inequality, *J. Inequal. Pure and Appl. Math.*, **7** (2), (2006) Art.60.
[3] R. E. Castillo and E. Trousselot, Reverse generalized Hölder and Minkowski type inequalities and their applications, *Bol. Mat.* **17** (2), 137-142 (2010).
[4] B. Sroysang, , More on reverses of Minkowski's integral inequality, Mathematica Aeterna, **3** (7), (2013) pp. 597-600.
[5] W. T. Sulaiman, Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities, *Int. J. Mod. Math. Sci.*, 2012, **1** (1), 14–24.
[6] C.-J. Zhao and W.-S. Cheung, On reverse Minkowski-type inequalities. *Mediterr. J. Math.* **12** (2015), no. 3, 1085–1094.

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