LOWER AND UPPER BOUNDS IN TERMS OF SECOND DERIVATIVE FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta \mathcal{D}''(w, \mu)(\beta) \\ &\leq \mathcal{D}(w, \mu)(B) - \mathcal{D}(w, \mu)(A) - D(\mathcal{D}(w, \mu))(A)(B - A) \\ &\leq \frac{1}{2}\Delta \mathcal{D}''(w, \mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta \mathcal{D}''(w, \mu)(\beta) \\ &\leq \int_0^1 \mathcal{D}(w, \mu)((1 - t)A + tB)\, dt - \mathcal{D}(w, \mu)\left(\frac{A + B}{2}\right) \\ &\leq \frac{1}{24}\Delta \mathcal{D}''(w, \mu)(\alpha), \end{aligned}$$

where $D(\mathcal{D}(w, \mu))$ is the Fréchet derivative of $\mathcal{D}(w, \mu)$ as an operator functions and $\mathcal{D}''(w, \mu)$ is the second derivative of $\mathcal{D}(w, \mu)$ as a real function.

We also have the integral inequalities for power $r \in (0, 1]$

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta(1 - r)(2 - r)\beta^{r-3} \leq \int_0^1 ((1 - t)A + tB)^{r-1} dt - \left(\frac{A + B}{2}\right)^{r-1} \\ &\leq \frac{1}{24}\Delta(1 - r)(2 - r)\alpha^{r-3}. \end{aligned}$$

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

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Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.2}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.1).

For some example of operator monotone functions see [3]-[5], [8], [9] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.3}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.4}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.6}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^r=\frac{\sin(r\pi)}{\pi}t\mathcal{D}(w_r)(t),\ t>0. \tag{1.7}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.8}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.9}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.10}$$

for $T>0$.

In this paper, we show among others that, if $\beta\geq A$, $B\geq\alpha>0$, and $0<\delta\leq(B-A)^2\leq\Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned}0&\leq\frac{1}{2}\delta\mathcal{D}''(w,\mu)(\beta)\\&\leq\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A)\\&\leq\frac{1}{2}\Delta\mathcal{D}''(w,\mu)(\alpha)\end{aligned}$$

and

$$\begin{aligned}0&\leq\frac{1}{24}\delta\mathcal{D}''(w,\mu)(\beta)\\&\leq\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\&\leq\frac{1}{24}\Delta\mathcal{D}''(w,\mu)(\alpha).\end{aligned}$$

We also have the integral inequalities for power $r\in(0,1]$

$$\begin{aligned}0&\leq\frac{1}{24}\delta(1-r)(2-r)\beta^{r-3}\leq\int_0^1((1-t)A+tB)^{r-1}dt-\left(\frac{A+B}{2}\right)^{r-1}\\&\leq\frac{1}{24}\Delta(1-r)(2-r)\alpha^{r-3}.\end{aligned}$$

2. Main Results

We consider the inverse function $\ell^{-1}(t)=t^{-1}$, $t>0$ and the corresponding operator inverse function $\ell^{-1}(A)=A^{-1}$ defined for operators $A>0$. We denote by $S(B)$ the class of all selfadjoint operators on H.

Lemma 1. *The operator inverse function ℓ^{-1} is Fréchet differentiable in $A>0$ and*

$$D\left(\ell^{-1}\right)(A)(V)=-A^{-1}VA^{-1} \tag{2.1}$$

for all $V\in S(H)$.

Proof. Let $A>0$ and $V\in S(H)$. Then there exists a small open interval $(-\delta,\delta)$ such that for $t\in(-\delta,\delta)$, $A+tV$ is invertible. Then

$$\begin{aligned}\ell^{-1}(A+tV)-\ell^{-1}(A)&=(A+tV)^{-1}-A^{-1}=(A+tV)^{-1}(A-(A+tV))A^{-1}\\&=-t(A+tV)^{-1}VA^{-1},\end{aligned}$$

which implies that

$$\frac{\ell^{-1}(A+tV)-\ell^{-1}(A)}{t}=-(A+tV)^{-1}VA^{-1},\ t\neq 0.$$

By taking the limit over $t\to 0$, we get

$$\begin{aligned}D\left(\ell^{-1}\right)(A)(V)&=\lim_{t\to 0}\frac{\ell^{-1}(A+tV)-\ell^{-1}(A)}{t}=-\lim_{t\to 0}(A+tV)^{-1}VA^{-1}\\&=-A^{-1}VA^{-1}\end{aligned}$$

and the statement is proved. □

Lemma 2. *The operator inverse function ℓ^{-1} is twice Fréchet differentiable in $A>0$ and*

$$D^{2}\left(\ell^{-1}\right)(A)(V,V)=2A^{-1}\left(VA^{-1}V\right)A^{-1}\geq 0 \tag{2.2}$$

for all $V\in S(H)$.

Proof. Let $A>0$ and $V\in S(H)$. Then there exists a small open interval $(-\delta,\delta)$ such that for $t\in(-\delta,\delta)$, $A+tV$ is invertible and put

$$\begin{aligned}U_t&:=D\left(\ell^{-1}\right)(A+tV)(V)-D\left(\ell^{-1}\right)(A)(V)\\&=A^{-1}VA^{-1}-(A+tV)^{-1}V(A+tV)^{-1}.\end{aligned}$$

If we multiply both sides of U_t with $A+tV$, then we get

$$\begin{aligned}(A+tV)U_t(A+tV)&=(A+tV)A^{-1}VA^{-1}(A+tV)-V\\&=\left(1+tVA^{-1}\right)V\left(1+tA^{-1}V\right)-V\\&=\left(V+tVA^{-1}V\right)\left(1+tA^{-1}V\right)-V\\&=V+tVA^{-1}V+tVA^{-1}V+t^{2}VA^{-1}VA^{-1}V-V\\&=2tVA^{-1}V+t^{2}\left(VA^{-1}\right)^{2}V\\&=t\left[2VA^{-1}V+t\left(VA^{-1}\right)^{2}V\right].\end{aligned}$$

By multiplying this equality both sides with $(A+tV)^{-1}$ we get

$$U_t=t(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^{2}V\right](A+tV)^{-1}.$$

Therefore

$$\begin{aligned}D^{2}\left(\ell^{-1}\right)(A)(V,V)&=\lim_{t\to 0}\frac{D\left(\ell^{-1}\right)(A+tV)(V)-D\left(\ell^{-1}\right)(A)(V)}{t}\\&=\lim_{t\to 0}\frac{t(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^{2}V\right](A+tV)^{-1}}{t}\\&=\lim_{t\to 0}(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^{2}V\right](A+tV)^{-1}\\&=2A^{-1}\left(VA^{-1}V\right)A^{-1}.\end{aligned}$$

Since $A^{-1}>0$, then by multiplying both sides by V we get $VA^{-1}V\geq 0$ and by multiplying both sides by A^{-1} we get $A^{-1}\left(VA^{-1}V\right)A^{-1}\geq 0$, and the proof of (2.2) is thus proved. □

Lemma 3. *For all* C, $E>0$ *we have the representation*

$$E^{-1}-C^{-1}=-C^{-1}(E-C)C^{-1} \\ +2\int_0^1(1-t)((1-t)C+tE)^{-1} \\ \times\Big((E-C)((1-t)C+tE)^{-1}(E-C)\Big)((1-t)C+tE)^{-1}dt. \tag{2.3}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E)=f(C)+D(f)(C)(E-C) \\ +\int_0^1(1-t)D^2(f)((1-t)C+tE)(E-C,E-C)\,dt \tag{2.4}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E\in\mathcal{O}$.

Therefore, if we write this formula for ℓ^{-1} and the class of strictly positive operators, we get

$$E^{-1}=C^{-1}-C^{-1}(E-C)C^{-1} \\ +2\int_0^1(1-t)((1-t)C+tE)^{-1} \\ \times\Big((E-C)((1-t)C+tE)^{-1}(E-C)\Big)((1-t)C+tE)^{-1}dt$$

and the representation (2.3) is proved. □

Lemma 4. *For all* $A>0$,

$$D(\mathcal{D}(w,\mu))(A)(V)=-\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda) \tag{2.5}$$

for all $V\in S(H)$.

Proof. By the definition of $\mathcal{D}(w,\mu)$ we have for t in a small open interval around 0 that

$$\mathcal{D}(w,\mu)(A+tV)-\mathcal{D}(w,\mu)(A) \\ =\int_0^\infty w(\lambda)\Big[(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}\Big]d\mu(\lambda) \\ =\int_0^\infty w(\lambda)\Big[(\lambda+A+tV)^{-1}(\lambda+A-\lambda-A-tV)(\lambda+A)^{-1}\Big]d\mu(\lambda) \\ =-t\int_0^\infty w(\lambda)\Big[(\lambda+A+tV)^{-1}V(\lambda+A)^{-1}\Big]d\mu(\lambda).$$

Therefore,

$$\lim_{t\to0}\frac{\mathcal{D}(w,\mu)(A+tV)-\mathcal{D}(w,\mu)(A)}{t} \\ =-\lim_{t\to0}\int_0^\infty w(\lambda)\Big[(\lambda+A+tV)^{-1}V(\lambda+A)^{-1}\Big]d\mu(\lambda) \\ =-\int_0^\infty w(\lambda)\Big[(\lambda+A)^{-1}V(\lambda+A)^{-1}\Big]d\mu(\lambda)$$

and the identity (2.5) is obtained. □

We have the following upper and lower bounds for the error term in approximating $\mathcal{D}(w,\mu)(B)$ by $\mathcal{D}(w,\mu)(A)+D(\mathcal{D}(w,\mu))(A)(B-A)$.

Theorem 2. *Assume that* $\beta\geq A\geq\alpha>0$, $B>0$ *and* $0<m\leq B-A\leq M$ *for some constants* α, β, m, M, *then*

$$\begin{aligned}0&\leq\frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(M+\beta)-\mathcal{D}(w,\mu)(\beta)-\mathcal{D}'(w,\mu)(\beta)M\right]\\&\leq\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A)\\&\leq\frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(m+\alpha)-\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}'(w,\mu)(\alpha)m\right].\end{aligned}\tag{2.6}$$

Proof. From the identity (2.3) we get for $E=\lambda+B$ and $C=\lambda+A$ that

$$\begin{aligned}(\lambda+B)^{-1}-(\lambda+A)^{-1}&=-(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}\\&\quad+2\int_0^1(1-t)(\lambda+(1-t)A+tB)^{-1}\\&\quad\times\Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big)\\&\quad\times(\lambda+(1-t)A+tB)^{-1}dt\end{aligned}\tag{2.7}$$

for $\lambda\geq0$.

If we multiply this equality by $w(\lambda)\geq0$ and integrate, then we get

$$\begin{aligned}&\int_0^\infty w(\lambda)(\lambda+B)^{-1}d\mu(\lambda)-\int_0^\infty w(\lambda)(\lambda+A)^{-1}d\mu(\lambda)\\&=-\int_0^\infty w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}\\&+2\int_0^\infty w(\lambda)\Big(\int_0^1(1-t)(\lambda+(1-t)A+tB)^{-1}\\&\times\Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big)\\&\times(\lambda+(1-t)A+tB)^{-1}dt\Big)d\mu(\lambda)\end{aligned}\tag{2.8}$$

and since, by (2.5),

$$-\int_0^\infty w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}=D(\mathcal{D}(w,\mu))(A)(B-A),$$

hence by (2.8) we derive the equality

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A)\\&=2\int_0^\infty w(\lambda)\Big(\int_0^1(1-t)(\lambda+(1-t)A+tB)^{-1}\\&\times\Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big)\\&\times(\lambda+(1-t)A+tB)^{-1}dt\Big)d\mu(\lambda)\quad(\geq0).\end{aligned}\tag{2.9}$$

For $\lambda\geq0$ and $t\in[0,1]$ we have

$$\lambda+(1-t)A+tB=\lambda+A+t(B-A),$$

which by the assumption from the theorem implies that

$$\lambda+\alpha+tm\leq\lambda+(1-t)A+tB\leq\lambda+\beta+tM$$

for $\lambda\geq 0$ and $t\in[0,1]$.

This is equivalent to

$$(\lambda+\beta+tM)^{-1}\leq(\lambda+(1-t)A+tB)^{-1}\leq(\lambda+\alpha+tm)^{-1} \tag{2.10}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply this both sides with $B-A$ we obtain

$$\begin{aligned}(\lambda+\beta+tM)^{-1}(B-A)^2 &\leq (B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \\ &\leq(\lambda+\alpha+tm)^{-1}(B-A)^2.\end{aligned} \tag{2.11}$$

Since $0<m\leq B-A\leq M$, hence

$$m^2(\lambda+\beta+tM)^{-1}\leq(\lambda+\beta+tM)^{-1}(B-A)^2$$

and

$$(\lambda+\alpha+tm)^{-1}(B-A)^2\leq M^2(\lambda+\alpha+tm)^{-1},$$

then

$$\begin{aligned}m^2(\lambda+\beta+tM)^{-1} &\leq(B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \\ &\leq M^2(\lambda+\alpha+tm)^{-1}\end{aligned} \tag{2.12}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

Further, if we multiply (2.12) by $(\lambda+(1-t)A+tB)^{-1}$ both sides, we get

$$\begin{aligned}&m^2(\lambda+\beta+tM)^{-1}(\lambda+(1-t)A+tB)^{-2} \\ &\leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\ &\times(B-A)(\lambda+(1-t)A+tB)^{-1} \\ &\leq M^2(\lambda+\alpha+tm)^{-1}(\lambda+(1-t)A+tB)^{-2}\end{aligned} \tag{2.13}$$

and since

$$(\lambda+\beta+tM)^{-2}\leq(\lambda+(1-t)A+tB)^{-2}\leq(\lambda+\alpha+tm)^{-2},$$

then by (2.13) we get

$$\begin{aligned}&m^2(\lambda+\beta+tM)^{-3} \\ &\leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\ &\times(B-A)(\lambda+(1-t)A+tB)^{-1} \\ &\leq M^2(\lambda+\alpha+tm)^{-3}\end{aligned} \tag{2.14}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

By multiplication with $(1-t)\geq 0$, $w(\lambda)\geq 0$ integration and by making use of the identity (2.9), we deduce

$$\begin{aligned}&2m^2\int_0^\infty w(\lambda)\int_0^1\left[(1-t)(\lambda+\beta+tM)^{-3}dt\right]d\mu(\lambda) \\ &\leq\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A) \\ &\leq 2M^2\int_0^\infty w(\lambda)\int_0^1\left[(1-t)(\lambda+\alpha+tm)^{-3}dt\right]d\mu(\lambda).\end{aligned} \tag{2.15}$$

Using the identity (2.9) for the scalars we have

$$
\begin{aligned}
(2.16)\qquad & 0\leq \mathcal{D}(w,\mu)(M+\beta)-\mathcal{D}(w,\mu)(\beta)-\mathcal{D}'(w,\mu)(\beta)M \\
&=2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)\beta+t(M+\beta))^{-1}\right. \\
&\times\Big((M+\beta-\beta)(\lambda+(1-t)\beta+t(M+\beta))^{-1}(M+\beta-\beta)\Big) \\
&\left.\times(\lambda+(1-t)\beta+t(M+\beta))^{-1}dt\right)d\mu(\lambda) \\
&=2M^2\int_0^\infty w(\lambda)\int_0^1\left[(1-t)(\lambda+\beta+tM)^{-3}dt\right]d\mu(\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.17)\qquad & \mathcal{D}(w,\mu)(m+\alpha)-\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}'(w,\mu)(\alpha)m \\
&=2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}\right. \\
&\times\Big((m+\alpha-\alpha)(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}(m+\alpha-\alpha)\Big) \\
&\left.\times(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}dt\right)d\mu(\lambda) \\
&=2m^2\int_0^\infty w(\lambda)\int_0^1\left[(1-t)(\lambda+\alpha+tm)^{-3}dt\right]d\mu(\lambda).
\end{aligned}
$$

By making use of (2.15)-(2.17), we derive (2.6). □

Corollary 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ with $f(0)=0$. If $\beta\geq A\geq\alpha>0$, $B>0$ and $0<m\leq B-A\leq M$, then*

$$
\begin{aligned}
(2.18)\qquad & 0\leq\frac{m^2}{M^2}\left[\frac{f(M+\beta)}{M+\beta}-\frac{f(\beta)}{\beta}-\frac{f'(\beta)\beta-f(\beta)}{\beta^2}M\right] \\
&\leq f(B)B^{-1}-\left(2-A^{-1}B\right)A^{-1}f(A)-A^{-1}D(f)(A)(B-A) \\
&\leq\frac{M^2}{m^2}\left[\frac{f(m+\alpha)}{m+\alpha}-\frac{f(\alpha)}{\alpha}-\frac{f'(\alpha)\alpha-f(\alpha)}{\alpha^2}m\right].
\end{aligned}
$$

Proof. We have

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)t-f(t)}{t^2},\ t>0.$$

We have

$$
\begin{aligned}
&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D\left(\ell^{-1}f\right)(A)(B-A) \\
&=f(B)B^{-1}-f(A)A^{-1} \\
&-\left[D\left(\ell^{-1}\right)(A)(B-A)f(A)+\ell^{-1}(A)D(f)(A)(B-A)\right] \\
&=f(B)B^{-1}-f(A)A^{-1}+A^{-1}(B-A)A^{-1}f(A) \\
&-A^{-1}D(f)(A)(B-A),
\end{aligned}
$$

$$\mathcal{D}(w,\mu)(M+\beta)-\mathcal{D}(w,\mu)(\beta)-\mathcal{D}'(w,\mu)(\beta)M$$
$$=\frac{f(M+\beta)}{M+\beta}-\frac{f(\beta)}{\beta}-\frac{f'(\beta)\beta-f(\beta)}{\beta^2}M$$

and

$$\mathcal{D}(w,\mu)(m+\alpha)-\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}'(w,\mu)(\alpha)m$$
$$=\frac{f(m+\alpha)}{m+\alpha}-\frac{f(\alpha)}{\alpha}-\frac{f'(\alpha)\alpha-f(\alpha)}{\alpha^2}m$$

and by (2.6) we get (2.18). □

From a different perspective, we also have:

Theorem 3. *Assume that* $\beta \geq A$, $B \geq \alpha > 0$, *and* $0 < \delta \leq (B-A)^2 \leq \Delta$, *then*

$$\begin{aligned}0 &\leq \frac{1}{2}\delta\mathcal{D}''(w,\mu)(\beta)\\ &\leq \mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A)\\ &\leq \frac{1}{2}\Delta\mathcal{D}''(w,\mu)(\alpha).\end{aligned} \tag{2.19}$$

Proof. For $\lambda \geq 0$ and $t \in [0,1]$ we have

$$\lambda+\alpha \leq \lambda+(1-t)A+tB \leq \lambda+\beta,$$

which implies that

$$(\lambda+\beta)^{-1} \leq (\lambda+(1-t)A+tB)^{-1} \leq (\lambda+\alpha)^{-1}. \tag{2.20}$$

If we multiply this both sides with $B-A$, then we obtain

$$\begin{aligned}(\lambda+\beta)^{-1}(B-A)^2 &\leq (B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\\ &\leq (\lambda+\alpha)^{-1}(B-A)^2.\end{aligned} \tag{2.21}$$

Since $0 < \delta \leq (B-A)^2 \leq \Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2 \geq \delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2 \leq (\lambda+\alpha)^{-1}\Delta$, then by (2.21) we get

$$\delta(\lambda+\beta)^{-1} \leq (B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \leq (\lambda+\alpha)^{-1}\Delta, \tag{2.22}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply this inequality both sides with $(\lambda+(1-t)A+tB)^{-1}$ we derive

$$\begin{aligned}&\delta(\lambda+\beta)^{-1}(\lambda+(1-t)A+tB)^{-2}\\ &\leq (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\ &\times(B-A)(\lambda+(1-t)A+tB)^{-1}\\ &\leq (\lambda+\alpha)^{-1}\Delta(\lambda+(1-t)A+tB)^{-2}\end{aligned} \tag{2.23}$$

and by (2.20) we further obtain the bounds

$$\begin{aligned}&\delta(\lambda+\beta)^{-3}\\ &\leq (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\ &\times(B-A)(\lambda+(1-t)A+tB)^{-1}\\ &\leq (\lambda+\alpha)^{-3}\Delta.\end{aligned} \tag{2.24}$$

If we multiply with $2w(\lambda)(1-t)$ and integrate, then we get

$$\begin{aligned}
&2\delta\int_0^\infty w(\lambda)(\lambda+\beta)^{-3}\left(\int_0^1 (1-t)\,dt\right)d\mu(\lambda)\\
&\leq 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)A+tB)^{-1}\right.\\
&\times\left((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\right)\\
&\times\left.(\lambda+(1-t)A+tB)^{-1}\,dt\right)d\mu(\lambda)\\
&\leq 2\Delta\int_0^\infty w(\lambda)(\lambda+\alpha)^{-3}\left(\int_0^1 (1-t)\,dt\right)d\mu(\lambda),
\end{aligned}$$

which, by the equality (2.9), is equivalent to

$$\begin{aligned}
&\delta\int_0^\infty w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda)\\
&\leq \mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B-A)\\
&\leq \Delta\int_0^\infty w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).
\end{aligned}\tag{2.25}$$

If we take the derivative in (1.5) over t then we get

$$\mathcal{D}'(w,\mu)(t)=-\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),\ t>0,$$

and

$$\mathcal{D}''(w,\mu)(t)=2\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0.$$

This gives

$$\begin{aligned}
\int_0^\infty \frac{w(\lambda)}{(\lambda+\alpha)^3}d\mu(\lambda) &= \frac{1}{2}\mathcal{D}''(w,\mu)(\alpha),\\
\int_0^\infty \frac{w(\lambda)}{(\lambda+\beta)^3}d\mu(\lambda) &= \frac{1}{2}\mathcal{D}''(w,\mu)(\beta)
\end{aligned}$$

and by (2.25) we get (2.19). □

Corollary 2. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *with* $f(0)=0$*. If* $\beta\geq A$, $B\geq\alpha>0$, *and* $0<\delta\leq(B-A)^2\leq\Delta$, *then*

$$\begin{aligned}
0&\leq\frac{1}{2}\delta\left(\frac{f''(\beta)\beta^2-2\beta f'(\beta)+2f(\beta)}{\beta^3}\right)\\
&\leq f(B)B^{-1}-\left(2-A^{-1}B\right)A^{-1}f(A)-A^{-1}D(f)(A)(B-A)\\
&\leq\frac{1}{2}\Delta\left(\frac{f''(\alpha)\alpha^2-2\alpha f'(\alpha)+2f(\alpha)}{\alpha^3}\right).
\end{aligned}\tag{2.26}$$

Proof. We have

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)\,t-f(t)}{t^2},\ t>0$$

and the second derivative

$$\begin{aligned}\mathcal{D}''(\ell,\mu)(t)&=\frac{(f'(t)\,t-f(t))'\,t^2-2t\,(f'(t)\,t-f(t))}{t^4}\\&=\frac{(f''(t)\,t+f'(t)-f'(t))\,t^2-2t\,(f'(t)\,t-f(t))}{t^4}\\&=\frac{f''(t)\,t^3-2t^2f'(t)+2tf(t)}{t^4}=\frac{f''(t)\,t^2-2tf'(t)+2f(t)}{t^3}.\end{aligned}$$

By using (2.19) we obtain (2.26). □

We also have the midpoint type inequalities:

Theorem 4. *Assume that* $\beta\geq A,\ B\geq\alpha>0$ *and* $0<\delta\leq(B-A)^2\leq\Delta$, *then*

$$\begin{aligned}0&\leq\frac{1}{24}\delta\mathcal{D}''(w,\mu)(\beta)\\&\leq\int_0^1\mathcal{D}(w,\mu)((1-t)\,A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\&\leq\frac{1}{24}\Delta\mathcal{D}''(w,\mu)(\alpha)\,.\end{aligned}\tag{2.27}$$

Proof. For $t\in[0,1]$ we have

$$\left((1-t)\,A+tB-\frac{A+B}{2}\right)^2=\left(t-\frac{1}{2}\right)^2(B-A)^2\,.$$

Since $0<\delta\leq(B-A)^2\leq\Delta$, hence

$$\left(t-\frac{1}{2}\right)^2\delta\leq\left((1-t)\,A+tB-\frac{A+B}{2}\right)^2\leq\left(t-\frac{1}{2}\right)^2\Delta$$

for all $t\in[0,1]$.

By utilising (2.19) we derive

$$\begin{aligned}0&\leq\frac{1}{2}\delta\mathcal{D}''(w,\mu)(\beta)\left(t-\frac{1}{2}\right)^2\\&\leq\mathcal{D}(w,\mu)((1-t)\,A+tB)-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\&\quad-D\left(\mathcal{D}(w,\mu)\right)\left(\frac{A+B}{2}\right)\left(t-\frac{1}{2}\right)(B-A)\\&\leq\frac{1}{2}\Delta\mathcal{D}''(w,\mu)(\alpha)\left(t-\frac{1}{2}\right)^2\end{aligned}\tag{2.28}$$

for all $t\in[0,1]$.

If we integrate this inequality over $t \in [0,1]$, then we obtain

$$
\begin{aligned}
0 &\le \frac{1}{2}\delta \mathcal{D}''(w,\mu)(\beta)\int_0^1 \left(t-\frac{1}{2}\right)^2 dt \\
&\le \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&- \left(\int_0^1 \left(t-\frac{1}{2}\right)dt\right) D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)(B-A) \\
&\le \frac{1}{2}\Delta \mathcal{D}''(w,\mu)(\alpha)\int_0^1 \left(t-\frac{1}{2}\right)^2 dt.
\end{aligned}
\tag{2.29}
$$

Since

$$\int_0^1 \left(t-\frac{1}{2}\right)^2 dt = \frac{1}{12} \text{ and } \int_0^1 \left(t-\frac{1}{2}\right)dt = 0,$$

hence by (2.29) we deduce (2.27). □

Corollary 3. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *with* $f(0)=0$. *If* $\beta \ge A$, $B \ge \alpha > 0$, *and* $0 < \delta \le (B-A)^2 \le \Delta$, *then*

$$
\begin{aligned}
0 &\le \frac{1}{24}\delta\left(\frac{f''(\beta)\beta^2 - 2\beta f'(\beta) + 2f(\beta)}{\beta^3}\right) \\
&\le \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt \\
&- \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right) \\
&\le \frac{1}{24}\Delta\left(\frac{f''(\alpha)\alpha^2 - 2\alpha f'(\alpha) + 2f(\alpha)}{\alpha^3}\right).
\end{aligned}
\tag{2.30}
$$

3. Examples for Power Function

We consider the representation obtained from (1.7) for the operator $T>0$ and the power $r\in(0,1]$,

$$T^{r-1} = \mathcal{D}(\tilde{w}_r)(T)$$

with the kernel $\tilde{w}_r(\lambda) := \frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$.

Assume that $\beta \ge A \ge \alpha > 0$, $B>0$ and $0 < m \le B-A \le M$ for some constants α, β, m, M, then by (2.6) we get

$$
\begin{aligned}
0 &\le \frac{m^2}{M^2}\left[(1-r)\beta^{r-2}M + (M+\beta)^{r-1} - \beta^{r-1}\right] \\
&\le B^{r-1} - A^{r-1} + \int_0^\infty \lambda^{r-1}(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}\,d\lambda \\
&\le \frac{M^2}{m^2}\left[(1-r)\alpha^{r-2}m + (m+\alpha)^{r-1} - \alpha^{r-1}\right].
\end{aligned}
\tag{3.1}
$$

Under the same conditions, we obtain by (2.19),

$$0 \leq \frac{1}{2}\delta (1-r)(2-r)\beta^{r-3} \leq B^{r-1} - A^{r-1} + \int_0^\infty \lambda^{r-1} (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\lambda \leq \frac{1}{2}\Delta (1-r)(2-r)\alpha^{r-3} \tag{3.2}$$

for $r \in (0, 1]$.

By (2.27) we also have the integral inequality

$$0 \leq \frac{1}{24}\delta (1-r)(2-r)\beta^{r-3} \leq \int_0^1 ((1-t)A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \leq \frac{1}{24}\Delta (1-r)(2-r)\alpha^{r-3}. \tag{3.3}$$

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