ERROR BOUNDS RELATED TO MIDPOINT AND TRAPEZOID RULES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta\mathcal{D}''(w, \mu)(\beta) \\ &\leq \int_0^1 \mathcal{D}(w, \mu)((1-t)A + tB)\,dt - \mathcal{D}(w, \mu)\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{24}\Delta\mathcal{D}''(w, \mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{12}\delta\mathcal{D}''(w, \mu)(\beta) \\ &\leq \frac{\mathcal{D}(w, \mu)(A) + \mathcal{D}(w, \mu)(B)}{2} - \int_0^1 \mathcal{D}(w, \mu)((1-t)A + tB)\,dt \\ &\leq \frac{1}{12}\Delta\mathcal{D}''(w, \mu)(\alpha), \end{aligned}$$

where $\mathcal{D}''(w, \mu)$ is the second derivative of $\mathcal{D}(w, \mu)$ as a real function.

We also have the integral inequalities for power $r \in (0, 1]$,

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta(1-r)(2-r)\beta^{r-3} \leq \int_0^1 ((1-t)A + tB)^{r-1}\,dt - \left(\frac{A+B}{2}\right)^{r-1} \\ &\leq \frac{1}{24}\Delta(1-r)(2-r)\alpha^{r-3}. \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{12}\delta(1-r)(2-r)\beta^{r-3} \leq \frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1}\,dt \\ &\leq \frac{1}{12}\Delta(1-r)(2-r)\alpha^{r-3}. \end{aligned}$$

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued

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continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.2}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.1).

For some example of operator monotone functions see [3]-[5], [8], [9] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.3}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.4}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.6}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^r=\frac{\sin(r\pi)}{\pi}t\mathcal{D}(w_r)(t),\ t>0. \tag{1.7}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t = (t-1)\,\mathcal{D}(w_{\ln})(t)\,,\ \ t>0. \tag{1.8}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1}\,d\mu(\lambda)\,, \tag{1.9}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1}\,d\lambda, \tag{1.10}$$

for $T>0$.

In this paper, we show among others that, if $\beta\ge A$, $B\ge\alpha>0$, and $0<\delta\le (B-A)^2\le\Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned}0 &\le \frac{1}{24}\delta\mathcal{D}''(w,\mu)(\beta)\\ &\le \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\ &\le \frac{1}{24}\Delta\mathcal{D}''(w,\mu)(\alpha)\end{aligned}$$

and

$$\begin{aligned}0 &\le \frac{1}{12}\delta\mathcal{D}''(w,\mu)(\beta)\\ &\le \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} - \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt\\ &\le \frac{1}{12}\Delta\mathcal{D}''(w,\mu)(\alpha)\,.\end{aligned}$$

We also have the integral inequalities for power $r\in(0,1]$,

$$\begin{aligned}0 &\le \frac{1}{24}\delta(1-r)(2-r)\beta^{r-3} \le \int_0^1 ((1-t)A+tB)^{r-1}\,dt - \left(\frac{A+B}{2}\right)^{r-1}\\ &\le \frac{1}{24}\Delta(1-r)(2-r)\alpha^{r-3}.\end{aligned}$$

and

$$\begin{aligned}0 &\le \frac{1}{12}\delta(1-r)(2-r)\beta^{r-3} \le \frac{A^{r-1}+B^{r-1}}{2} - \int_0^1 ((1-t)A+tB)^{r-1}\,dt\\ &\le \frac{1}{12}\Delta(1-r)(2-r)\alpha^{r-3}.\end{aligned}$$

2. Preliminary Results

We have the following representation of the Fréchet derivative:

Lemma 1. *For all $A>0$,*

$$D(\mathcal{D}(w,\mu))(A)(V) = -\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}\,d\mu(\lambda) \tag{2.1}$$

for all $V\in S(H)$, the class of all selfajoint operators on H.

Proof. By the definition of $\mathcal{D}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}
&\mathcal{D}(w,\mu)(A+tV)-\mathcal{D}(w,\mu)(A)\\
&=\int_0^\infty w(\lambda)\left[(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\left[(\lambda+A+tV)^{-1}(\lambda+A-\lambda-A-tV)(\lambda+A)^{-1}\right]d\mu(\lambda)\\
&=-t\int_0^\infty w(\lambda)\left[(\lambda+A+tV)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{D}(w,\mu)(A+tV)-\mathcal{D}(w,\mu)(A)}{t}\\
&=-\lim_{t\to 0}\int_0^\infty w(\lambda)\left[(\lambda+A+tV)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)\\
&=-\int_0^\infty w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is obtained. □

The second Fréchet derivative can be represented as follows:

Lemma 2. *For all $A>0$,*

$$\begin{aligned}
&D^2(\mathcal{D}(w,\mu))(A)(V,V)\\
&=2\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}\tag{2.2}$$

for all $V\in S(H)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2(\mathcal{D}(w,\mu))(A)(V,V)\\
&=\lim_{t\to 0}\frac{D(\mathcal{D}(w,\mu))(A+tV)(V)-D(\mathcal{D}(w,\mu))(A)(V)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D(\mathcal{D}(w,\mu))(A+tV)(V)\\
&=-\int_0^\infty w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D(\mathcal{D}(w,\mu))(A+tV)(V)-D(\mathcal{D}(w,\mu))(A)(V)\\
&=-\int_0^\infty w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda)\\
&+\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\\
&\times\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda \geq 0$ and t as above,

$$U_{t,\lambda} := (\lambda + A)^{-1} V (\lambda + A)^{-1} - (\lambda + A + tV)^{-1} V (\lambda + A + tV)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda + A + tV$, the we get

$$\begin{aligned}
&(\lambda + A + tV) U_{t,\lambda} (\lambda + A + tV) \\
&= (\lambda + A + tV)(\lambda + A)^{-1} V (\lambda + A)^{-1} (\lambda + A + tV) - V \\
&= \left(1 + tV (\lambda + A)^{-1}\right) V \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= \left(V + tV (\lambda + A)^{-1} V\right) \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= V + tV (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V \\
&+ t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V - V \\
&= 2tV (\lambda + A)^{-1} V + t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V \\
&= t \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right].
\end{aligned} \tag{2.3}$$

If we multiply the equality by $(\lambda + A + tV)^{-1}$ both sides, we get for $t \neq 0$

$$\begin{aligned}
\frac{U_{t,\lambda}}{t} &= (\lambda + A + tV)^{-1} \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right] \\
&\times (\lambda + A + tV)^{-1}.
\end{aligned} \tag{2.4}$$

If we take the limit over $t \to 0$ in, then we get

$$\lim_{t \to 0} \left(\frac{U_{t,\lambda}}{t}\right) = 2 (\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1}.$$

Therefore, by the properties of limit under the sign of integral, we derive

$$\begin{aligned}
&\lim_{t \to 0} \frac{D(\mathcal{D}(w,\mu))(A + tV)(V) - D(\mathcal{D}(w,\mu))(A)(V)}{t} \\
&= \int_0^\infty w(\lambda) \lim_{t \to 0} \left(\frac{U_{t,\lambda}}{t}\right) d\mu(\lambda) \\
&= 2 \int_0^\infty w(\lambda) (\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda)
\end{aligned}$$

and the representation (2.2) is obtained. □

We have the following representation for the transform $\mathcal{D}(w,\mu)$:

Theorem 2. *For all A, $B > 0$ we have*

$$\begin{aligned}
&\mathcal{D}(w,\mu)(B) \\
&= \mathcal{D}(w,\mu)(A) - \int_0^\infty w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda) \\
&+ 2 \int_0^1 (1 - t) \left[\int_0^\infty w(\lambda) (\lambda + (1 - t) A + tB)^{-1} (B - A)\right. \\
&\left.\times (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} d\mu(\lambda)\right] dt.
\end{aligned} \tag{2.5}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E)=f(C)+D(f)(C)(E-C)+\int_0^1(1-t)D^2(f)((1-t)C+tE)(E-C,E-C)\,dt \tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and $C,\ E\in\mathcal{O}$.

If we write (2.6) for $\mathcal{D}(w,\mu)$ and $A,\ B>0$, then we get

$$\mathcal{D}(w,\mu)(B)=\mathcal{D}(w,\mu)(A)+D(\mathcal{D}(w,\mu))(A)(B-A)+\int_0^1(1-t)D^2(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A,B-A)\,dt$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). □

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B}:[0,1]\to\mathbb{R}$ defined by

$$f_{A,B}(t):=f((1-t)A+tB),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on $[0,1]$,*

$$\frac{df_{A,B}(t)}{dt}=D(f)((1-t)A+tB)(B-A) \tag{2.7}$$

and

$$\frac{d^2f_{A,B}(t)}{dt^2}=D^2(f)((1-t)A+tB)(B-A,B-A) \tag{2.8}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq0$,

$$\begin{aligned}&\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h\to0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt}&=\lim_{h\to0}\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\lim_{h\to0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\&=D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (2.7).

Similarly,

$$\begin{aligned}&\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\\&=\frac{D(f)((1-(t+h))A+(t+h)B)(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\\&=\frac{D(f)((1-t)A+tB+h(B-A))(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}\frac{d^2 f_{A,B}(t)}{dt^2}&=\lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\right\}\\&=D^2(f)((1-t)A+tB)(B-A,B-A),\end{aligned}$$

which proves (2.8). □

For the transform $\mathcal{D}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\begin{aligned}\mathcal{D}(w,\mu)_{A,B}(t)&:=\mathcal{D}(w,\mu)((1-t)A+tB)\\&=\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}$$

where $A,\ B>0$ and $t\in[0,1]$.

Corollary 1. *For all* $A,\ B>0$ *and* $t\in[0,1]$,

$$\begin{aligned}\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}&=D(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A)\\&=-\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\quad\times(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}\\&=D^2(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A,B-A)\\&=2\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}\tag{2.10}$$

We observe that if $f(t)=\mathcal{D}(w,\mu)(t),\ t>0$, in Lemma 3, then by the representations from Lemma 1 and Lemma 2 we obtain the desired equalities (2.9) and (2.10).

3. Main Results

We have the following identity for the midpoint rule:

Theorem 3. *For all A, $B>0$ we have the identity*

$$\begin{aligned}
&\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right. \\
&\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.
\end{aligned}\tag{3.1}$$

Proof. From (2.5) we have for $B=E>0$ and $A=C>0$ that

$$\begin{aligned}
&\mathcal{D}(w,\mu)(E) \\
&=\mathcal{D}(w,\mu)(C)-\int_0^\infty w(\lambda)(\lambda+C)^{-1}(E-C)(\lambda+C)^{-1}d\mu(\lambda) \\
&+2\int_0^1(1-s)\left[\int_0^\infty w(\lambda)(\lambda+(1-s)C+sE)^{-1}(E-C)\right. \\
&\left.\times(\lambda+(1-s)C+sE)^{-1}(E-C)(\lambda+(1-s)C+sE)^{-1}d\mu(\lambda)\right]ds,
\end{aligned}$$

which implies for $E=(1-t)A+tB$, $t\in[0,1]$ and $C=\frac{A+B}{2}$, that

$$\begin{aligned}
&\mathcal{D}(w,\mu)((1-t)A+tB) \\
&=\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&-\left(t-\frac{1}{2}\right)\int_0^\infty w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1}d\mu(\lambda) \\
&+2\left(t-\frac{1}{2}\right)^2\int_0^1(1-s) \\
&\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds.
\end{aligned}\tag{3.2}$$

If we integrate (3.2) over $t \in [0,1]$, then we get

$$\begin{aligned}
&\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\
&= \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&- \int_0^1 \left(t-\frac{1}{2}\right) dt \\
&\times \int_0^\infty w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1} d\mu(\lambda) \\
&+ 2\int_0^1 \left(t-\frac{1}{2}\right)^2 \left\{\int_0^1 (1-s)\right. \\
&\times \left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times \left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} d\mu(\lambda)\right] ds\right\} dt
\end{aligned}$$

and since $\int_0^1 \left(t-\frac{1}{2}\right) dt = 0$, hence the identity (3.1) is proved. □

Corollary 2. *Assume that $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then*

$$\begin{aligned}
0 &\leq \frac{1}{24}\delta\mathcal{D}''(w,\mu)(\beta) \\
&\leq \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&\leq \frac{1}{24}\Delta\mathcal{D}''(w,\mu)(\alpha).
\end{aligned} \tag{3.3}$$

Proof. Since $\beta \geq A$, $B \geq \alpha > 0$, hence

$$\lambda+\alpha \leq \lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB) \leq \lambda+\beta,$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

This implies that

$$(\lambda+\beta)^{-1} \leq \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} \leq (\lambda+\alpha)^{-1} \tag{3.4}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply this both sides with $B-A$, then we obtain

$$\begin{aligned}
&(\lambda+\beta)^{-1}(B-A)^2 \\
&\leq (B-A)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\leq (\lambda+\alpha)^{-1}(B-A)^2
\end{aligned} \tag{3.5}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

Since $0 < \delta \leq (B-A)^2 \leq \Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2 \geq \delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2 \leq (\lambda+\alpha)^{-1}\Delta$, then by (3.5)

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1} \\
&\leq (B-A)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\leq \Delta(\lambda+\alpha)^{-1}
\end{aligned} \tag{3.6}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply both sides with $\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}$ we derive

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-2} \\
&\leq \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} \\
&\leq \Delta(\lambda+\alpha)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-2}
\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

By utilising (3.4) we further obtain the bounds

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3} \\
&\leq \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} \\
&\leq \Delta(\lambda+\alpha)^{-3}
\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply by $2w(\lambda)\left(t-\frac{1}{2}\right)^2(1-s)\geq 0$ and integrate, then we get

$$
\begin{aligned}
(3.7)\qquad & 2\delta\int_0^\infty w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2dt\int_0^1(1-s)\,ds \\
&\leq 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right. \\
&\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt \\
&\leq 2\Delta\int_0^\infty w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2dt\int_0^1(1-s)\,ds
\end{aligned}
$$

and by the identity (3.1) and the fact that

$$\int_0^1\left(t-\frac{1}{2}\right)^2dt=\frac{1}{12}\text{ and }\int_0^1(1-s)\,ds=\frac{1}{2},$$

we obtain

$$
\begin{aligned}
(3.8)\qquad & \frac{1}{12}\delta\int_0^\infty w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda) \\
&\leq\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&\leq\frac{1}{12}\Delta\int_0^\infty w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda).
\end{aligned}
$$

If we take the derivative in (1.5) over t, then we get

$$\mathcal{D}'(w,\mu)(t)=-\int_0^\infty\frac{w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),\ t>0,$$

and

$$\mathcal{D}''(w,\mu)(t)=2\int_0^\infty\frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0.$$

This gives

$$
\begin{aligned}
\int_0^\infty\frac{w(\lambda)}{(\lambda+\alpha)^3}d\mu(\lambda) &= \frac{1}{2}\mathcal{D}''(w,\mu)(\alpha), \\
\int_0^\infty\frac{w(\lambda)}{(\lambda+\beta)^3}d\mu(\lambda) &= \frac{1}{2}\mathcal{D}''(w,\mu)(\beta)
\end{aligned}
$$

and by (3.2) we obtain (3.3). □

We have the following identity for the trapezoid rule:

Theorem 4. *For all* A, $B>0$ *we have the identity*

$$\begin{aligned}(3.9)\quad & \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\ & =\int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ & \left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}& \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\ & =\frac{1}{2}\left[t(1-t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^1-\int_0^1(1-2t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt\right] \\ & =\int_0^1\left(t-\frac{1}{2}\right)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt \\ & =\left(t-\frac{1}{2}\right)\mathcal{D}(w,\mu)_{A,B}(t)\bigg|_0^1-\int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt \\ & =\frac{1}{2}\left[\mathcal{D}(w,\mu)_{A,B}(1)+\mathcal{D}(w,\mu)_{A,B}(0)\right]-\int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt,\end{aligned}$$

that gives the identity

$$\begin{aligned}(3.10)\quad & \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\ & =\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt.\end{aligned}$$

By (2.10) we have

$$\begin{aligned}(3.11)\quad & \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\ & =\int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ & \left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}$$

By making use of (3.10) and (3.11). □

We have:

Corollary 3. *Assume that* $\beta\geq A$, $B\geq\alpha>0$ *and* $0<\delta\leq(B-A)^2\leq\Delta$, *then*

$$\begin{aligned}(3.12)\quad 0 & \leq \frac{1}{12}\delta\mathcal{D}''(w,\mu)(\beta) \\ & \leq \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt \\ & \leq \frac{1}{12}\Delta\mathcal{D}''(w,\mu)(\alpha).\end{aligned}$$

Proof. As in the proof of Corollary 2 we have

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3} \\
&\leq(\lambda+(1-t)A+tB)^{-1}(B-A) \\
&\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\leq\Delta(\lambda+\alpha)^{-3}
\end{aligned} \tag{3.13}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply by $t(1-t)w(\lambda)\geq 0$ and integrate, then we get

$$\begin{aligned}
&\delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda) \\
&\leq\int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt \\
&\leq\Delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).
\end{aligned} \tag{3.14}$$

Since

$$\int_0^1 t(1-t)\,dt=\frac{1}{6},$$

$$\int_0^\infty\frac{w(\lambda)}{(\lambda+\alpha)^3}d\mu(\lambda)=\frac{1}{2}\mathcal{D}''(w,\mu)(\alpha)$$

and

$$\int_0^\infty\frac{w(\lambda)}{(\lambda+\beta)^3}d\mu(\lambda)=\frac{1}{2}\mathcal{D}''(w,\mu)(\beta),$$

then by (3.14) we derive (3.12). □

We have an alternative identity for the midpoint rule:

Theorem 5. *For all A, $B>0$ we have the identity*

$$\begin{aligned}
&\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=\int_0^{1/2}t^2\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt \\
&+\int_{1/2}^1(t-1)^2\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt.
\end{aligned} \tag{3.15}$$

Proof. Using integration by parts for Bochner's integral, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= \frac{1}{2}\left[t^2 \frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^{1/2} - 2\int_0^{1/2} t\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt\right] \\
&= \frac{1}{8}\frac{d\mathcal{D}(w,\mu)_{A,B}(1/2)}{dt} - \int_0^{1/2} t\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt \\
&= \frac{1}{8}\frac{d\mathcal{D}(w,\mu)_{A,B}(1/2)}{dt} \\
&- \left[t\mathcal{D}(w,\mu)_{A,B}(t)\Big|_0^{1/2} - \int_0^{1/2}\mathcal{D}(w,\mu)_{A,B}(t)\,dt\right] \\
&= \frac{1}{8}\frac{d\mathcal{D}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{D}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{D}(w,\mu)_{A,B}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{1/2}^{1} (t-1)^2 \frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= \frac{1}{2}\left[(t-1)^2 \frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}\bigg|_{1/2}^{1} - 2\int_{1/2}^{1} (t-1)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt\right] \\
&= -\frac{1}{8}\frac{d\mathcal{D}(w,\mu)_{A,B}(1/2)}{dt} \\
&- \left[(t-1)\mathcal{D}(w,\mu)_{A,B}(t)\Big|_{1/2}^{1} - \int_{1/2}^{1}\mathcal{D}(w,\mu)_{A,B}(t)\,dt\right] \\
&= -\frac{1}{8}\frac{d\mathcal{D}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{D}(w,\mu)_{A,B}(1/2) + \int_{1/2}^{1}\mathcal{D}(w,\mu)_{A,B}(t)\,dt.
\end{aligned}$$

If we add these two equalities, then we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt + \frac{1}{2}\int_{1/2}^{1} (t-1)^2 \frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\mathcal{D}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{D}(w,\mu)_{A,B}(t)\,dt + \int_{1/2}^{1}\mathcal{D}(w,\mu)_{A,B}(t)\,dt \\
&= \int_0^{1}\mathcal{D}(w,\mu)_{A,B}(t)\,dt - \mathcal{D}(w,\mu)_{A,B}(1/2).
\end{aligned} \tag{3.16}$$

By (2.10) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2 \frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= \int_0^{1/2} t^2 \left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned} & \frac{1}{2}\int_{1/2}^{1}(t-1)^2\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \\ & =\int_{1/2}^{1}(t-1)^2\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ & \left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt. \end{aligned} \tag{3.18}$$

By employing (3.16)-(3.18) we derive the desired result (3.15). □

Remark 1. *By making use of the identity (3.15) one can obtain the same upper and lower bounds for the midpoint rule as those in Corollary 2.*

4. Examples for Operator Monotone Functions

We have the following result:

Proposition 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ with $f(0)=0$. If $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$, then*

$$\begin{aligned} 0 & \leq\frac{1}{24}\delta\left(\frac{f''(\beta)\beta^2-2\beta f'(\beta)+2f(\beta)}{\beta^3}\right) \\ & \leq\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)dt \\ & -\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right) \\ & \leq\frac{1}{24}\Delta\left(\frac{f''(\alpha)\alpha^2-2\alpha f'(\alpha)+2f(\alpha)}{\alpha^3}\right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 0 & \leq\frac{1}{12}\delta\left(\frac{f''(\beta)\beta^2-2\beta f'(\beta)+2f(\beta)}{\beta^3}\right) \\ & \leq\frac{A^{-1}f(A)+B^{-1}f(B)}{2} \\ & -\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)dt \\ & \leq\frac{1}{12}\Delta\left(\frac{f''(\alpha)\alpha^2-2\alpha f'(\alpha)+2f(\alpha)}{\alpha^3}\right). \end{aligned} \tag{4.2}$$

Proof. We have

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\geq0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)t-f(t)}{t^2},\ t>0$$

and the second derivative

$$\mathcal{D}''(\ell,\mu)(t) = \frac{(f'(t)t - f(t))' t^2 - 2t(f'(t)t - f(t))}{t^4}$$
$$= \frac{f''(t)t^3 - 2t^2 f'(t) + 2tf(t)}{t^4} = \frac{f''(t)t^2 - 2tf'(t) + 2f(t)}{t^3}.$$

By using (3.3) we obtain (4.1) and by (3.12) we obtain (4.2). □

Consider $f(t) = t^r$, $r \in (0,1]$. Then for $t > 0$,

$$\frac{f''(t)t^2 - 2tf'(t) + 2f(t)}{t^3} = \frac{r(r-1)t^r - 2rt^r + 2t^r}{t^3} = (1-r)(2-3)t^{r-3}$$

If $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then by Proposition 4.1 we get

$$0 \leq \frac{1}{24}\delta(1-r)(2-r)\beta^{r-3} \leq \int_0^1 ((1-t)A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \leq \frac{1}{24}\Delta(1-r)(2-r)\alpha^{r-3} \tag{4.3}$$

and

$$0 \leq \frac{1}{12}\delta(1-r)(2-r)\beta^{r-3} \leq \frac{A^{r-1}+B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1} dt \leq \frac{1}{12}\Delta(1-r)(2-r)\alpha^{r-3}. \tag{4.4}$$

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