SECOND DERIVATIVE LIPSCHITZ TYPE INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\| \\
&\leq \|B-A\|^2 \\
&\times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}
\end{aligned}$$

where $D(\mathcal{D}(w,\mu))$ is the Fréchet derivative of $\mathcal{D}(w,\mu)$ as a function of operator and $\mathcal{D}''(w,\mu)$ is the second derivative of $\mathcal{D}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r \in (0,1]$ and A, $B \geq m > 0$,

$$\begin{aligned}
&\left\| \int_0^1 ((1-t)A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \right\| \\
&\leq \frac{1}{24}(1-r)(2-r)m^{r-3}\|B-A\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1} dt \right\| \\
&\leq \frac{1}{12}(1-r)(2-r)m^{r-3}\|B-A\|^2.
\end{aligned}$$

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [16] had given a definitive characterization of operator monotone functions as follows, see for instance [5, p. 144-145]:

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Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda) \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.1).

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [14]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [11] and [13].

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

for any $A,B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [15], the following inequality holds

$$\||A|-|B|\|\leq\frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.2}$$

for any $A,B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\|A-B\|_{HS} \tag{1.3}$$

for any $A,B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\|\leq a_1\|A-B\|+a_2\|A-B\|^2+O\left(\|A-B\|^3\right) \tag{1.4}$$

where

$$a_1=\left\|A^{-1}\right\|\|A\| \text{ and } a_2=\left\|A^{-1}\right\|+\left\|A^{-1}\right\|^3\|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\|\leq f'(a)\|A-B\| \tag{1.5}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,B\geq a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.6}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.7}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.8}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.8) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.9}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.10}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t - 1) \mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.11}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.12}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.13}$$

for $T > 0$.

In this paper, we show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned}
&\left\| \mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A) \right\| \\
&\leq \|B-A\|^2 \\
&\times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) \text{ if } m_1 = m_2 = m, \end{cases}
\end{aligned}$$

where $D(\mathcal{D}(w,\mu))$ is the Fréchet derivative of $\mathcal{D}(w,\mu)$ as a function of operator and $\mathcal{D}''(w,\mu)$ is the second derivative of $\mathcal{D}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r \in (0,1]$ and A, $B \geq m > 0$,

$$\begin{aligned}
&\left\| \int_0^1 ((1-t)A + tB)^{r-1} dt - \left(\frac{A+B}{2}\right)^{r-1} \right\| \\
&\leq \frac{1}{24}(1-r)(2-r)m^{r-3}\|B-A\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{A^{r-1} + B^{r-1}}{2} - \int_0^1 ((1-t)A + tB)^{r-1} dt \right\| \\
&\leq \frac{1}{12}(1-r)(2-r)m^{r-3}\|B-A\|^2.
\end{aligned}$$

2. Preliminary Results

We have the following representation of the Fréchet derivative:

Lemma 1. *For all $A > 0$,*

$$D(\mathcal{D}(w,\mu))(A)(V) = -\int_0^\infty w(\lambda)(\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda) \tag{2.1}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. By the definition of $\mathcal{D}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}
&\mathcal{D}(w,\mu)(A+tV) - \mathcal{D}(w,\mu)(A) \\
&= \int_0^\infty w(\lambda)\left[(\lambda + A + tV)^{-1} - (\lambda + A)^{-1}\right] d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[(\lambda + A + tV)^{-1}(\lambda + A - \lambda - A - tV)(\lambda + A)^{-1}\right] d\mu(\lambda) \\
&= -t\int_0^\infty w(\lambda)\left[(\lambda + A + tV)^{-1} V (\lambda + A)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{D}(w,\mu)(A+tV)-\mathcal{D}(w,\mu)(A)}{t}\\
&=-\lim_{t\to 0}\int_0^\infty w(\lambda)\left[(\lambda+A+tV)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)\\
&=-\int_0^\infty w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is obtained. □

The second Fréchet derivative can be represented as follows:

Lemma 2. *For all $A>0$,*

$$\begin{aligned}
(2.2)\qquad &D^2(\mathcal{D}(w,\mu))(A)(V,V)\\
&=2\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}$$

for all $V\in S(H)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2(\mathcal{D}(w,\mu))(A)(V,V)\\
&=\lim_{t\to 0}\frac{D(\mathcal{D}(w,\mu))(A+tV)(V)-D(\mathcal{D}(w,\mu))(A)(V)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D(\mathcal{D}(w,\mu))(A+tV)(V)\\
&=-\int_0^\infty w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D(\mathcal{D}(w,\mu))(A+tV)(V)-D(\mathcal{D}(w,\mu))(A)(V)\\
&=-\int_0^\infty w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda)\\
&+\int_0^\infty w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\\
&\times\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda\geq 0$ and t as above,

$$U_{t,\lambda}:=(\lambda+A)^{-1}V(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda + A + tV$, the we get

$$
\begin{aligned}
&(\lambda + A + tV) U_{t,\lambda} (\lambda + A + tV) \\
&= (\lambda + A + tV) (\lambda + A)^{-1} V (\lambda + A)^{-1} (\lambda + A + tV) - V \\
&= \left(1 + tV (\lambda + A)^{-1}\right) V \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= \left(V + tV (\lambda + A)^{-1} V\right) \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= V + tV (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V \\
&+ t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V - V \\
&= 2tV (\lambda + A)^{-1} V + t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V \\
&= t \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right].
\end{aligned} \tag{2.3}
$$

If we multiply the equality by $(\lambda + A + tV)^{-1}$ both sides, we get for $t \neq 0$

$$
\begin{aligned}
\frac{U_{t,\lambda}}{t} &= (\lambda + A + tV)^{-1} \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right] \\
&\times (\lambda + A + tV)^{-1}.
\end{aligned} \tag{2.4}
$$

If we take the limit over $t \to 0$ in, then we get

$$
\lim_{t \to 0} \left(\frac{U_{t,\lambda}}{t}\right) = 2 (\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1}.
$$

Therefore, by the properties of limit under the sign of integral, we get

$$
\begin{aligned}
&\lim_{t \to 0} \frac{D(\mathcal{D}(w,\mu))(A + tV)(V) - D(\mathcal{D}(w,\mu))(A)(V)}{t} \\
&= \int_0^\infty w(\lambda) \lim_{t \to 0} \left(\frac{U_{t,\lambda}}{t}\right) d\mu(\lambda) \\
&= 2 \int_0^\infty w(\lambda) (\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda)
\end{aligned}
$$

and the representation (2.2) is obtained. □

We have the following representation for the transform $\mathcal{D}(w,\mu)$:

Theorem 2. *For all A, $B > 0$ we have*

$$
\begin{aligned}
&\mathcal{D}(w,\mu)(B) \\
&= \mathcal{D}(w,\mu)(A) - \int_0^\infty w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda) \\
&+ 2 \int_0^1 (1 - t) \left[\int_0^\infty w(\lambda) (\lambda + (1 - t) A + tB)^{-1} (B - A)\right. \\
&\left. \times (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} d\mu(\lambda)\right] dt.
\end{aligned} \tag{2.5}
$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [6, p. 112],

$$f(E) = f(C) + D(f)(C)(E-C) + \int_0^1 (1-t) D^2(f)((1-t)C + tE)(E-C, E-C)\,dt \tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E \in \mathcal{O}$.

If we write (2.6) for $\mathcal{D}(w,\mu)$ and A, $B > 0$, we get

$$\mathcal{D}(w,\mu)(B) = \mathcal{D}(w,\mu)(A) + D(\mathcal{D}(w,\mu))(A)(B-A) + \int_0^1 (1-t) D^2(\mathcal{D}(w,\mu))((1-t)A + tB)(B-A, B-A)\,dt$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). □

3. Main Results

We have the following Lipschitz type inequality:

Theorem 3. *Assume that* $A \geq m_1 > 0$, $B \geq m_2 > 0$, *then*

$$\begin{aligned} &\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\| \\ &\leq \|B-A\|^2 \\ &\times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_2) - \mathcal{D}(w,\mu)(m_1) - (m_2-m_1)\mathcal{D}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \frac{1}{2}\mathcal{D}''(w,\mu)(m) \quad \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.1}$$

Proof. From (2.5) we get

$$\begin{aligned} &\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A)\| \\ &\leq 2\int_0^1 (1-t)\left[\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}(B-A)\right.\right. \\ &\quad \left.\left.\times (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\right\| d\mu(\lambda)\right] dt \\ &\leq 2\|B-A\|^2 \int_0^1 (1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 d\mu(\lambda)\right) dt. \end{aligned} \tag{3.2}$$

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\|((1-t)A + tB + \lambda)^{-1}\right\|^3 \leq ((1-t)m_1 + tm_2 + \lambda)^{-3} \tag{3.3}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (3.3) we derive

$$\begin{aligned} &\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt \\ &\le \int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-3} d\mu(\lambda)\right)dt \\ &= \frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right. \\ &\times ((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1) \\ &\left.\times ((1-t)m_1+tm_2+\lambda)^{-1}\right) d\mu(\lambda)\,dt. \end{aligned} \tag{3.4}$$

From (2.5) we have for $m_2 > m_1$ that

$$\begin{aligned} &\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)+(m_2-m_1)\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda) \\ &= 2\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right. \\ &\times ((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1) \\ &\left.\times ((1-t)m_1+tm_2+\lambda)^{-1}\right) d\mu(\lambda)\Big]\,dt. \end{aligned} \tag{3.5}$$

Also

$$\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda) = -\mathcal{D}'(w,\mu)(m_1)$$

and then by (3.5) we get

$$\begin{aligned} &\frac{1}{2(m_2-m_1)^2}\left[\mathcal{D}(w,\mu)(m_2)-\mathcal{D}(w,\mu)(m_1)-(m_2-m_1)\mathcal{D}'(w,\mu)(m_1)\right] \\ &= \frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right. \\ &\left.\times ((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}\right) d\mu(\lambda)\,dt. \end{aligned} \tag{3.6}$$

By utilising (3.2) and (3.4)-(3.6) we derive (3.1).

The case $m_2 < m_1$ goes in a similar way and we also obtain (3.1).

Assume that $m_2 = m_1 > 0$. Let $\epsilon > 0$. Then $B+\epsilon \ge m+\epsilon > m$. By the first inequality for $m_2 = m+\epsilon$ and $m_1 = m$, we have

$$\begin{aligned} &\left\|\mathcal{D}(w,\mu)(B+\epsilon)-\mathcal{D}(w,\mu)(A)-D(\mathcal{D}(w,\mu))(A)(B+\epsilon-A)\right\| \\ &\le \|B+\epsilon-A\|^2\frac{1}{\epsilon^2}\left[\mathcal{D}(w,\mu)(m+\epsilon)-\mathcal{D}(w,\mu)(m)-\epsilon\mathcal{D}'(w,\mu)(m)\right]. \end{aligned} \tag{3.7}$$

By Taylor's expansion theorem with the Lagrange remainder we have

$$\mathcal{D}(w,\mu)(m+\epsilon)-\mathcal{D}(w,\mu)(m)-\epsilon\mathcal{D}'(w,\mu)(m) = \frac{1}{2}\epsilon^2\mathcal{D}''(w,\mu)(\zeta_\epsilon)$$

with $m+\epsilon > \zeta_\epsilon > m$. Therefore

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon^2}\left[\mathcal{D}(w,\mu)(m+\epsilon)-\mathcal{D}(w,\mu)(m)-\epsilon\mathcal{D}'(w,\mu)(m)\right] = \frac{1}{2}\mathcal{D}''(w,\mu)(m)$$

and by taking the limit $\epsilon \to 0+$ in (3.7) then we get

$$\begin{aligned}&\left\|\mathcal{D}\left(w,\mu\right)\left(B\right)-\mathcal{D}\left(w,\mu\right)\left(A\right)-D\left(\mathcal{D}\left(w,\mu\right)\right)\left(A\right)\left(B-A\right)\right\|\\&\leq\frac{1}{2}\left\|B-A\right\|^{2}\mathcal{D}''\left(w,\mu\right)\left(m\right)\end{aligned}$$

and the second part of (3.1) is proved. □

The case of operator monotone function is as follows:

Corollary 1. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function with* $f(0)=0$. *If* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\begin{aligned}&\left\|f\left(B\right)B^{-1}-\left(2-A^{-1}B\right)A^{-1}f\left(A\right)-A^{-1}D\left(f\right)\left(A\right)\left(B-A\right)\right\|\\&\leq\left\|B-A\right\|^{2}\\&\times\begin{cases}\frac{1}{(m_2-m_1)^2}\left[\frac{f(m_2)}{m_2}-\frac{f(m_1)}{m_1}-\left(m_2-m_1\right)\frac{f'(m_1)m_1-f(m_1)}{m_1^2}\right]\\ \text{if } m_1\neq m_2,\\ \\ \frac{1}{2}\frac{f''(m)m^2-2mf'(m)+2f(m)}{m^3}\ \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{3.8}$$

Proof. We have

$$\mathcal{D}\left(\ell,\mu\right)\left(t\right)=\frac{f\left(t\right)}{t}-b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'\left(\ell,\mu\right)\left(t\right)=\frac{f'\left(t\right)t-f\left(t\right)}{t^{2}},\ t>0$$

and the second derivative

$$\begin{aligned}\mathcal{D}''\left(\ell,\mu\right)\left(t\right)&=\frac{\left(f'\left(t\right)t-f\left(t\right)\right)'t^{2}-2t\left(f'\left(t\right)t-f\left(t\right)\right)}{t^{4}}\\&=\frac{\left(f''\left(t\right)t+f'\left(t\right)-f'\left(t\right)\right)t^{2}-2t\left(f'\left(t\right)t-f\left(t\right)\right)}{t^{4}}\\&=\frac{f''\left(t\right)t^{3}-2t^{2}f'\left(t\right)+2tf\left(t\right)}{t^{4}}=\frac{f''\left(t\right)t^{2}-2tf'\left(t\right)+2f\left(t\right)}{t^{3}}.\end{aligned}$$

We have

$$\begin{aligned}&\mathcal{D}\left(w,\mu\right)\left(B\right)-\mathcal{D}\left(w,\mu\right)\left(A\right)-D\left(\ell^{-1}f\right)\left(A\right)\left(B-A\right)\\&=f\left(B\right)B^{-1}-f\left(A\right)A^{-1}\\&-\left[D\left(\ell^{-1}\right)\left(A\right)\left(B-A\right)f\left(A\right)+\ell^{-1}\left(A\right)D\left(f\right)\left(A\right)\left(B-A\right)\right]\\&=f\left(B\right)B^{-1}-f\left(A\right)A^{-1}+A^{-1}\left(B-A\right)A^{-1}f\left(A\right)\\&-A^{-1}D\left(f\right)\left(A\right)\left(B-A\right),\end{aligned}$$

since $D\left(\ell^{-1}\right)\left(A\right)\left(B-A\right)=-A^{-1}\left(B-A\right)A^{-1}$.

Also

$$\begin{aligned}&\mathcal{D}\left(w,\mu\right)\left(m_2\right)-\mathcal{D}\left(w,\mu\right)\left(m_1\right)-\left(m_2-m_1\right)\mathcal{D}'\left(w,\mu\right)\left(m_1\right)\\&=\frac{f\left(m_2\right)}{m_2}-\frac{f\left(m_1\right)}{m_1}-\left(m_2-m_1\right)\frac{f'\left(m_1\right)m_1-f\left(m_1\right)}{m_1^{2}}.\end{aligned}$$

By making use of (3.1) we deduce (3.8). □

We consider the representation obtained from (1.9) for the operator $T>0$ and the power $r\in(0,1]$,

$$T^{r-1}=\mathcal{D}\left(\tilde{w}_r\right)(T)$$

with the kernel $\tilde{w}_r(\lambda):=\frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$.

From (3.1) we get for $A\geq m_1>0$, $B\geq m_2>0$ and $r\in(0,1]$ that

$$\left\|B^{r-1}-A^{r-1}+\int_0^\infty\lambda^{r-1}(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\lambda\right\| \tag{3.9}$$
$$\leq\|B-A\|^2\begin{cases}\frac{(1-r)(m_2-m_1)m_1^{r-2}-m_1^{r-1}+m_2^{r-1}}{(m_2-m_1)^2} & \text{if } m_1\neq m_2,\\ \\ \frac{1}{2}(1-r)(2-r)m^{r-3} & \text{if } m_1=m_2=m.\end{cases}$$

We have the following error bounds for operator Jensen's gap related to the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{D}(w,\mu)):=\sum_{k=1}^n p_k\mathcal{D}(w,\mu)(A_k)-\mathcal{D}(w,\mu)\left(\sum_{k=1}^n p_kA_k\right).$$

Theorem 4. *Assume that $A_i\geq m>0$ for $i\in\{1,...,n\}$ and consider the probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$, then*

$$\|J(\mathbf{A},\mathbf{p},\mathcal{D}(w,\mu))\|\leq\frac{1}{2}\mathcal{D}''(w,\mu)(m)\sum_{k=1}^n p_k\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2 \tag{3.10}$$
$$\leq\frac{1}{2}\mathcal{D}''(w,\mu)(m)\sum_{k=1}^n\sum_{j=1}^n p_jp_k\|A_k-A_j\|^2$$
$$\leq\frac{1}{2}\mathcal{D}''(w,\mu)(m)\max_{k,j\in\{1,...,n\}}\|A_k-A_j\|^2.$$

Proof. From (3.1) we get

$$\left\|\mathcal{D}(w,\mu)(A_k)-\mathcal{D}(w,\mu)\left(\sum_{j=1}^n p_jA_j\right)\right. \tag{3.11}$$
$$\left.-D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^n p_jA_j\right)\left(A_k-\sum_{j=1}^n p_jA_j\right)\right\|$$
$$\leq\frac{1}{2}\mathcal{D}''(w,\mu)(m)\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2$$

for all $k\in\{1,...,n\}$.

If we multiply this inequality by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\sum_{k=1}^{n}\left\| p_k\mathcal{D}(w,\mu)(A_k) - p_k\mathcal{D}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(p_kA_k - \sum_{j=1}^{n}p_jA_j\right)\right\| \leq \frac{1}{2}\mathcal{D}''(w,\mu)(m)\sum_{k=1}^{n}p_k\left\|A_k - \sum_{j=1}^{n}p_jA_j\right\|^2 . \tag{3.12}$$

By making use of the triangle inequality for norms, we also have

$$\begin{aligned}&\sum_{k=1}^{n}\left\| p_k\mathcal{D}(w,\mu)(A_k) - p_k\mathcal{D}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(p_kA_k - \sum_{j=1}^{n}p_jA_j\right)\right\| \\ &\geq \left\|\sum_{k=1}^{n}p_k\mathcal{D}(w,\mu)(A_k) - \sum_{k=1}^{n}p_k\mathcal{D}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right) - D(\mathcal{D}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(\sum_{k=1}^{n}p_kA_k - \sum_{j=1}^{n}p_jA_j\right)\right\| \\ &= \left\|\sum_{k=1}^{n}p_k\mathcal{D}(w,\mu)(A_k) - \mathcal{D}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right\| .\end{aligned} \tag{3.13}$$

By utilising (3.12) and (3.13) we deduce the first part of (3.10). The rest is obvious. □

Remark 1. *From (3.10) we can obtain the following norm inequalities for power* $r \in (0, 1]$,

$$\begin{aligned}&\left\|\sum_{k=1}^{n}p_kA_k^{r-1} - \mathcal{D}(w,\mu)\left(\sum_{k=1}^{n}p_kA_k\right)^{r-1}\right\| \\ &\leq \frac{1}{2}(1-r)(2-r)m^{r-3}\sum_{k=1}^{n}p_k\left\|A_k - \sum_{j=1}^{n}p_jA_j\right\|^2 \\ &\leq \frac{1}{2}(1-r)(2-r)m^{r-3}\sum_{k=1}^{n}\sum_{j=1}^{n}p_jp_k\left\|A_k - A_j\right\|^2 \\ &\leq \frac{1}{2}(1-r)(2-r)m^{r-3}\max_{k,j\in\{1,...,n\}}\left\|A_k - A_j\right\|^2 ,\end{aligned} \tag{3.14}$$

where $A_i \geq m > 0$ *for* $i \in \{1, ..., n\}$ *and the probability density* n*-tuple* $\mathbf{p} = (p_1, ..., p_n)$.

4. Midpoint and Trapezoid Inequalities

We have the following midpoint norm inequality:

Theorem 5. *If A, $B \geq m > 0$ for some constant m, then*

$$\left\| \int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \leq \frac{1}{24}\mathcal{D}''(w,\mu)(m)\|B-A\|^2. \tag{4.1}$$

Proof. From (3.1) we have for all $t\in[0,1]$ and A, $B \geq m > 0$,

$$\begin{aligned}&\left\|\mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right. \\ &\left.-D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\| \\ &\leq \frac{1}{2}\mathcal{D}''(w,\mu)(m)\left\|(1-t)A+tB-\frac{A+B}{2}\right\|^2 \\ &= \frac{1}{2}\mathcal{D}''(w,\mu)(m)\left(t-\frac{1}{2}\right)^2\|B-A\|^2.\end{aligned} \tag{4.2}$$

If we integrate this inequality, we get

$$\begin{aligned}&\int_0^1\left\|\mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right. \\ &\left.-D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\|dt \\ &\leq \frac{1}{2}\mathcal{D}''(w,\mu)(m)\|B-A\|^2\int_0^1\left(t-\frac{1}{2}\right)^2dt \\ &= \frac{1}{24}\mathcal{D}''(w,\mu)(m)\|B-A\|^2.\end{aligned} \tag{4.3}$$

Using the properties of norm and integral, we also have

$$\begin{aligned}&\int_0^1\left\|\mathcal{D}(w,\mu)((1-t)A+tB) - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right. \\ &\left.-D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\|dt \\ &\geq \left\|\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right. \\ &\left.-\left(\int_0^1\left(t-\frac{1}{2}\right)dt\right)D(\mathcal{D}(w,\mu))\left(\frac{A+B}{2}\right)(B-A)\right\| \\ &= \left\|\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|.\end{aligned} \tag{4.4}$$

By employing (4.3) and (4.4) we derive the desired result (4.1). □

Corollary 2. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function with* $f(0)=0$. *If* $A,\ B\geq m>0$, *then*

$$\left\|\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right)\right\| \tag{4.5}$$
$$\leq\frac{f''(m)m^2-2mf'(m)+2f(m)}{24m^3}\|B-A\|^2.$$

The proof follows by (4.1) for

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Remark 2. *If* $A,\ B\geq m>0$, *then for* $r\in(0,1]$ *we get by (4.5) that*

$$\left\|\int_0^1((1-t)A+tB)^{r-1}\,dt-\left(\frac{A+B}{2}\right)^{r-1}\right\| \tag{4.6}$$
$$\leq\frac{1}{24}(1-r)(2-r)m^{r-3}\|B-A\|^2.$$

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B}:[0,1]\to\mathbb{R}$ defined by

$$f_{A,B}(t):=f((1-t)A+tB),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by* f *is twice Fréchet differentiable in each* $A>0$, *then for* $B>0$ *we have that* $f_{A,B}$ *is twice differentiable on* $[0,1]$,

$$\frac{df_{A,B}(t)}{dt}=D(f)((1-t)A+tB)(B-A) \tag{4.7}$$

and

$$\frac{d^2f_{A,B}(t)}{dt^2}=D^2(f)((1-t)A+tB)(B-A,B-A) \tag{4.8}$$

for $t\in[0,1]$, *where in* 0 *and* 1 *the derivatives are the right and left derivatives.*

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\frac{f_{A,B}(t+h)-f(t)}{h}$$
$$=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}$$
$$=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt} &= \lim_{h\to 0}\frac{f_{A,B}(t+h)-f(t)}{h}\\ &= \lim_{h\to 0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\ &= D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (4.7).

Similarly,

$$\begin{aligned}&\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\\ &=\frac{D(f)((1-(t+h))A+(t+h)B)(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\\ &=\frac{D(f)((1-t)A+tB+h(B-A))(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}\frac{d^2 f_{A,B}(t)}{dt^2} &= \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\right\}\\ &= D^2(f)((1-t)A+tB)(B-A,B-A),\end{aligned}$$

which proves (4.8). □

For the transform $\mathcal{D}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\begin{aligned}\mathcal{D}(w,\mu)_{A,B}(t) &:= \mathcal{D}(w,\mu)((1-t)A+tB)\\ &= \int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}$$

where $A,\ B>0$ and $t\in[0,1]$.

Corollary 3. *For all $A,\ B>0$ and $t\in[0,1]$,*

$$\begin{aligned}\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt} &= D(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A)\\ &= -\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\ &\quad\times(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}\\ &= D^2(\mathcal{D}(w,\mu))((1-t)A+tB)(B-A,B-A)\\ &= 2\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\ &\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}\tag{4.10}$$

We observe that if $f(t) = \mathcal{D}(w,\mu)(t)$, $t>0$, in Lemma 3, then by the representations from Lemma 1 and Lemma 2 we obtain the desired equalities (4.9) and (4.10).

We have the following identity for the trapezoid rule:

Lemma 4. *For all A, $B>0$ we have the identity*

$$\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt \tag{4.11}$$
$$=\int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.$$

Proof. Using integration by parts for the Bochner integral, we have

$$\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt$$
$$=\frac{1}{2}\left[t(1-t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}\Big|_0^1-\int_0^1(1-2t)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt\right]$$
$$=\int_0^1\left(t-\frac{1}{2}\right)\frac{d\mathcal{D}(w,\mu)_{A,B}(t)}{dt}dt$$
$$=\left(t-\frac{1}{2}\right)\mathcal{D}(w,\mu)_{A,B}(t)\Big|_0^1-\int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt$$
$$=\frac{1}{2}\left[\mathcal{D}(w,\mu)_{A,B}(1)+\mathcal{D}(w,\mu)_{A,B}(0)\right]-\int_0^1\mathcal{D}(w,\mu)_{A,B}(t)\,dt,$$

that gives the identity

$$\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt \tag{4.12}$$
$$=\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt.$$

By (4.12) we have

$$\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{D}(w,\mu)_{A,B}(t)}{dt^2}dt \tag{4.13}$$
$$=\int_0^1 t(1-t)\left[\int_0^\infty w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.$$

By making use of (4.12) and (4.13) we deduce (4.11). □

We can state now the corresponding trapezoid norm inequality:

Theorem 6. *If A, $B \geq m > 0$ for some constant m, then*

$$\left\| \frac{\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B)}{2} - \int_0^1 \mathcal{D}(w,\mu)((1-t)A + tB)\,dt \right\| \leq \frac{1}{12}\mathcal{D}''(w,\mu)(m)\|B-A\|^2. \tag{4.14}$$

Proof. By taking the norm in (4.11), we obtain

$$\left\| \frac{\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B)}{2} - \int_0^1 \mathcal{D}(w,\mu)((1-t)A + tB)\,dt \right\| \leq \|B-A\|^2 \int_0^1 t(1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt. \tag{4.15}$$

Since A, $B \geq m > 0$, then for $\lambda \geq 0$ and $t \in [0,1]$,

$$\lambda + (1-t)A + tB \geq \lambda + m,$$

which implies that

$$(\lambda + (1-t)A + tB)^{-1} \leq (\lambda + m)^{-1}.$$

This implies that

$$\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 \leq (\lambda + m)^{-3}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

By multiplying this inequality by $t(1-t)w(\lambda) \geq 0$ and integrating we get

$$\begin{aligned} &\int_0^1 t(1-t)\left(\int_0^\infty w(\lambda)\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt \\ &\leq \left(\int_0^1 t(1-t)\,dt\right)\left(\int_0^\infty w(\lambda)(\lambda+m)^{-3}\,d\mu(\lambda)\right) \\ &= \frac{1}{6}\int_0^\infty w(\lambda)(\lambda+m)^{-3}\,d\mu(\lambda). \end{aligned} \tag{4.16}$$

Taking the derivative over t twice in (1.8), we get

$$\mathcal{D}''(w,\mu)(t) := 2\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^3}\,d\mu(\lambda),\ t > 0,$$

that gives

$$\int_0^\infty w(\lambda)(\lambda+m)^{-3}\,d\mu(\lambda) = \frac{1}{2}\mathcal{D}''(w,\mu)(m)$$

and by (4.15) and (4.16) we derive (4.14). □

Corollary 4. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$. If A, $B \geq m > 0$, then*

$$\left\| \frac{A^{-1}f(A) + B^{-1}f(B)}{2} - \int_0^1 ((1-t)A + tB)^{-1} f((1-t)A + tB)\,dt \right\| \leq \frac{f''(m)m^2 - 2mf'(m) + 2f(m)}{12m^3}\|B-A\|^2. \tag{4.17}$$

The proof follows by (4.14) for

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Remark 3. *If $A,\ B\geq m>0$, then for $r\in(0,1]$ we get by (4.5) that*

$$\left\|\frac{A^{r-1}+B^{r-1}}{2}-\int_0^1\left((1-t)A+tB\right)^{r-1}dt\right\| \leq\frac{1}{12}(1-r)(2-r)m^{r-3}\|B-A\|^2. \tag{4.18}$$

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