BOUNDS FOR JENSEN'S GAP OF AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, for $T > 0$ and $x \in H$ with $\|x\| = 1$, we have the representation for the Jensen's gap

$$\begin{aligned}
&\langle \mathcal{D}(w, \mu)(T)x, x\rangle - \mathcal{D}(w, \mu)(\langle Tx, x\rangle) \\
&= 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)\Big\langle(\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1}\right. \\
&\times \Big((T - \langle Tx, x\rangle 1_H)(\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1}(T - \langle Tx, x\rangle 1_H)\Big) \\
&\left.\times (\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1} x, x\Big\rangle dt\right) d\mu(\lambda) \\
&\geq 0.
\end{aligned}$$

Assume that $M \geq T \geq m > 0$ for some constants M, m, then for $x \in H$ with $\|x\| = 1$,

$$\begin{aligned}
0 &\leq \left(\|Tx\|^2 - \langle Tx, x\rangle^2\right)\mathcal{D}''(w, \mu)(M) \\
&\leq \langle \mathcal{D}(w, \mu)(T)x, x\rangle - \mathcal{D}(w, \mu)(\langle Tx, x\rangle) \\
&\leq \left(\|Tx\|^2 - \langle Tx, x\rangle^2\right)\mathcal{D}''(w, \mu)(m),
\end{aligned}$$

where $\mathcal{D}''(w, \mu)$ is the second derivative of $\mathcal{D}(w, \mu)$ as a functions of real variable.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot, \cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [10] (see also [7, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [10]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Jensen's gap.

RGMIA Res. Rep. Coll. 23 (2020), Art. 104, 13 pp. Received 03/08/20

The following result that provides a reverse of the Mond & Pečarić has been obtained in [3]:

Theorem 2 (Dragomir, 2008, [3])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$.

A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$. We have the following representation of operator monotone functions [9], see for instance [1, p. 144-145]:

Theorem 3. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.2}$$

where $a \in \mathbb{R}$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.3}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.2).

For some example of operator monotone functions see [4]-[6], [11], [12] and the references therein.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 4. *A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2\lambda}{t + \lambda} d\mu(\lambda), \tag{1.4}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0, \infty)$ such that (1.3) holds. If f is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f'_+(0)$, the right derivative, in (1.2).

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.5}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.6}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.7}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.7) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.8}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^r = \frac{\sin(r\pi)}{\pi} t\mathcal{D}(w_r)(t), \ t > 0. \tag{1.9}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.10}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.11}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.12}$$

for $T > 0$.

We show among others that, for $T > 0$ and $x \in H$ with $\|x\| = 1$, we have the representation for the Jensen's gap

$$\begin{aligned}
&\langle \mathcal{D}(w, \mu)(T)x, x\rangle - \mathcal{D}(w, \mu)(\langle Tx, x\rangle) \\
&= 2\int_0^\infty w(\lambda) \left(\int_0^1 (1-t) \Big\langle (\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1} \right. \\
&\times \Big((T - \langle Tx, x\rangle 1_H)(\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1} (T - \langle Tx, x\rangle 1_H) \Big) \\
&\left. \times (\lambda + (1-t)\langle Tx, x\rangle 1_H + tT)^{-1} x, x \Big\rangle dt \right) d\mu(\lambda) \\
&\geq 0.
\end{aligned}$$

Assume that $M \geq T \geq m > 0$ for some constants M, m, then for $x \in H$ with $\|x\| = 1$,

$$\begin{aligned} 0 &\leq \left(\|Tx\|^2 - \langle Tx, x\rangle^2\right) \mathcal{D}''(w,\mu)(M) \\ &\leq \langle \mathcal{D}(w,\mu)(T)x, x\rangle - \mathcal{D}(w,\mu)(\langle Tx, x\rangle) \\ &\leq \left(\|Tx\|^2 - \langle Tx, x\rangle^2\right) \mathcal{D}''(w,\mu)(m). \end{aligned}$$

2. Representation Results

We consider the inverse function $\ell^{-1}(t) = t^{-1}$, $t > 0$ and the corresponding operator inverse function $\ell^{-1}(A) = A^{-1}$ defined for operators $A > 0$.

Lemma 1. *The operator inverse function ℓ^{-1} is Fréchet differentiable in $A > 0$ and*

$$D\left(\ell^{-1}\right)(A)(V) = -A^{-1}VA^{-1} \tag{2.1}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. Let $A > 0$ and $V \in S(H)$. Then there exists a small open interval $(-\delta, \delta)$ such that for $t \in (-\delta, \delta)$, $A + tV$ is invertible. Then

$$\begin{aligned} \ell^{-1}(A+tV) - \ell^{-1}(A) &= (A+tV)^{-1} - A^{-1} = (A+tV)^{-1}(A - (A+tV))A^{-1} \\ &= -t(A+tV)^{-1}VA^{-1}, \end{aligned}$$

which implies that

$$\frac{\ell^{-1}(A+tV) - \ell^{-1}(A)}{t} = -(A+tV)^{-1}VA^{-1}, \ t \neq 0.$$

By taking the limit over $t \to 0$, we get

$$\begin{aligned} D\left(\ell^{-1}\right)(A)(V) &= \lim_{t\to 0} \frac{\ell^{-1}(A+tV) - \ell^{-1}(A)}{t} = -\lim_{t\to 0}(A+tV)^{-1}VA^{-1} \\ &= -A^{-1}VA^{-1} \end{aligned}$$

and the statement is proved. □

Lemma 2. *The operator inverse function ℓ^{-1} is twice Fréchet differentiable in $A > 0$ and*

$$D^2\left(\ell^{-1}\right)(A)(V,V) = 2A^{-1}\left(VA^{-1}V\right)A^{-1} \geq 0 \tag{2.2}$$

for all $V \in S(H)$.

Proof. Let $A > 0$ and $V \in S(H)$. Then there exists a small open interval $(-\delta, \delta)$ such that for $t \in (-\delta, \delta)$, $A + tV$ is invertible and put

$$\begin{aligned} U_t &:= D\left(\ell^{-1}\right)(A+tV)(V) - D\left(\ell^{-1}\right)(A)(V) \\ &= A^{-1}VA^{-1} - (A+tV)^{-1}V(A+tV)^{-1}. \end{aligned}$$

If we multiply both sides of U_t with $A+tV$, then we get

$$\begin{aligned}(A+tV)\,U_t\,(A+tV) &= (A+tV)\,A^{-1}VA^{-1}\,(A+tV)-V\\ &= \left(1+tVA^{-1}\right)V\left(1+tA^{-1}V\right)-V\\ &= \left(V+tVA^{-1}V\right)\left(1+tA^{-1}V\right)-V\\ &= V+tVA^{-1}V+tVA^{-1}V+t^2VA^{-1}VA^{-1}V-V\\ &= 2tVA^{-1}V+t^2\left(VA^{-1}\right)^2V\\ &= t\left[2VA^{-1}V+t\left(VA^{-1}\right)^2V\right].\end{aligned}$$

By multiplying this equality both sides with $(A+tV)^{-1}$ we get

$$U_t = t\,(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^2V\right](A+tV)^{-1}.$$

Therefore

$$\begin{aligned}D^2\left(\ell^{-1}\right)(A)\,(V,V) &= \lim_{t\to 0}\frac{D\left(\ell^{-1}\right)(A+tV)\,(V)-D\left(\ell^{-1}\right)(A)\,(V)}{t}\\ &= \lim_{t\to 0}\frac{t\,(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^2V\right](A+tV)^{-1}}{t}\\ &= \lim_{t\to 0}(A+tV)^{-1}\left[2VA^{-1}V+t\left(VA^{-1}\right)^2V\right](A+tV)^{-1}\\ &= 2A^{-1}\left(VA^{-1}V\right)A^{-1}.\end{aligned}$$

Since $A^{-1}>0$, then by multiplying both sides by V we get $VA^{-1}V\geq 0$ and by multiplying both sides by A^{-1} we get $A^{-1}\left(VA^{-1}V\right)A^{-1}\geq 0$, and the proof of (2.2) is thus proved. □

Lemma 3. *For all C, $E>0$ we have the representation*

$$\begin{aligned}E^{-1}-C^{-1} &= -C^{-1}\,(E-C)\,C^{-1}\\ &\quad +2\int_0^1(1-t)\,((1-t)\,C+tE)^{-1}\\ &\quad \times\Big((E-C)\,((1-t)\,C+tE)^{-1}\,(E-C)\Big)\,((1-t)\,C+tE)^{-1}\,dt.\end{aligned}\tag{2.3}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$\begin{aligned}f(E) &= f(C)+D(f)(C)(E-C)\\ &\quad +\int_0^1(1-t)\,D^2(f)\,((1-t)\,C+tE)\,(E-C,E-C)\,dt\end{aligned}\tag{2.4}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E\in\mathcal{O}$.

Therefore, if we write this formula for ℓ^{-1} and the class of strictly positive operators, we get

$$\begin{aligned}E^{-1} &= C^{-1} - C^{-1}(E-C)C^{-1} \\ &+ 2\int_0^1 (1-t)((1-t)C+tE)^{-1} \\ &\times \Big((E-C)((1-t)C+tE)^{-1}(E-C)\Big)((1-t)C+tE)^{-1}dt\end{aligned}$$

and the representation (2.3) is proved. □

We have the following representation result for $\mathcal{D}(w,\mu)$:

Lemma 4. *For all A, $B>0$, we have the representation*

$$\begin{aligned}&\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) - D(\mathcal{D}(w,\mu))(A)(B-A) \\ &= 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)A+tB)^{-1}\right. \\ &\times \Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big) \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}dt\right)d\mu(\lambda) \\ &\geq 0.\end{aligned} \tag{2.5}$$

Proof. From the identity (2.3) we get for $E=\lambda+B$ and $C=\lambda+A$ that

$$\begin{aligned}(\lambda+B)^{-1} - (\lambda+A)^{-1} &= -(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \\ &+ 2\int_0^1 (1-t)(\lambda+(1-t)A+tB)^{-1} \\ &\times \Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big) \\ &\times(\lambda+(1-t)A+tB)^{-1}dt\end{aligned} \tag{2.6}$$

for $\lambda\geq 0$.

If we multiply this equality by $w(\lambda)\geq 0$ and integrate, then we get

$$\begin{aligned}&\int_0^\infty w(\lambda)(\lambda+B)^{-1}d\mu(\lambda) - \int_0^\infty w(\lambda)(\lambda+A)^{-1}d\mu(\lambda) \\ &= -\int_0^\infty w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \\ &+ 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)A+tB)^{-1}\right. \\ &\times \Big((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\Big) \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}dt\right)d\mu(\lambda)\end{aligned} \tag{2.7}$$

and since, by (2),

$$-\int_0^\infty w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} = D(\mathcal{D}(w,\mu))(A)(B-A),$$

hence by (2.7) we derive the equality in (2.7).

We observe that for all $\lambda \geq 0$ and $t \in [0,1]$, $(\lambda + (1-t)A + tB)^{-1} > 0$. If we multiply this inequality both sides by $(B-A)$ we derive

$$(B-A)(\lambda + (1-t)A + tB)^{-1}(B-A) \geq 0.$$

Further, if we multiply this inequality both sides by $(\lambda + (1-t)A + tB)^{-1}$ we get

$$\begin{aligned}&(1-t)(\lambda + (1-t)A + tB)^{-1}\\ &\times \Big((B-A)(\lambda + (1-t)A + tB)^{-1}(B-A)\Big)(\lambda + (1-t)A + tB)^{-1}\\ &\geq 0\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$. If we multiply this inequality with $w(\lambda) \geq 0$ and integrate, then we obtain the inequality in (2.5). □

For a positive operator $T > 0$ and $x \in H$ with $\|x\| = 1$, we define the Jensen's gap related to the transform $\mathcal{D}(w,\mu)$

$$J(\mathcal{D}(w,\mu), T, x) := \langle \mathcal{D}(w,\mu)(T)x, x\rangle - \mathcal{D}(w,\mu)(\langle Tx, x\rangle).$$

Theorem 5. *For $T > 0$ and $x \in H$ with $\|x\| = 1$, we have the representation*

$$\begin{aligned}&J(\mathcal{D}(w,\mu), T, x)\\ &= 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)\Big\langle(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}\right.\\ &\times \Big((T - \langle Tx,x\rangle 1_H)(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}(T - \langle Tx,x\rangle 1_H)\Big)\\ &\left.\times (\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}x, x\Big\rangle dt\right) d\mu(\lambda)\\ &\geq 0.\end{aligned} \tag{2.8}$$

Proof. If we take in (2.5) $B = T$ and $A = u1_H$ with $u > 0$ then we get

$$\begin{aligned}&\mathcal{D}(w,\mu)(T) - \mathcal{D}(w,\mu)(u1_H) - D(\mathcal{D}(w,\mu))(u1_H)(T - u1_H)\\ &= 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)(\lambda + (1-t)u1_H + tT)^{-1}\right.\\ &\times \Big((T - u1_H)(\lambda + (1-t)u1_H + tT)^{-1}(T - u1_H)\Big)\\ &\left.\times (\lambda + (1-t)u1_H + tT)^{-1} dt\right) d\mu(\lambda)\\ &\geq 0.\end{aligned} \tag{2.9}$$

Let $x \in H$ with $\|x\| = 1$ and take the inner product in (2.9) to get the scalar identity

$$\begin{aligned}&\langle \mathcal{D}(w,\mu)(T)x, x\rangle - \mathcal{D}(w,\mu)(u) - D(\mathcal{D}(w,\mu))(u1_H)(\langle Tx,x\rangle - u)\\ &= 2\int_0^\infty w(\lambda)\left(\int_0^1 (1-t)\Big\langle(\lambda + (1-t)u1_H + tT)^{-1}\right.\\ &\times \Big((T - u1_H)(\lambda + (1-t)u1_H + tT)^{-1}(T - u1_H)\Big)\\ &\left.\times (\lambda + (1-t)u1_H + tT)^{-1}x, x\Big\rangle dt\right) d\mu(\lambda)\\ &\geq 0\end{aligned} \tag{2.10}$$

for all $u > 0$.

Now, if we take $u=\langle Tx,x\rangle$ in (2.10) and take into account that

$$D\left(\mathcal{D}\left(w,\mu\right)\right)\left(\langle Tx,x\rangle 1_{H}\right)\left(\langle Tx,x\rangle-\langle Tx,x\rangle\right)=0,$$

then by (2.10) we derive (2.8). □

Corollary 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ that has the representation (1.2). Then for $T>0$ and $x\in H$ with $\|x\|=1$, we have the representation*

$$\begin{aligned}(2.11)\quad &\left\langle f\left(T\right)T^{-1}x,x\right\rangle-f\left(\langle Tx,x\rangle\right)\langle Tx,x\rangle^{-1}-f\left(0\right)\left(\left\langle T^{-1}x,x\right\rangle-\langle Tx,x\rangle^{-1}\right)\\ &=2\int_{0}^{\infty}\lambda\left(\int_{0}^{1}\left(1-t\right)\left\langle\left(\lambda+\left(1-t\right)\langle Tx,x\rangle 1_{H}+tT\right)^{-1}\right.\right.\\ &\times\left(\left(T-\langle Tx,x\rangle 1_{H}\right)\left(\lambda+\left(1-t\right)\langle Tx,x\rangle 1_{H}+tT\right)^{-1}\left(T-\langle Tx,x\rangle 1_{H}\right)\right)\\ &\times\left.\left.\left(\lambda+\left(1-t\right)\langle Tx,x\rangle 1_{H}+tT\right)^{-1}x,x\right\rangle dt\right)d\mu\left(\lambda\right)\\ &\geq 0.\end{aligned}$$

Proof. From (1.2) we have the representation

$$\mathcal{D}\left(\ell,\mu\right)\left(t\right)=\frac{f\left(t\right)-f\left(0\right)}{t}-b,$$

where $b\geq 0$, $\ell\left(\lambda\right)=\lambda$ and μ a positive measure on $(0,\infty)$.

Since for $T>0$ and $x\in H$ with $\|x\|=1$,

$$\begin{aligned}&\langle\mathcal{D}\left(\ell,\mu\right)\left(T\right)x,x\rangle-\mathcal{D}\left(\ell,\mu\right)\left(\langle Tx,x\rangle\right)\\ &=\left\langle f\left(T\right)T^{-1}x,x\right\rangle-f\left(0\right)\left\langle T^{-1}x,x\right\rangle-\frac{f\left(\langle Tx,x\rangle\right)-f\left(0\right)}{\langle Tx,x\rangle}\\ &=\left\langle f\left(T\right)T^{-1}x,x\right\rangle-f\left(\langle Tx,x\rangle\right)\langle Tx,x\rangle^{-1}-f\left(0\right)\left(\left\langle T^{-1}x,x\right\rangle-\langle Tx,x\rangle^{-1}\right),\end{aligned}$$

hence by (2.8) we obtain (2.11). □

Remark 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then by (2.11) we get*

$$(2.12)\qquad\left\langle f\left(T\right)T^{-1}x,x\right\rangle\langle Tx,x\rangle-f\left(\langle Tx,x\rangle\right)\geq f\left(0\right)\left(\left\langle T^{-1}x,x\right\rangle\langle Tx,x\rangle-1\right)$$

for $T>0$ and $x\in H$ with $\|x\|=1$.

In particular, for $f(0)\geq 0$, we get

$$\left\langle f\left(T\right)T^{-1}x,x\right\rangle\langle Tx,x\rangle\geq f\left(\langle Tx,x\rangle\right),\ x\in H\ \textit{with}\ \|x\|=1.$$

Remark 2. *We consider the representation obtained from (1.9) for the operator $T>0$ and the power $r\in(0,1]$,*

$$T^{r-1}=\mathcal{D}\left(\tilde{w}_{r}\right)\left(T\right)$$

with the kernel $\tilde{w}_r(\lambda) := \frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$. *Then for* $T > 0$ *and* $x \in H$ *with* $\|x\| = 1$, *we have the representation*

$$\begin{aligned}(2.13)\quad & \left\langle T^{r-1}x, x\right\rangle - \langle Tx, x\rangle^{r-1} \\ & = 2\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\left(\int_0^1 (1-t)\left\langle(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}\right.\right. \\ & \times \Big((T - \langle Tx,x\rangle 1_H)(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}(T - \langle Tx,x\rangle 1_H)\Big) \\ & \left.\left.\times (\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}x, x\right\rangle dt\right) d\lambda \\ & \geq 0.\end{aligned}$$

Corollary 2. *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$ *and has the representation (1.4). Then for* $T > 0$ *and* $x \in H$ *with* $\|x\| = 1$, *we have the representation*

$$\begin{aligned}(2.14)\quad & \left\langle f(T)T^{-2}x, x\right\rangle - f(\langle Tx,x\rangle)\langle Tx,x\rangle^{-2} - f(0)\left(\left\langle T^{-2}x,x\right\rangle - \langle Tx,x\rangle^{-2}\right) \\ & - f'_+(0)\left(\left\langle T^{-1}x,x\right\rangle - \langle Tx,x\rangle^{-1}\right) \\ & = 2\int_0^\infty \lambda\left(\int_0^1 (1-t)\left\langle(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}\right.\right. \\ & \times \Big((T - \langle Tx,x\rangle 1_H)(\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}(T - \langle Tx,x\rangle 1_H)\Big) \\ & \left.\left.\times (\lambda + (1-t)\langle Tx,x\rangle 1_H + tT)^{-1}x, x\right\rangle dt\right) d\lambda \\ & \geq 0.\end{aligned}$$

Proof. From the representation (1.4) we have

$$\mathcal{D}(\ell,\mu)(t) = \frac{f(t) - f(0) - f'_+(0)t}{t^2} - ct,$$

where $c \geq 0$, $\ell(\lambda) = \lambda$ and μ a positive measure on $(0,\infty)$.

Since for $T > 0$ and $x \in H$ with $\|x\| = 1$,

$$\begin{aligned}& \langle \mathcal{D}(\ell,\mu)(T)x, x\rangle - \mathcal{D}(\ell,\mu)(\langle Tx,x\rangle) \\ & = \left\langle f(T)T^{-2}x,x\right\rangle - f(0)\left\langle T^{-2}x,x\right\rangle - f'_+(0)\left\langle T^{-1}x,x\right\rangle - c\langle Tx,x\rangle \\ & - \frac{f(\langle Tx,x\rangle) - f(0) - f'_+(0)\langle Tx,x\rangle}{\langle Tx,x\rangle^2} + c\langle Tx,x\rangle \\ & = \left\langle f(T)T^{-2}x,x\right\rangle - f(\langle Tx,x\rangle)\langle Tx,x\rangle^{-2} - f(0)\left(\left\langle T^{-2}x,x\right\rangle - \langle Tx,x\rangle^{-2}\right) \\ & - f'_+(0)\left(\left\langle T^{-1}x,x\right\rangle - \langle Tx,x\rangle^{-1}\right)\end{aligned}$$

hence by (2.8) we obtain (2.14). □

Remark 3. *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$. *Then for* $T > 0$ *and* $x \in H$ *with* $\|x\| = 1$,

$$\begin{aligned}(2.15)\quad & \left\langle f(T)T^{-2}x,x\right\rangle - f(\langle Tx,x\rangle)\langle Tx,x\rangle^{-2} - f(0)\left(\left\langle T^{-2}x,x\right\rangle - \langle Tx,x\rangle^{-2}\right) \\ & \geq f'_+(0)\left(\left\langle T^{-1}x,x\right\rangle - \langle Tx,x\rangle^{-1}\right).\end{aligned}$$

In particular, for $f'_+(0) \geq 0$, *we get*

$$\left\langle f(T)T^{-2}x,x\right\rangle - f(\langle Tx,x\rangle)\langle Tx,x\rangle^{-2} \geq f(0)\left(\left\langle T^{-2}x,x\right\rangle - \langle Tx,x\rangle^{-2}\right) \tag{2.16}$$

for $T>0$ *and* $x\in H$ *with* $\|x\|=1$.

3. Upper and Lower Bounds

We have the following upper and lower bounds for the Jensen's gap:

Theorem 6. *Assume that* $M\geq T\geq m>0$ *for some constants* M, m. *Then for* $x\in H$ *with* $\|x\|=1$,

$$\begin{aligned} 0 &\leq \left(\|Tx\|^2-\langle Tx,x\rangle^2\right)\mathcal{D}''(w,\mu)(M) \leq J(\mathcal{D}(w,\mu),T,x) \\ &\leq \left(\|Tx\|^2-\langle Tx,x\rangle^2\right)\mathcal{D}''(w,\mu)(m). \end{aligned} \tag{3.1}$$

Proof. Assume that $x\in H$ with $\|x\|=1$. Observe that

$$0<\lambda+m\leq\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\leq\lambda+M$$

for $\lambda\geq0$ and $t\in[0,1]$.

This implies that

$$0<(\lambda+M)^{-1}\leq(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}\leq(\lambda+m)^{-1}$$

for $\lambda\geq0$ and $t\in[0,1]$.

If we multiply both sides by $T-\langle Tx,x\rangle 1_H$ we get

$$\begin{aligned} 0 &< (\lambda+M)^{-1}(T-\langle Tx,x\rangle 1_H)^2 \\ &\leq (T-\langle Tx,x\rangle 1_H)(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}(T-\langle Tx,x\rangle 1_H) \\ &\leq (\lambda+m)^{-1}(T-\langle Tx,x\rangle 1_H)^2. \end{aligned}$$

Further, if we multiply this inequality both sides by $(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}$, we get

$$\begin{aligned} 0 &< (\lambda+M)^{-1}(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}(T-\langle Tx,x\rangle 1_H)^2 \\ &\quad\times(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1} \\ &\leq (\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}(T-\langle Tx,x\rangle 1_H) \\ &\quad\times(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1} \\ &\quad\times(T-\langle Tx,x\rangle 1_H)(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1} \\ &\leq (\lambda+m)^{-1}(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1}(T-\langle Tx,x\rangle 1_H)^2 \\ &\quad\times(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT)^{-1} \end{aligned} \tag{3.2}$$

for $\lambda\geq0$ and $t\in[0,1]$.

Using the commutativity property of the continuous functional calculus for one selfadjoint operator T we have

$$
\begin{aligned}
&(\lambda+M)^{-2}\left(T-\langle Tx,x\rangle 1_H\right)^2 \\
&\leq \left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}\left(T-\langle Tx,x\rangle 1_H\right)^2 \\
&\times\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1} \\
&\leq (\lambda+m)^{-2}\left(T-\langle Tx,x\rangle 1_H\right)^2
\end{aligned} \tag{3.3}
$$

for $\lambda \geq 0$ and $t \in [0,1]$.

By (3.2) and (3.3) we get

$$
\begin{aligned}
&(\lambda+M)^{-3}\left(T-\langle Tx,x\rangle 1_H\right)^2 \\
&\leq \left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}\left(T-\langle Tx,x\rangle 1_H\right) \\
&\times\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1} \\
&\times\left(T-\langle Tx,x\rangle 1_H\right)\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1} \\
&\leq (\lambda+m)^{-3}\left(T-\langle Tx,x\rangle 1_H\right)^2
\end{aligned} \tag{3.4}
$$

for $\lambda \geq 0$ and $t \in [0,1]$.

If we take the inner product for $x \in H$ with $\|x\|=1$ in (3.4), then we get

$$
\begin{aligned}
&(\lambda+M)^{-3}\left\langle\left(T-\langle Tx,x\rangle 1_H\right)^2 x,x\right\rangle \\
&\leq \Big\langle\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}\left(T-\langle Tx,x\rangle 1_H\right) \\
&\times\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1} \\
&\times\left(T-\langle Tx,x\rangle 1_H\right)\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}x,x\Big\rangle \\
&\leq (\lambda+m)^{-3}\left\langle\left(T-\langle Tx,x\rangle 1_H\right)^2 x,x\right\rangle .
\end{aligned} \tag{3.5}
$$

If we multiply this inequality by $2(1-t)\,w(\lambda) \geq 0$ and integrate, then

$$
\begin{aligned}
&2\left\langle\left(T-\langle Tx,x\rangle 1_H\right)^2 x,x\right\rangle \int_0^\infty w(\lambda)(\lambda+M)^{-3}\,d\mu(\lambda) \\
&\leq 2\int_0^\infty w(\lambda)\Big(\int_0^1 (1-t)\Big\langle\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1} \\
&\times\Big(\left(T-\langle Tx,x\rangle 1_H\right)\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}\left(T-\langle Tx,x\rangle 1_H\right)\Big) \\
&\times\left(\lambda+(1-t)\langle Tx,x\rangle 1_H+tT\right)^{-1}x,x\Big\rangle\,dt\Big)\,d\mu(\lambda) \\
&\leq 2\left\langle\left(T-\langle Tx,x\rangle 1_H\right)^2 x,x\right\rangle \int_0^\infty w(\lambda)(\lambda+m)^{-3}\,d\mu(\lambda)
\end{aligned} \tag{3.6}
$$

for $x \in H$ with $\|x\|=1$.

Using (2.8) we get

$$2\left\langle \left(T-\left\langle Tx,x\right\rangle 1_H\right)^2 x,x\right\rangle \int_0^\infty w(\lambda)(\lambda+M)^{-3}\,d\mu(\lambda) \tag{3.7}$$
$$\le J(\mathcal{D}(w,\mu),T,x)$$
$$\le 2\left\langle \left(T-\left\langle Tx,x\right\rangle 1_H\right)^2 x,x\right\rangle \int_0^\infty w(\lambda)(\lambda+m)^{-3}\,d\mu(\lambda)$$

for $x\in H$ with $\|x\|=1$.

Observe that for $x\in H$ with $\|x\|=1$,

$$\begin{aligned}\left\langle \left(T-\left\langle Tx,x\right\rangle 1_H\right)^2 x,x\right\rangle &= \left\langle \left(T^2-2\left\langle Tx,x\right\rangle T+\left\langle Tx,x\right\rangle^2\right)x,x\right\rangle\\ &= \left\langle T^2x-2\left\langle Tx,x\right\rangle Tx+\left\langle Tx,x\right\rangle^2 x,x\right\rangle\\ &= \left\langle T^2x,x\right\rangle-\left\langle Tx,x\right\rangle^2=\|Tx\|^2-\left\langle Tx,x\right\rangle^2.\end{aligned}$$

If we take the derivative in (1.7) over t then we get

$$\mathcal{D}'(w,\mu)(t)=-\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),\ t>0,$$

and

$$\mathcal{D}''(w,\mu)(t)=2\int_0^\infty \frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0.$$

This gives

$$\begin{aligned}\int_0^\infty \frac{w(\lambda)}{(\lambda+M)^3}d\mu(\lambda)&=\frac{1}{2}\mathcal{D}''(w,\mu)(M),\\ \int_0^\infty \frac{w(\lambda)}{(\lambda+m)^3}d\mu(\lambda)&=\frac{1}{2}\mathcal{D}''(w,\mu)(m).\end{aligned}$$

Finally, by making use of (3.7) we derive the desired result (3.1). □

Corollary 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ that has the representation (1.2). Then for $T>0$ and $x\in H$ with $\|x\|=1$,*

$$0\le\left(\|Tx\|^2-\left\langle Tx,x\right\rangle^2\right)\frac{f''(M)M^2-2Mf'(M)+2f(M)}{M^3} \tag{3.8}$$
$$\le\left\langle f(T)T^{-1}x,x\right\rangle-f(\left\langle Tx,x\right\rangle)\left\langle Tx,x\right\rangle^{-1}$$
$$-f(0)\left(\left\langle T^{-1}x,x\right\rangle-\left\langle Tx,x\right\rangle^{-1}\right)$$
$$\le\left(\|Tx\|^2-\left\langle Tx,x\right\rangle^2\right)\frac{f''(m)m^2-2mf'(m)+2f(m)}{m^3}.$$

Proof. We have

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)}{t}-b,\ t>0,$$

where $b\ge0$ and μ is a positive measure on $(0,\infty)$.

The derivative of this function is

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)t-f(t)}{t^2},\ t>0$$

and the second derivative

$$\mathcal{D}''(\ell,\mu)(t) = \frac{\left(f'(t)\,t - f(t)\right)' t^2 - 2t\left(f'(t)\,t - f(t)\right)}{t^4}$$
$$= \frac{f''(t)\,t^3 - 2t^2 f'(t) + 2tf(t)}{t^4} = \frac{f''(t)\,t^2 - 2tf'(t) + 2f(t)}{t^3}.$$

By (3.1) we then get (3.8). □

Remark 4. *If we write the inequality (3.8) for $f(t) = t^r$, $r \in (0,1]$, then by (3.8) we get*

$$0 \le (1-r)(2-r)\left(\|Tx\|^2 - \langle Tx,x\rangle^2\right) M^{r-3} \le \left\langle T^{r-1}x,x\right\rangle - \langle Tx,x\rangle^{r-1} \le (1-r)(2-r)\left(\|Tx\|^2 - \langle Tx,x\rangle^2\right) m^{r-3}, \tag{3.9}$$

for $x \in H$ with $\|x\| = 1$.

References

[1] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] R. Coleman, *Calculus on Normed Vector Spaces*, Springer, New York, 2012.

[3] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces. *J. Inequal. Appl.* **2010**, Art. ID 496821, 15 pp

[4] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[5] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[6] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[7] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[8] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[9] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[10] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19** (1993), 405-420.

[11] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[12] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.