LOWER AND UPPER BOUNDS FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO JENSEN'S GAP

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq -\frac{1}{2}\delta\mathcal{M}''(w, \mu)(\beta) \\ &\leq D(\mathcal{M}(w, \mu))(A)(B - A) + \mathcal{M}(w, \mu)(A) - \mathcal{M}(w, \mu)(B) \\ &\leq -\frac{1}{2}\Delta\mathcal{M}''(w, \mu)(\alpha), \end{aligned}$$

where $D(\mathcal{M}(w, \mu))$ is the Fréchet derivative of $\mathcal{M}(w, \mu)$ as an operator function and $\mathcal{M}''(w, \mu)$ is the second derivative of $\mathcal{M}(w, \mu)$ as a real function.

Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$. If $\beta \geq A_k \geq \alpha > 0$, for $k \in \{1, ..., n\}$ and $0 < \delta \leq \left(A_j - \sum_{k=1}^n p_k A_k\right)^2 \leq \Delta$ for all $j \in \{1, ..., n\}$ for some constants α, β, δ, Δ, then

$$0 \leq -\frac{1}{2}\delta f''(\beta) \leq f\left(\sum_{k=1}^n p_k A_k\right) - \sum_{k=1}^n p_k f(A_k) \leq -\frac{1}{2}\Delta f''(\alpha)$$

for any probability density n-tuple $\mathbf{p} = (p_1, ..., p_n)$.

Some applications for power and logarithmic functions are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

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for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in\mathbb{R}$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, *then* $a=f(0)$ *in (1.9).*

For some example of operator monotone functions see [3]-[5], [8], [9] and the references therein.

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.11}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)&=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.12}$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$, then

$$\mathcal{M}(w,\mu)(t)=\int_0^\infty w(\lambda)\,d\mu(\lambda)-\mathcal{D}(\ell w,\mu)(t), \tag{1.13}$$

where $\ell(t)=t$, $t>0$.

Consider the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq 0$$

and

$$\int_0^\infty w(\lambda)\,d\lambda=\int_0^\infty\exp(-a\lambda)\,d\lambda=\frac{1}{a},$$

where

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})(t)=t\mathcal{D}(w,\mu)(t)=tE_1(at)\exp(at),\ t\geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty\frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.13) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.13) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.14}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

for $T>0$.

We show among others that, if $\beta\geq A,\ B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$ for some constants $\alpha,\ \beta,\ \delta,\ \Delta$, then

$$\begin{aligned}0&\leq-\frac{1}{2}\delta\mathcal{M}''(w,\mu)(\beta)\\&\leq D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B)\\&\leq-\frac{1}{2}\Delta\mathcal{M}''(w,\mu)(\alpha).\end{aligned}$$

Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A_k\geq\alpha>0$, for $k\in\{1,...,n\}$ and $0<\delta\leq\left(A_j-\sum_{k=1}^n p_kA_k\right)^2\leq\Delta$ for all $j\in\{1,...,n\}$ for some constants $\alpha,\ \beta,\ \delta,\ \Delta$, then

$$0\leq-\frac{1}{2}\delta f''(\beta)\leq f\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_kf(A_k)\leq-\frac{1}{2}\Delta f''(\alpha)$$

for any probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$.

Some applications for power and logarithmic functions are also given.

2. Upper and Lower Bounds

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w,\mu))$:

Lemma 1. *For all $A>0$,*

$$D(\mathcal{M}(w,\mu))(A)(V)=\int_0^\infty\lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda) \tag{2.1}$$

for all $V\in S(H)$, the class of all selfadjoint operators on H.

Proof. By the definition of $\mathcal{M}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}&\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(A+tV+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty\lambda w(\lambda)\left[(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty\lambda w(\lambda)\left[(\lambda+A)^{-1}(\lambda+A+tV-\lambda-A)(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\&=t\int_0^\infty\lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)}{t}\\
&=\lim_{t\to 0}\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is obtained. □

For the case of second Fréchet derivative $D^2(\mathcal{M}(w,\mu))$, we have the representation:

Lemma 2. *For all $A>0$,*

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=-2\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}\tag{2.2}$$

for all $V\in S(H)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=\lim_{t\to 0}\frac{D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda)\\
&-\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\\
&\times\left[(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda\geq 0$ and t as above,

$$U_{t,\lambda}:=(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda+A+tV$, then we get

$$
\begin{aligned}
&(\lambda+A+tV)\,U_{t,\lambda}\,(\lambda+A+tV) \\
&= V-(\lambda+A+tV)\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,(\lambda+A+tV) \\
&= V-\left(1+tV\,(\lambda+A)^{-1}\right)V\left(1+t\,(\lambda+A)^{-1}\,V\right) \\
&= V-\left(V+tV\,(\lambda+A)^{-1}\,V\right)\left(1+t\,(\lambda+A)^{-1}\,V\right) \\
&= V-V-tV\,(\lambda+A)^{-1}\,V-tV\,(\lambda+A)^{-1}\,V \\
&-t^2V\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V \\
&= -2tV\,(\lambda+A)^{-1}\,V+t^2V\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V \\
&= -t\left[2V\,(\lambda+A)^{-1}\,V+tV\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V\right].
\end{aligned}
\tag{2.3}
$$

If we multiply the equality by $(\lambda+A+tV)^{-1}$ both sides, we get for $t\neq 0$

$$
\begin{aligned}
\frac{U_{t,\lambda}}{t} &= -(\lambda+A+tV)^{-1}\left[2V\,(\lambda+A)^{-1}\,V+tV\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V\right] \\
&\times(\lambda+A+tV)^{-1}.
\end{aligned}
\tag{2.4}
$$

If we take the limit over $t\to 0$ in, then we get

$$\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)=-2\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}.$$

Therefore, by the properties of limit under the sign of integral, we derive

$$
\begin{aligned}
&\lim_{t\to 0}\frac{D\left(\mathcal{M}\left(w,\mu\right)\right)(A+tV)\,(V)-D\left(\mathcal{M}\left(w,\mu\right)\right)(A)\,(V)}{t} \\
&=\int_0^\infty \lambda w(\lambda)\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)d\mu(\lambda) \\
&=-2\int_0^\infty \lambda w(\lambda)\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,V\,(\lambda+A)^{-1}\,d\mu(\lambda)
\end{aligned}
$$

and the representation (2.2) is obtained. □

We have the following representation for the transform $\mathcal{M}(w,\mu)$:

Lemma 3. *For all* A, $B>0$,

$$
\begin{aligned}
&\mathcal{M}(w,\mu)\,(B) \\
&=\mathcal{M}(w,\mu)\,(A)+\int_0^\infty \lambda w(\lambda)\,(\lambda+A)^{-1}\,(B-A)\,(\lambda+A)^{-1}\,d\mu(\lambda) \\
&-2\int_0^1(1-t)\left[\int_0^\infty \lambda w(\lambda)\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)\right. \\
&\left.\times(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)\,(\lambda+(1-t)\,A+tB)^{-1}\,d\mu(\lambda)\right]dt.
\end{aligned}
\tag{2.5}
$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E) = f(C) + D(f)(C)(E - C) + \int_0^1 (1-t) D^2(f)((1-t)C + tE)(E - C, E - C)\, dt \tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E \in \mathcal{O}$.

If we write (2.6) for $\mathcal{M}(w,\mu)$ and A, $B > 0$, we get

$$\mathcal{M}(w,\mu)(B) = \mathcal{M}(w,\mu)(A) + D(\mathcal{M}(w,\mu))(A)(B - A) + \int_0^1 (1-t) D^2(\mathcal{M}(w,\mu))((1-t)A + tB)(B - A, B - A)\, dt$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). □

Theorem 2. *Assume that* $\beta \geq A \geq \alpha > 0$, $B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, β, m, M, *then*

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[\mathcal{M}'(w,\mu)(\beta) M + \mathcal{M}(w,\mu)(\beta) - \mathcal{M}(w,\mu)(M + \beta)\right] \\ &\leq D(\mathcal{M}(w,\mu))(A)(B - A) + \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)(B) \\ &\leq \frac{M^2}{m^2}\left[\mathcal{M}'(w,\mu)(\alpha) m + \mathcal{M}(w,\mu)(\alpha) - \mathcal{M}(w,\mu)(m + \alpha)\right]. \end{aligned} \tag{2.7}$$

Proof. For $\lambda \geq 0$ and $t \in [0,1]$ we have

$$\lambda + (1-t)A + tB = \lambda + A + t(B - A),$$

which by the assumption from the theorem implies that

$$\lambda + \alpha + tm \leq \lambda + (1-t)A + tB \leq \lambda + \beta + tM$$

for $\lambda \geq 0$ and $t \in [0,1]$.

This is equivalent to

$$(\lambda + \beta + tM)^{-1} \leq (\lambda + (1-t)A + tB)^{-1} \leq (\lambda + \alpha + tm)^{-1} \tag{2.8}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply this both sides with $B - A$ we obtain

$$\begin{aligned} (\lambda + \beta + tM)^{-1}(B - A)^2 &\leq (B - A)(\lambda + (1-t)A + tB)^{-1}(B - A) \\ &\leq (\lambda + \alpha + tm)^{-1}(B - A)^2. \end{aligned} \tag{2.9}$$

Since $0 < m \leq B - A \leq M$, hence

$$m^2(\lambda + \beta + tM)^{-1} \leq (\lambda + \beta + tM)^{-1}(B - A)^2$$

and

$$(\lambda + \alpha + tm)^{-1}(B - A)^2 \leq M^2(\lambda + \alpha + tm)^{-1},$$

then

$$\begin{aligned} m^2(\lambda + \beta + tM)^{-1} &\leq (B - A)(\lambda + (1-t)A + tB)^{-1}(B - A) \\ &\leq M^2(\lambda + \alpha + tm)^{-1} \end{aligned} \tag{2.10}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

Further, if we multiply (2.10) by $(\lambda+(1-t)A+tB)^{-1}$ both sides, we get

$$\begin{aligned}(2.11)\qquad & m^2(\lambda+\beta+tM)^{-1}(\lambda+(1-t)A+tB)^{-2}\\ & \leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\ & \times(B-A)(\lambda+(1-t)A+tB)^{-1}\\ & \leq M^2(\lambda+\alpha+tm)^{-1}(\lambda+(1-t)A+tB)^{-2}\end{aligned}$$

and since

$$(\lambda+\beta+tM)^{-2}\leq(\lambda+(1-t)A+tB)^{-2}\leq(\lambda+\alpha+tm)^{-2},$$

then by (2.11) we get

$$\begin{aligned}(2.12)\qquad & m^2(\lambda+\beta+tM)^{-3}\\ & \leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\ & \times(B-A)(\lambda+(1-t)A+tB)^{-1}\\ & \leq M^2(\lambda+\alpha+tm)^{-3}\end{aligned}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

By multiplication with $(1-t)\geq 0$, $\lambda w(\lambda)\geq 0$ integration and by making use of the identity (2.5), we deduce

$$\begin{aligned}(2.13)\qquad & 2m^2\int_0^\infty\lambda w(\lambda)\int_0^1\left[(1-t)(\lambda+\beta+tM)^{-3}dt\right]d\mu(\lambda)\\ & \leq D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B)\\ & \leq 2M^2\int_0^\infty\lambda w(\lambda)\int_0^1\left[(1-t)(\lambda+\alpha+tm)^{-3}dt\right]d\mu(\lambda).\end{aligned}$$

Using the identity (2.5) for scalars we have

$$\begin{aligned}(2.14)\qquad & 0\leq\mathcal{M}'(w,\mu)(\beta)M+\mathcal{M}(w,\mu)(\beta)-\mathcal{M}(w,\mu)(M+\beta)\\ & =2\int_0^\infty\lambda w(\lambda)\left(\int_0^1(1-t)(\lambda+(1-t)\beta+t(M+\beta))^{-1}\right.\\ & \times\Big((M+\beta-\beta)(\lambda+(1-t)\beta+t(M+\beta))^{-1}(M+\beta-\beta)\Big)\\ & \left.\times(\lambda+(1-t)\beta+t(M+\beta))^{-1}dt\right)d\mu(\lambda)\\ & =2M^2\int_0^\infty\lambda w(\lambda)\int_0^1\left[(1-t)(\lambda+\beta+tM)^{-3}dt\right]d\mu(\lambda)\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}'(w,\mu)(\alpha)\,m+\mathcal{M}(w,\mu)(\alpha)-\mathcal{M}(w,\mu)(m+\alpha) \\
&=2\int_0^\infty \lambda w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}\right. \\
&\times\Big((m+\alpha-\alpha)(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}(m+\alpha-\alpha)\Big) \\
&\left.\times(\lambda+(1-t)\alpha+t(m+\alpha))^{-1}dt\right)d\mu(\lambda) \\
&=2m^2\int_0^\infty \lambda w(\lambda)\int_0^1\Big[(1-t)(\lambda+\alpha+tm)^{-3}dt\Big]d\mu(\lambda)\,.
\end{aligned}\tag{2.15}$$

By making use of (2.13)-(2.15), we derive (2.7). □

From a different perspective, we also have:

Theorem 3. *Assume that* $\beta\geq A$, $B\geq\alpha>0$ *and* $0<\delta\leq(B-A)^2\leq\Delta$ *for some constants* α, β, δ, Δ, *then*

$$\begin{aligned}
0&\leq-\frac{1}{2}\delta\mathcal{M}''(w,\mu)(\beta) \\
&\leq D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B) \\
&\leq-\frac{1}{2}\Delta\mathcal{M}''(w,\mu)(\alpha)\,.
\end{aligned}\tag{2.16}$$

Proof. For $\lambda\geq 0$ and $t\in[0,1]$ we have

$$\lambda+\alpha\leq\lambda+(1-t)A+tB\leq\lambda+\beta,$$

which implies that

$$(\lambda+\beta)^{-1}\leq(\lambda+(1-t)A+tB)^{-1}\leq(\lambda+\alpha)^{-1}\,.\tag{2.17}$$

If we multiply this both sides with $B-A$, then we obtain

$$\begin{aligned}
(\lambda+\beta)^{-1}(B-A)^2&\leq(B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \\
&\leq(\lambda+\alpha)^{-1}(B-A)^2\,.
\end{aligned}\tag{2.18}$$

Since $0<\delta\leq(B-A)^2\leq\Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2\geq\delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2\leq(\lambda+\alpha)^{-1}\Delta$, then by (2.18) we get

$$\delta(\lambda+\beta)^{-1}\leq(B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\leq(\lambda+\alpha)^{-1}\Delta,\tag{2.19}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply this inequality both sides with $(\lambda+(1-t)A+tB)^{-1}$ we derive

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1}(\lambda+(1-t)A+tB)^{-2} \\
&\leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\times(B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\leq(\lambda+\alpha)^{-1}\Delta(\lambda+(1-t)A+tB)^{-2}
\end{aligned}\tag{2.20}$$

and by (2.17) we further obtain the bounds

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3} \\
&\leq (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\times (B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\leq (\lambda+\alpha)^{-3}\Delta.
\end{aligned} \tag{2.21}$$

If we multiply with $2\lambda w(\lambda)(1-t)$ and integrate, then we get

$$\begin{aligned}
&2\delta\int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3}\left(\int_0^1 (1-t)\,dt\right)d\mu(\lambda) \\
&\leq 2\int_0^\infty \lambda w(\lambda)\left(\int_0^1 (1-t)(\lambda+(1-t)A+tB)^{-1}\right. \\
&\times \left((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\right) \\
&\times \left.(\lambda+(1-t)A+tB)^{-1}\,dt\right)d\mu(\lambda) \\
&\leq 2\Delta\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3}\left(\int_0^1 (1-t)\,dt\right)d\mu(\lambda),
\end{aligned}$$

which, by the equality (2.5), is equivalent to

$$\begin{aligned}
&\delta\int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda) \\
&\leq D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B) \\
&\leq \Delta\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).
\end{aligned} \tag{2.22}$$

If we take the derivative in (1.12) over t then we get

$$\mathcal{M}'(w,\mu)(t)=\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),\ t>0,$$

and

$$\mathcal{M}''(w,\mu)(t)=-2\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0.$$

This gives

$$\begin{aligned}
\int_0^\infty \frac{w(\lambda)}{(\lambda+\alpha)^3}d\mu(\lambda) &= -\frac{1}{2}\mathcal{M}''(w,\mu)(\alpha), \\
\int_0^\infty \frac{w(\lambda)}{(\lambda+\beta)^3}d\mu(\lambda) &= -\frac{1}{2}\mathcal{M}''(w,\mu)(\beta)
\end{aligned}$$

and by (2.22) we get (2.16). □

We have the following representation of *operator Jensen's gap* for the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{M}(w,\mu)):=\mathcal{M}(w,\mu)\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_k\mathcal{M}(w,\mu)(A_k).$$

Corollary 1. *Assume that* $\beta \geq A_k \geq \alpha > 0$, *for* $k \in \{1, ..., n\}$ *and* $0 < \delta \leq \left(A_j - \sum_{k=1}^{n} p_k A_k\right)^2 \leq \Delta$ *for all* $j \in \{1, ..., n\}$ *for some constants* α, β, δ, Δ, *then*

$$0 \leq -\frac{1}{2}\delta\mathcal{M}''(w,\mu)(\beta) \leq J(\mathbf{A}, \mathbf{p}, \mathcal{M}(w,\mu)) \leq -\frac{1}{2}\Delta\mathcal{M}''(w,\mu)(\alpha), \tag{2.23}$$

for any probability density n*-tuple* $\mathbf{p} = (p_1, ..., p_n)$.

Proof. From (2.16) we have

$$\begin{aligned} 0 &\leq -\frac{1}{2}\delta\mathcal{M}''(w,\mu)(\beta) \\ &\leq D(\mathcal{M}(w,\mu))\left(\sum_{k=1}^{n} p_k A_k\right)\left(A_j - \sum_{k=1}^{n} p_k A_k\right) \\ &\quad + \mathcal{M}(w,\mu)\left(\sum_{k=1}^{n} p_k A_k\right) - \mathcal{M}(w,\mu)(A_j) \\ &\leq -\frac{1}{2}\Delta\mathcal{M}''(w,\mu)(\alpha) \end{aligned} \tag{2.24}$$

for all $j \in \{1, ..., n\}$.

If we multiply by $p_j \geq 0$ and sum over j from 1 to n we get

$$\begin{aligned} 0 &\leq -\frac{1}{2}\delta\sum_{j=1}^{n} p_j\mathcal{M}''(w,\mu)(\beta) \\ &\leq D(\mathcal{M}(w,\mu))\left(\sum_{k=1}^{n} p_k A_k\right)\left(\sum_{j=1}^{n} p_j p_j A_j - \sum_{j=1}^{n} p_j \sum_{k=1}^{n} p_k A_k\right) \\ &\quad + \sum_{j=1}^{n} p_j\mathcal{M}(w,\mu)\left(\sum_{k=1}^{n} p_k A_k\right) - \sum_{j=1}^{n} p_j\mathcal{M}(w,\mu)(A_j) \\ &\leq -\frac{1}{2}\Delta\sum_{j=1}^{n} p_j\mathcal{M}''(w,\mu)(\alpha), \end{aligned}$$

which is equivalent to (2.23). □

3. Examples

We have:

Proposition 1. *Assume that the function* $f : (0, \infty) \to \mathbb{R}$ *is operator monotone in* $(0, \infty)$. *If* $\beta \geq A \geq \alpha > 0$, $B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, β, m, M, *then*

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[f'(\beta)M + f(\beta) - f(M+\beta)\right] \\ &\leq D(f)(A)(B-A) + f(A) - f(B) \\ &\leq \frac{M^2}{m^2}\left[f'(\alpha)m + f(\alpha) - f(m+\alpha)\right]. \end{aligned} \tag{3.1}$$

Proof. From (1.9) we have

$$\mathcal{M}(\ell,\mu)(t) = f(t) - a - bt,$$

where $a\in\mathbb{R}$, $\ell(\lambda)=\lambda$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

We have

$$\begin{aligned}&D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B)\\&=D(f-a-b\ell)(A)(B-A)+f(A)-a-bA-f(B)+a+bB\\&=D(f)(A)(B-A)+f(A)-f(B)\end{aligned}$$

$$\begin{aligned}&\mathcal{M}'(w,\mu)(\beta)M+\mathcal{M}(w,\mu)(\beta)-\mathcal{M}(w,\mu)(M+\beta)\\&=(f'(\beta)-b)M+f(\beta)-a-b\beta-f(M+\beta)+a+b(M+\beta)\\&=f'(\beta)M+f(\beta)-f(M+\beta)\end{aligned}$$

and similarly

$$\begin{aligned}&\mathcal{M}'(w,\mu)(\alpha)m+\mathcal{M}(w,\mu)(\alpha)-\mathcal{M}(w,\mu)(m+\alpha)\\&=f'(\alpha)m+f(\alpha)-f(m+\alpha).\end{aligned}$$

By utilising (2.7) we derive (3.1). □

Proposition 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$, then*

$$0\leq-\frac{1}{2}\delta f''(\beta)\leq D(f)(A)(B-A)+f(A)-f(B)\leq-\frac{1}{2}\Delta f''(\alpha). \tag{3.2}$$

The proof follows by (2.16) for the monotonic transform

$$\mathcal{M}(\ell,\mu)(t)=f(t)-a-bt,$$

where $a\in\mathbb{R}$, $\ell(\lambda)=\lambda$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Corollary 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A_k\geq\alpha>0$ for $k\in\{1,...,n\}$ and $0<\delta\leq\left(A_j-\sum_{k=1}^n p_kA_k\right)^2\leq\Delta$ for all $j\in\{1,...,n\}$, then*

$$0\leq-\frac{1}{2}\delta f''(\beta)\leq f\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_kf(A_k)\leq-\frac{1}{2}\Delta f''(\alpha) \tag{3.3}$$

for any probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$.

If we consider the operator monotone function $f(t)=t^r$, $r\in(0,1]$, then we obtain the power inequalities

$$0\leq\frac{1}{2}r(1-r)\delta\beta^{r-2}\leq\left(\sum_{k=1}^n p_kA_k\right)^r-\sum_{k=1}^n p_kA_k^r\leq\frac{1}{2}r(1-r)\Delta\alpha^{r-2}, \tag{3.4}$$

provided that $\beta\geq A_k\geq\alpha>0$, for $k\in\{1,...,n\}$ and $0<\delta\leq\left(A_j-\sum_{k=1}^n p_kA_k\right)^2\leq\Delta$ for all $j\in\{1,...,n\}$.

With these assumptions we also have the logarithmic inequalities

$$0\leq\frac{\delta}{2\beta}\leq\ln\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_k\ln A_k\leq\frac{\Delta}{2\alpha}. \tag{3.5}$$

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