ERROR BOUNDS RELATED TO MIDPOINT AND TRAPEZOID RULES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq -\frac{1}{24}\delta\mathcal{M}''(w, \mu)(\beta) \\ &\leq \mathcal{M}(w, \mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w, \mu)((1-t)A + tB)\, dt \\ &\leq -\frac{1}{24}\Delta\mathcal{M}''(w, \mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq -\frac{1}{12}\delta\mathcal{M}''(w, \mu)(\beta) \\ &\leq \int_0^1 \mathcal{M}(w, \mu)((1-t)A + tB)\, dt - \frac{\mathcal{M}(w, \mu)(A) + \mathcal{M}(w, \mu)(B)}{2} \\ &\leq -\frac{1}{12}\Delta\mathcal{D}''(w, \mu)(\alpha), \end{aligned}$$

where $\mathcal{M}''(w, \mu)$ is the second derivative of $\mathcal{M}(w, \mu)$ as a real function.

Applications for power function and logarithm are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

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for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.2}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.3}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.3) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1) \mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.6}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.9}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$\text{(OC)}\qquad f\left((1-\lambda)A+\lambda B\right)\leq(\geq)(1-\lambda)f(A)+\lambda f(B)$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda),\tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.11).

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t),\ t>0.\tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)&=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned}\tag{1.13}$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$, then

$$\mathcal{M}(w,\mu)(t)=\int_0^\infty w(\lambda)\,d\mu(\lambda)-\mathcal{D}(\ell w,\mu)(t),\tag{1.14}$$

where $\ell(t)=t$, $t>0$.

Consider the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq0$ and $a>0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq0$$

and

$$\int_0^\infty w(\lambda)\,d\lambda=\int_0^\infty\exp(-a\lambda)\,d\lambda=\frac{1}{a},$$

where

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})(t)=t\mathcal{D}(w,\mu)(t)=tE_1(at)\exp(at),\ t\geq0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty\frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.14) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.14) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda).\tag{1.15}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

for $T>0$.

In this paper, we show among others that, if $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned}0&\leq-\frac{1}{24}\delta\mathcal{M}''(w,\mu)(\beta)\\&\leq\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)-\int_0^1\mathcal{M}(w,\mu)((1-t)A+tB)\,dt\\&\leq-\frac{1}{24}\Delta\mathcal{M}''(w,\mu)(\alpha)\end{aligned}$$

and

$$\begin{aligned}0&\leq-\frac{1}{12}\delta\mathcal{M}''(w,\mu)(\beta)\\&\leq\int_0^1\mathcal{M}(w,\mu)((1-t)A+tB)\,dt-\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}\\&\leq-\frac{1}{12}\Delta\mathcal{D}''(w,\mu)(\alpha).\end{aligned}$$

Applications for power function and logarithm are also provided.

2. Some Representations

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w,\mu))$:

Lemma 1. *For all $A>0$,*

$$D(\mathcal{M}(w,\mu))(A)(V)=\int_0^\infty\lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\tag{2.1}$$

for all $V\in S(H)$, the class of all selfadjoint operators on H.

Proof. By the definition of $\mathcal{M}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}
&\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)\\
&=\int_0^\infty w(\lambda)\left[1-\lambda(A+tV+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}(\lambda+A+tV-\lambda-A)(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\
&=t\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)}{t}\\
&=\lim_{t\to 0}\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is obtained. □

For the case of second Fréchet derivative $D^2(\mathcal{M}(w,\mu))$, we have the representation:

Lemma 2. *For all* $A>0$,

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=-2\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}\tag{2.2}$$

for all $V\in S(H)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=\lim_{t\to 0}\frac{D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda)\\
&-\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\\
&\times\left[(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda\geq 0$ and t as above,

$$U_{t,\lambda}:=(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda+A+tV$, then we get

$$\begin{aligned}
&(\lambda+A+tV)U_{t,\lambda}(\lambda+A+tV)\\
&=V-(\lambda+A+tV)(\lambda+A)^{-1}V(\lambda+A)^{-1}(\lambda+A+tV)\\
&=V-\left(1+tV(\lambda+A)^{-1}\right)V\left(1+t(\lambda+A)^{-1}V\right)\\
&=V-\left(V+tV(\lambda+A)^{-1}V\right)\left(1+t(\lambda+A)^{-1}V\right)\\
&=V-V-tV(\lambda+A)^{-1}V-tV(\lambda+A)^{-1}V\\
&-t^2V(\lambda+A)^{-1}V(\lambda+A)^{-1}V\\
&=-2tV(\lambda+A)^{-1}V+t^2V(\lambda+A)^{-1}V(\lambda+A)^{-1}V\\
&=-t\left[2V(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V(\lambda+A)^{-1}V\right].
\end{aligned}\tag{2.3}$$

If we multiply the equality by $(\lambda+A+tV)^{-1}$ both sides, we get for $t\neq 0$

$$\begin{aligned}
\frac{U_{t,\lambda}}{t}&=-(\lambda+A+tV)^{-1}\left[2V(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V(\lambda+A)^{-1}V\right]\\
&\times(\lambda+A+tV)^{-1}.
\end{aligned}\tag{2.4}$$

If we take the limit over $t\to 0$ in, then we get

$$\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)=-2(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}.$$

Therefore, by the properties of limit under the sign of integral, we get

$$\begin{aligned}
&\lim_{t\to 0}\frac{D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)}{t}\\
&=\int_0^\infty \lambda w(\lambda)\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right)d\mu(\lambda)\\
&=-2\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}$$

and the representation (2.2) is obtained. □

We have the following representation for the transform $\mathcal{M}(w,\mu)$:

Theorem 3. *For all A, $B>0$ we have*

$$\begin{aligned}\mathcal{M}(w,\mu)(B) &\\ &= \mathcal{M}(w,\mu)(A) + \int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda) \\ &- 2\int_0^1 (1-t)\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt. \end{aligned} \tag{2.5}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E) = f(C) + D(f)(C)(E-C) + \int_0^1 (1-t)D^2(f)((1-t)C+tE)(E-C,E-C)\,dt \tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E\in\mathcal{O}$.

If we write (2.6) for $\mathcal{M}(w,\mu)$ and A, $B>0$, we get

$$\begin{aligned}\mathcal{M}(w,\mu)(B) &= \mathcal{M}(w,\mu)(A) + D(\mathcal{M}(w,\mu))(A)(B-A) \\ &+ \int_0^1 (1-t)D^2(\mathcal{M}(w,\mu))((1-t)A+tB)(B-A,B-A)\,dt\end{aligned}$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). □

We have the following representation of operator Jensen's gap for the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{M}(w,\mu)) := \mathcal{M}(w,\mu)\left(\sum_{k=1}^n p_kA_k\right) - \sum_{k=1}^n p_k\mathcal{M}(w,\mu)(A_k).$$

Theorem 4. *We have the representation*

$$\begin{aligned} J(\mathbf{A},\mathbf{p},\mathcal{M}(w,\mu)) &\\ &= 2\sum_{k=1}^n p_k \int_0^\infty \lambda w(\lambda)\left(\int_0^1 (1-t)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j + tA_k\right)^{-1}\right. \\ &\times\left(A_k - \sum_{j=1}^n p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j+tA_k\right)^{-1} \\ &\times\left.\left(A_k-\sum_{j=1}^n p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^n p_jA_j+tA_k\right)^{-1}dt\right)d\mu(\lambda) \\ &\geq 0\end{aligned} \tag{2.7}$$

for the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$. This also shows that $\mathcal{M}(w,\mu)$ is operator concave on $(0,\infty)$.

Proof. From the identity (2.5) we get

$$\begin{aligned}
& D\left(\mathcal{M}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n}p_jA_j\right)\left(A_k-\sum_{j=1}^{n}p_jA_j\right) \\
& +\mathcal{M}\left(w,\mu\right)\left(\sum_{j=1}^{n}p_jA_j\right)-\mathcal{M}\left(w,\mu\right)\left(A_k\right) \\
& =2\int_0^\infty w\left(\lambda\right)\left(\int_0^1\left(1-t\right)\left(\lambda+\left(1-t\right)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}\right. \\
& \times\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+\left(1-t\right)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1} \\
& \left.\times\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+\left(1-t\right)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}dt\right)d\mu\left(\lambda\right) \\
& \geq 0
\end{aligned}$$

for all $k\in\{1,...,n\}$.

If we multiply this inequality with $p_k\geq 0$, take into account that $\sum_{k=1}^{n}p_k=1$ and

$$\begin{aligned}
& \sum_{k=1}^{n}p_kD\left(\mathcal{M}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n}p_jA_j\right)\left(A_k-\sum_{j=1}^{n}p_jA_j\right) \\
& =D\left(\mathcal{M}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n}p_jA_j\right)\left(\sum_{k=1}^{n}p_kA_k-\sum_{j=1}^{n}p_jA_j\right) \\
& =D\left(\mathcal{M}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n}p_jA_j\right)(0)=0,
\end{aligned}$$

then we obtain the desired result (2.7). □

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B}:[0,1]\rightarrow\mathbb{R}$ defined by

$$f_{A,B}\left(t\right):=f\left(\left(1-t\right)A+tB\right),\ t\in\left[0,1\right].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on $[0,1]$,*

$$\frac{df_{A,B}\left(t\right)}{dt}=D\left(f\right)\left(\left(1-t\right)A+tB\right)\left(B-A\right)\tag{2.8}$$

and

$$\frac{d^2f_{A,B}\left(t\right)}{dt^2}=D^2\left(f\right)\left(\left(1-t\right)A+tB\right)\left(B-A,B-A\right)\tag{2.9}$$

for $t\in[0,1]$, *where in* 0 *and* 1 *the derivatives are the right and left derivatives.*

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}&\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt}&=\lim_{h\to 0}\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\lim_{h\to 0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\&=D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (2.8).

Similarly,

$$\begin{aligned}&\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\\&=\frac{D(f)((1-(t+h))A+(t+h)B)(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\\&=\frac{D(f)((1-t)A+tB+h(B-A))(B-A)-D(f)((1-t)A+tB)(B-A)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}\frac{d^2f_{A,B}(t)}{dt^2}&=\lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\right\}\\&=D^2(f)((1-t)A+tB)(B-A,B-A),\end{aligned}$$

which proves (2.9). □

For the transform $\mathcal{M}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\mathcal{M}(w,\mu)_{A,B}(t):=\mathcal{M}(w,\mu)((1-t)A+tB)$$

where $A,\ B>0$ and $t\in[0,1]$.

Corollary 1. *For all* $A,\ B>0$ *and* $t\in[0,1]$,

$$\begin{aligned}\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}&=D(\mathcal{M}(w,\mu))((1-t)A+tB)(B-A)\\&=\int_0^\infty\lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\times(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}(2.11)\qquad &\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}\\&=D^2(\mathcal{M}(w,\mu))((1-t)A+tB)(B-A,B-A)\\&=-2\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}$$

We observe that if $f(t)=\mathcal{M}(w,\mu)(t)$, $t>0$, in Lemma 3, then by the representations from Lemma 1 and Lemma 2 we obtain the desired equalities (2.10) and (2.11).

3. Midpoint and Trapezoid Inequalities

We have the following identity for the midpoint rule:

Theorem 5. *For all A, $B>0$ we have the identity*

$$\begin{aligned}(3.1)\qquad &\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)-\int_0^1\mathcal{M}(w,\mu)((1-t)A+tB)\,dt\\&=2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right.\\&\times\left[\int_0^\infty\lambda w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right.\\&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.\end{aligned}$$

Proof. From (2.5) we have for $B=E>0$ and $A=C>0$ that

$$\begin{aligned}&\mathcal{M}(w,\mu)(E)\\&=\mathcal{M}(w,\mu)(C)+\int_0^\infty\lambda w(\lambda)(\lambda+C)^{-1}(E-C)(\lambda+C)^{-1}d\mu(\lambda)\\&-2\int_0^1(1-s)\left[\int_0^\infty\lambda w(\lambda)(\lambda+(1-s)C+sE)^{-1}(E-C)\right.\\&\times\left.(\lambda+(1-s)C+sE)^{-1}(E-C)(\lambda+(1-s)C+sE)^{-1}d\mu(\lambda)\right]ds,\end{aligned}$$

which implies for $E = (1-t)A + tB$, $t \in [0,1]$ and $C = \frac{A+B}{2}$, that

$$
\begin{aligned}
&\mathcal{M}(w,\mu)((1-t)A+tB) \\
&= \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \\
&+ \left(t-\frac{1}{2}\right)\int_0^\infty \lambda w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1} d\mu(\lambda) \\
&- 2\left(t-\frac{1}{2}\right)^2 \int_0^1 (1-s) \\
&\times \left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} d\mu(\lambda)\right] ds.
\end{aligned}
\tag{3.2}
$$

If we integrate (3.2) over $t \in [0,1]$, then we get

$$
\begin{aligned}
&\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt \\
&= \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \\
&+ \int_0^1 \left(t-\frac{1}{2}\right) dt \\
&\times \int_0^\infty \lambda w(\lambda)\left(\lambda+\frac{A+B}{2}\right)^{-1}(B-A)\left(\lambda+\frac{A+B}{2}\right)^{-1} d\mu(\lambda) \\
&- 2\int_0^1 \left(t-\frac{1}{2}\right)^2 \left\{\int_0^1 (1-s)\right. \\
&\times \left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times \left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\times \left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1} d\mu(\lambda)\right] ds\right\} dt
\end{aligned}
$$

and since $\int_0^1 \left(t-\frac{1}{2}\right) dt = 0$, hence the identity (3.1) is proved. □

Corollary 2. *Assume that* $\beta \geq A$, $B \geq \alpha > 0$ *and* $0 < \delta \leq (B-A)^2 \leq \Delta$ *for some constants* α, β, δ, Δ, *then*

$$(3.3)\qquad \begin{aligned} 0 &\leq -\frac{1}{24}\delta\mathcal{M}''(w,\mu)(\beta) \\ &\leq \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt \\ &\leq -\frac{1}{24}\Delta\mathcal{M}''(w,\mu)(\alpha). \end{aligned}$$

Proof. Since $\beta \geq A$, $B \geq \alpha > 0$, hence

$$\lambda + \alpha \leq \lambda + (1-s)\frac{A+B}{2} + s((1-t)A+tB) \leq \lambda + \beta,$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

This implies that

$$(3.4)\qquad (\lambda+\beta)^{-1} \leq \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A+tB)\right)^{-1} \leq (\lambda+\alpha)^{-1}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply this both sides with $B-A$, then we obtain

$$(3.5)\qquad \begin{aligned} &(\lambda+\beta)^{-1}(B-A)^2 \\ &\leq (B-A)\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A+tB)\right)^{-1}(B-A) \\ &\leq (\lambda+\alpha)^{-1}(B-A)^2 \end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

Since $0 < \delta \leq (B-A)^2 \leq \Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2 \geq \delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2 \leq (\lambda+\alpha)^{-1}\Delta$, then by (3.5)

$$(3.6)\qquad \begin{aligned} &\delta(\lambda+\beta)^{-1} \\ &\leq (B-A)\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A+tB)\right)^{-1}(B-A) \\ &\leq \Delta(\lambda+\alpha)^{-1} \end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply both sides with $\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}$ we derive

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-2}\\
&\leq\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}\\
&\leq\Delta(\lambda+\alpha)^{-1}\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-2}
\end{aligned}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

By utilising (3.4) we further obtain the bounds

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3}\\
&\leq\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}\\
&\leq\Delta(\lambda+\alpha)^{-3}
\end{aligned}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply by $2\lambda w(\lambda)\left(t-\frac{1}{2}\right)^2(1-s)\geq 0$ and integrate, then we get

$$\begin{aligned}
&2\delta\int_0^\infty\lambda w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2dt\int_0^1(1-s)\,ds\\
&\leq 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right.\\
&\times\left[\int_0^\infty\lambda w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right.\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt\\
&\leq 2\Delta\int_0^\infty\lambda w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2dt\int_0^1(1-s)\,ds
\end{aligned}\tag{3.7}$$

and by the identity (3.1) and the fact that

$$\int_0^1 \left(t-\frac{1}{2}\right)^2 dt = \frac{1}{12} \text{ and } \int_0^1 (1-s)\,ds = \frac{1}{2}$$

we obtain

$$\begin{aligned}
&\frac{1}{12}\delta\int_0^{\infty} \lambda w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda) \\
&\leq \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt \\
&\leq \frac{1}{12}\Delta\int_0^{\infty} \lambda w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).
\end{aligned} \tag{3.8}$$

If we take the derivative in (1.6) over t, then we get

$$\mathcal{M}'(w,\mu)(t) = \int_0^{\infty} \frac{\lambda w(\lambda)}{(\lambda+t)^2}\,d\mu(\lambda),\ t>0,$$

and

$$\mathcal{M}''(w,\mu)(t) = -2\int_0^{\infty} \frac{\lambda w(\lambda)}{(\lambda+t)^3}\,d\mu(\lambda),\ t>0.$$

This gives

$$\begin{aligned}
\int_0^{\infty} \frac{\lambda w(\lambda)}{(\lambda+\alpha)^3}\,d\mu(\lambda) &= -\frac{1}{2}\mathcal{M}''(w,\mu)(\alpha), \\
\int_0^{\infty} \frac{\lambda w(\lambda)}{(\lambda+\beta)^3}\,d\mu(\lambda) &= -\frac{1}{2}\mathcal{M}''(w,\mu)(\beta)
\end{aligned}$$

and by (3.2) we obtain (3.3). □

We have the following identity for the trapezoid rule:

Theorem 6. *For all A, $B>0$ we have the identity*

$$\begin{aligned}
&\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} \\
&= \int_0^1 t(1-t)\left[\int_0^{\infty} \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt.
\end{aligned} \tag{3.9}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= \frac{1}{2}\left[t(1-t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\Big|_0^1 - \int_0^1 (1-2t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right] \\
&= \int_0^1 \left(t-\frac{1}{2}\right)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt \\
&= \left(t-\frac{1}{2}\right)\mathcal{M}(w,\mu)_{A,B}(t)\Big|_0^1 - \int_0^1 \mathcal{M}(w,\mu)_{A,B}(t)\,dt \\
&= \frac{1}{2}\left[\mathcal{M}(w,\mu)_{A,B}(1)+\mathcal{M}(w,\mu)_{A,B}(0)\right] - \int_0^1 \mathcal{M}(w,\mu)_{A,B}(t)\,dt,
\end{aligned}$$

that gives the identity

$$\begin{aligned}
&\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} - \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt \\
&= \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt.
\end{aligned} \tag{3.10}$$

By (2.11) we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.
\end{aligned} \tag{3.11}$$

By making use of (3.10) and (3.11) we obtain (3.9). □

We have:

Corollary 3. *Assume that* $\beta \geq A,\ B \geq \alpha > 0,$ *and* $0 < \delta \leq (B-A)^2 \leq \Delta,$ *then*

$$\begin{aligned}
0 &\leq -\frac{1}{12}\delta\mathcal{M}''(w,\mu)(\beta) \\
&\leq \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} \\
&\leq -\frac{1}{12}\Delta\mathcal{M}''(w,\mu)(\alpha).
\end{aligned} \tag{3.12}$$

Proof. As in the proof of Corollary 2 we have

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3} \\
&\leq (\lambda+(1-t)A+tB)^{-1}(B-A) \\
&\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\
&\leq \Delta(\lambda+\alpha)^{-3}
\end{aligned} \tag{3.13}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply by $t(1-t)\lambda w(\lambda)\geq 0$ and integrate, then we get

$$\begin{aligned}(3.14)\qquad & \delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty \lambda w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda)\\ &\leq \int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt\\ &\leq \Delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).\end{aligned}$$

Since

$$\int_0^1 t(1-t)\,dt=\frac{1}{6},$$

$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\alpha)^3}\,d\mu(\lambda)=-\frac{1}{2}\mathcal{M}''(w,\mu)(\alpha)$$

and

$$\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+\beta)^3}\,d\mu(\lambda)=-\frac{1}{2}\mathcal{M}''(w,\mu)(\beta),$$

then by (3.14) we derive (3.12). □

We have an alternative identity for the midpoint rule:

Theorem 7. *For all A, $B>0$ we have the identity*

$$\begin{aligned}(3.15)\qquad & \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)-\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt\\ &=\int_0^{1/2} t^2\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt\\ &+\int_{1/2}^1 (t-1)^2\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt.\end{aligned}$$

Proof. Using integration by parts for Bochner's integral, we have

$$\begin{aligned}&\frac{1}{2}\int_0^{1/2} t^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt\\ &=\frac{1}{2}\left[t^2\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^{1/2}-2\int_0^{1/2} t\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \int_0^{1/2} t\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt \\
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} \\
&- \left[t\mathcal{M}(w,\mu)_{A,B}(t)\Big|_0^{1/2} - \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)\,dt \right] \\
&= \frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_{1/2}^{1}(t-1)^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= \frac{1}{2}\left[(t-1)^2\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\Bigg|_{1/2}^{1} - 2\int_{1/2}^{1}(t-1)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt \right] \\
&= -\frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} \\
&- \left[(t-1)\mathcal{M}(w,\mu)_{A,B}(t)\Big|_{1/2}^{1} - \int_{1/2}^{1}\mathcal{M}(w,\mu)_{A,B}(t)\,dt \right] \\
&= -\frac{1}{8}\frac{d\mathcal{M}(w,\mu)_{A,B}(1/2)}{dt} - \frac{1}{2}\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_{1/2}^{1}\mathcal{M}(w,\mu)_{A,B}(t)\,dt.
\end{aligned}
$$

If we add these two equalities, then we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt + \frac{1}{2}\int_{1/2}^{1}(t-1)^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\mathcal{M}(w,\mu)_{A,B}(1/2) + \int_0^{1/2}\mathcal{M}(w,\mu)_{A,B}(t)\,dt \\
&+ \int_{1/2}^{1}\mathcal{M}(w,\mu)_{A,B}(t)\,dt \\
&= \int_0^1 \mathcal{M}(w,\mu)_{A,B}(t)\,dt - \mathcal{M}(w,\mu)_{A,B}(1/2).
\end{aligned} \tag{3.16}
$$

By (2.11) we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^{1/2} t^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \\
&= -\int_0^{1/2} t^2\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt
\end{aligned} \tag{3.17}
$$

and

$$(3.18)\quad \frac{1}{2}\int_{1/2}^{1}(t-1)^2\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt$$
$$=-\int_{1/2}^{1}(t-1)^2\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.$$

By employing (3.16)-(3.18) we derive the desired result (3.15). □

Remark 1. *By making use of the identity (3.15) one can obtain the same upper and lower bounds for the midpoint rule as those in Corollary 2.*

4. Some Examples

The case of operator monotone functions is as follows:

Proposition 1. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.9), then for* A, $B>0$,

$$(4.1)\quad f(B)=f(A)+b(B-A)+\int_0^\infty \lambda^2(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda)$$
$$-2\int_0^1(1-t)\left[\int_0^\infty \lambda^2(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.$$

Proof. From (1.9) we get

$$\mathcal{M}(\ell,\mu)(t)=f(t)-a-bt,$$

where $a\in\mathbb{R}$, $\ell(\lambda)=\lambda$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}(\ell,\mu)(B)=f(B)-a-bB,\ \mathcal{M}(\ell,\mu)(A)=f(A)-a-bA$$

and by (2.5) we derive

$$f(B)-a-bB$$
$$=f(A)-a-bA+\int_0^\infty \lambda^2(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda)$$
$$-2\int_0^1(1-t)\left[\int_0^\infty \lambda^2(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt,$$

which is equivalent to (4.1). □

The case of operator monotone functions for the Jensen's gap is as follows:

Proposition 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then,*

$$
\begin{aligned}
(4.2)\qquad & f\left(\sum_{k=1}^{n}p_kA_k\right)-\sum_{k=1}^{n}p_kf\left(A_k\right)\\
&=2\int_0^{\infty}\lambda^2\sum_{k=1}^{n}p_k\left(\int_0^1(1-t)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}\right.\\
&\times\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}\\
&\times\left.\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}dt\right)d\mu\left(\lambda\right)\\
&\geq 0
\end{aligned}
$$

for the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$.

The proof follows by Theorem 4 applied for

$$\mathcal{M}\left(\ell,\mu\right)(t)=f\left(t\right)-a-bt,$$

where $a\in\mathbb{R}$, $\ell\left(\lambda\right)=\lambda$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

We have the following midpoint and trapezoid inequalities for operator monotone functions:

Proposition 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$, then*

$$
\begin{aligned}
(4.3)\qquad 0&\leq-\frac{1}{24}\delta f''\left(\beta\right)\leq f\left(\frac{A+B}{2}\right)-\int_0^1 f\left((1-t)A+tB\right)dt\\
&\leq-\frac{1}{24}\Delta f''\left(\alpha\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.4)\qquad 0&\leq-\frac{1}{12}\delta f''\left(\beta\right)\leq\int_0^1 f\left((1-t)A+tB\right)dt-\frac{f\left(A\right)+f\left(B\right)}{2}\\
&\leq-\frac{1}{12}\Delta f''\left(\alpha\right).
\end{aligned}
$$

Proof. From (1.9) we get

$$\mathcal{M}\left(\ell,\mu\right)(t)=f\left(t\right)-a-bt,$$

where $a\in\mathbb{R}$, $\ell\left(\lambda\right)=\lambda$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}\left(w,\mu\right)\left(\frac{A+B}{2}\right)=f\left(\frac{A+B}{2}\right)-a-b\frac{A+B}{2},$$

$$\frac{\mathcal{M}\left(w,\mu\right)(A)+\mathcal{M}\left(w,\mu\right)(B)}{2}=\frac{f\left(A\right)+f\left(B\right)}{2}-a-b\frac{A+B}{2},$$

$$\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt = \int_0^1 f((1-t)A+tB)\,dt - a - b\frac{A+B}{2}$$

and by Corollary 2 and 3 we derive (4.3) and (4.4). □

Remark 2. *If $\beta \geq A,\ B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$, then for $r \in (0,1]$ we have the power inequalities*

$$0 \leq \frac{1}{24} r(1-r)\delta\beta^{r-2} \leq \left(\frac{A+B}{2}\right)^r - \int_0^1 ((1-t)A+tB)^r\,dt \leq \frac{1}{24} r(1-r)\Delta\alpha^{r-2} \tag{4.5}$$

and

$$0 \leq \frac{1}{12} r(1-r)\delta\beta^{r-2} \leq \int_0^1 ((1-t)A+tB)^r\,dt - \frac{A^r+B^r}{2} \leq \frac{1}{12} r(1-r)\Delta\alpha^{r-2}. \tag{4.6}$$

We also have the logarithmic inequalities

$$0 \leq \frac{\delta}{24\beta} \leq \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln((1-t)A+tB)\,dt \leq \frac{\Delta}{24\alpha} \tag{4.7}$$

and

$$0 \leq \frac{\delta}{12\beta} \leq \int_0^1 \ln((1-t)A+tB)\,dt - \frac{\ln A + \ln B}{2} \leq \frac{\Delta}{12\alpha}, \tag{4.8}$$

if $\beta \geq A,\ B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$.

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