SECOND DERIVATIVE LIPSCHITZ TYPE INEQUALITIES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned}
&\|D(\mathcal{M}(w,\mu))(A)(B-A) + \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)(B)\| \\
&\leq \|B-A\|^2 \\
&\times \begin{cases} \frac{(m_2-m_1)\mathcal{M}'(w,\mu)(m_1)+\mathcal{M}(w,\mu)(m_1)-\mathcal{M}(w,\mu)(m_2)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \\ \left(-\frac{1}{2}\mathcal{M}''(w,\mu)(m)\right) & \text{if } m_1 = m_2 = m, \end{cases}
\end{aligned}$$

where $D(\mathcal{M}(w,\mu))$ is the Fréchet derivative of $\mathcal{M}(w,\mu)$ as a function of operator and $\mathcal{M}'(w,\mu)$, $\mathcal{M}''(w,\mu)$ are the first and the second derivative of $\mathcal{M}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r \in (0,1]$ and A, $B \geq m > 0$,

$$\left\| \int_0^1 ((1-t)A + tB)^r dt - \left(\frac{A+B}{2}\right)^r \right\| \leq \frac{1}{24} r(1-r) m^{r-2} \|B-A\|^2$$

and

$$\left\| \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r dt \right\| \leq \frac{1}{12} r(1-r) m^{r-2} \|B-A\|^2.$$

Similar logarithmic inequalities are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [16] had given a definitive characterization of operator monotone functions as follows, see for instance [5, p. 144-145]:

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Theorem 1. *A function $f:(0,\infty)\rightarrow\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda) \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.1).

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\rightarrow\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [14]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [11] and [13].

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

for any A, $B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [15], the following inequality holds

$$\||A|-|B|\|\leq\frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.2}$$

for any A, $B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\|A-B\|_{HS} \tag{1.3}$$

for any A, $B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\|\leq a_1\|A-B\|+a_2\|A-B\|^2+O\left(\|A-B\|^3\right) \tag{1.4}$$

where

$$a_1=\left\|A^{-1}\right\|\|A\| \text{ and } a_2=\left\|A^{-1}\right\|+\left\|A^{-1}\right\|^3\|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\|\leq f'(a)\|A-B\| \tag{1.5}$$

where f is an *operator monotone function* on $(0,\infty)$ and A, $B\geq a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [5, p. 145]

$$t^{r}=\frac{\sin(r\pi)}{\pi}t\int_{0}^{\infty}\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.6}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_{0}^{u}\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_{0}^{\infty}\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_{0}^{\infty}\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.7}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.8}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.8) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.9}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_{r}(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_{r})(t),\ t>0. \tag{1.10}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.11}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.12}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.13}$$

for $T>0$.

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \ t > 0. \tag{1.14}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t) &= \int_0^\infty w(\lambda) t (t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(t + \lambda - \lambda)(t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[1 - \lambda (t + \lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{1.15}$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w, \mu)(t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(\ell w, \mu)(t), \tag{1.16}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t + \lambda} d\lambda = E_1(at)\exp(at), \ t \geq 0$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a},$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w, \mu)(t) = tE_1(at)\exp(at), \ t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a}, \mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t + \lambda} d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda) d\lambda - \mathcal{D}(\ell e_{-a}, \mu)(t), \ t > 0$$

and the equality (1.16) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then $\int_0^\infty w_r(\lambda) d\lambda = \infty$ and the equality (1.16) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w, \mu)(T) = T\mathcal{D}(w, \mu)(T) = \int_0^\infty w(\lambda) \left[1 - \lambda (T + \lambda)^{-1}\right] d\mu(\lambda). \tag{1.17}$$

This gives the representation

$$T^{r}=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_{r},\mu)(T),$$

for $T>0$.

In this paper, we show among others that, if $A\geq m_{1}>0$, $B\geq m_{2}>0$, then

$$\begin{aligned}&\left\|D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B)\right\|\\&\leq\|B-A\|^{2}\\&\times\begin{cases}\frac{(m_{2}-m_{1})\mathcal{M}'(w,\mu)(m_{1})+\mathcal{M}(w,\mu)(m_{1})-\mathcal{M}(w,\mu)(m_{2})}{(m_{2}-m_{1})^{2}} & \text{if } m_{1}\neq m_{2},\\ \\ \left(-\frac{1}{2}\mathcal{M}''(w,\mu)(m)\right) & \text{if } m_{1}=m_{2}=m,\end{cases}\end{aligned}$$

where $D(\mathcal{M}(w,\mu))$ is the Fréchet derivative of $\mathcal{M}(w,\mu)$ as a function of operator and $\mathcal{M}'(w,\mu)$, $\mathcal{M}''(w,\mu)$ are the first and the second derivative of $\mathcal{M}(w,\mu)$ as a real function.

We also prove the norm integral inequalities for power $r\in(0,1]$ and A, $B\geq m>0$,

$$\left\|\int_{0}^{1}((1-t)A+tB)^{r}dt-\left(\frac{A+B}{2}\right)^{r}\right\|\leq\frac{1}{24}r(1-r)m^{r-2}\|B-A\|^{2}$$

and

$$\left\|\frac{A^{r}+B^{r}}{2}-\int_{0}^{1}((1-t)A+tB)^{r}dt\right\|\leq\frac{1}{12}r(1-r)m^{r-2}\|B-A\|^{2}.$$

Similar logarithmic inequalities are also provided.

2. Main Results

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w,\mu))$:

Lemma 1. *For all $A>0$,*

$$D(\mathcal{M}(w,\mu))(A)(V)=\int_{0}^{\infty}\lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\tag{2.1}$$

for all $V\in S(H)$, the class of all selfadjoint operators on H.

Proof. By the definition of $\mathcal{M}(w,\mu)$ we have for t in a small open interval around 0 that

$$\begin{aligned}&\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)\\&=\int_{0}^{\infty}w(\lambda)\left[1-\lambda(A+tV+\lambda)^{-1}\right]d\mu(\lambda)-\int_{0}^{\infty}w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_{0}^{\infty}\lambda w(\lambda)\left[(\lambda+A)^{-1}-(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\&=\int_{0}^{\infty}\lambda w(\lambda)\left[(\lambda+A)^{-1}(\lambda+A+tV-\lambda-A)(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\&=t\int_{0}^{\infty}\lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{M}(w,\mu)(A+tV)-\mathcal{M}(w,\mu)(A)}{t}\\
&=\lim_{t\to 0}\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A+tV)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\left[(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is obtained. □

For the case of second Fréchet derivative $D^2(\mathcal{M}(w,\mu))$, we have the representation:

Lemma 2. *For all $A>0$,*

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=-2\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)
\end{aligned}\tag{2.2}$$

for all $V\in S(H)$.

Proof. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}
&D^2(\mathcal{M}(w,\mu))(A)(V,V)\\
&=\lim_{t\to 0}\frac{D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)}{t}.
\end{aligned}$$

Observe, by (2.1), that we have for t in a small open interval around 0

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&D(\mathcal{M}(w,\mu))(A+tV)(V)-D(\mathcal{M}(w,\mu))(A)(V)\\
&=\int_0^\infty \lambda w(\lambda)(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}d\mu(\lambda)\\
&-\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}d\mu(\lambda)\\
&=\int_0^\infty \lambda w(\lambda)\\
&\times\left[(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Define for $\lambda\ge 0$ and t as above,

$$U_{t,\lambda}:=(\lambda+A+tV)^{-1}V(\lambda+A+tV)^{-1}-(\lambda+A)^{-1}V(\lambda+A)^{-1}.$$

If we multiply both sides of $U_{t,\lambda}$ with $\lambda + A + tV$, then we get

$$
\begin{aligned}
(2.3)\qquad & (\lambda + A + tV)\, U_{t,\lambda}\, (\lambda + A + tV) \\
&= V - (\lambda + A + tV)\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, (\lambda + A + tV) \\
&= V - \left(1 + tV\, (\lambda + A)^{-1}\right) V \left(1 + t\, (\lambda + A)^{-1}\, V\right) \\
&= V - \left(V + tV\, (\lambda + A)^{-1}\, V\right) \left(1 + t\, (\lambda + A)^{-1}\, V\right) \\
&= V - V - tV\, (\lambda + A)^{-1}\, V - tV\, (\lambda + A)^{-1}\, V \\
&- t^2 V\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V \\
&= -2tV\, (\lambda + A)^{-1}\, V + t^2 V\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V \\
&= -t\left[2V\, (\lambda + A)^{-1}\, V + tV\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V\right].
\end{aligned}
$$

If we multiply the equality by $(\lambda + A + tV)^{-1}$ both sides, we get for $t \neq 0$

$$
\begin{aligned}
(2.4)\quad \frac{U_{t,\lambda}}{t} &= -\,(\lambda + A + tV)^{-1}\left[2V\, (\lambda + A)^{-1}\, V + tV\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V\right] \\
&\times (\lambda + A + tV)^{-1}.
\end{aligned}
$$

If we take the limit over $t \to 0$ in, then we get

$$
\lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right) = -2\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}.
$$

Therefore, by the properties of limit under the sign of integral, we get

$$
\begin{aligned}
& \lim_{t\to 0} \frac{D\left(\mathcal{M}(w,\mu)\right)(A + tV)(V) - D\left(\mathcal{M}(w,\mu)\right)(A)(V)}{t} \\
&= \int_0^\infty \lambda w(\lambda) \lim_{t\to 0}\left(\frac{U_{t,\lambda}}{t}\right) d\mu(\lambda) \\
&= -2\int_0^\infty \lambda w(\lambda)\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, V\, (\lambda + A)^{-1}\, d\mu(\lambda)
\end{aligned}
$$

and the representation (2.2) is obtained. □

We have the following representation for the transform $\mathcal{M}(w,\mu)$:

Lemma 3. *For all* A, $B > 0$,

$$
\begin{aligned}
(2.5)\qquad & \mathcal{M}(w,\mu)(B) \\
&= \mathcal{M}(w,\mu)(A) + \int_0^\infty \lambda w(\lambda)\, (\lambda + A)^{-1}\, (B - A)\, (\lambda + A)^{-1}\, d\mu(\lambda) \\
&- 2\int_0^1 (1 - t)\left[\int_0^\infty \lambda w(\lambda)\, (\lambda + (1 - t)\, A + tB)^{-1}\, (B - A)\right. \\
&\left. \times (\lambda + (1 - t)\, A + tB)^{-1}\, (B - A)\, (\lambda + (1 - t)\, A + tB)^{-1}\, d\mu(\lambda)\right] dt.
\end{aligned}
$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [6, p. 112],

$$f(E) = f(C) + D(f)(C)(E - C) + \int_0^1 (1-t) D^2(f)((1-t)C + tE)(E - C, E - C)\, dt \tag{2.6}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and $C,\ E \in \mathcal{O}$.

If we write (2.6) for $\mathcal{M}(w,\mu)$ and $A,\ B > 0$, we get

$$\mathcal{M}(w,\mu)(B) = \mathcal{M}(w,\mu)(A) + D(\mathcal{M}(w,\mu))(A)(B-A) + \int_0^1 (1-t) D^2(\mathcal{M}(w,\mu))((1-t)A + tB)(B-A, B-A)\, dt$$

and by the representations (2.1) and (2.2) we obtain the desired result (2.5). □

We have the following Lipschitz type inequality:

Theorem 2. *Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\begin{aligned} &\|D(\mathcal{M}(w,\mu))(A)(B-A) + \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)(B)\| \\ &\leq \|B-A\|^2 \\ &\times \begin{cases} \frac{(m_2-m_1)\mathcal{M}'(w,\mu)(m_1) + \mathcal{M}(w,\mu)(m_1) - \mathcal{M}(w,\mu)(m_2)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \left(-\frac{1}{2}\mathcal{M}''(w,\mu)(m)\right) & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{2.7}$$

Proof. From (2.5) we get

$$\begin{aligned} &\|D(\mathcal{M}(w,\mu))(A)(B-A) + \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)(B)\| \\ &\leq 2\int_0^1 (1-t) \left[\int_0^\infty \lambda w(\lambda) \left\|(\lambda + (1-t)A + tB)^{-1}(B-A)\right.\right. \\ &\times \left.\left.(\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\right\| d\mu(\lambda)\right] dt \\ &\leq 2\|B-A\|^2 \\ &\times \int_0^1 (1-t) \left(\int_0^\infty \lambda w(\lambda) \left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 d\mu(\lambda)\right) dt. \end{aligned} \tag{2.8}$$

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\|((1-t)A + tB + \lambda)^{-1}\right\|^3 \leq ((1-t)m_1 + tm_2 + \lambda)^{-3} \tag{2.9}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (2.9) we derive

$$\begin{aligned}(2.10)\quad &\int_0^1 (1-t)\left(\int_0^\infty \lambda w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt\\ &\le \int_0^1 (1-t)\left(\int_0^\infty \lambda w(\lambda)((1-t)m_1+tm_2+\lambda)^{-3} d\mu(\lambda)\right)dt\\ &= \frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\left(\int_0^\infty \lambda w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}\right.\\ &\times (m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\ &\times \left.((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\,dt.\end{aligned}$$

From (2.5) we have for $m_2>m_1$ that

$$\begin{aligned}(2.11)\quad &\mathcal{M}(w,\mu)(m_1)-\mathcal{M}(w,\mu)(m_2)-(m_2-m_1)\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda)\\ &= 2\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right.\\ &\times ((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\ &\times \left.((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\Big]\,dt.\end{aligned}$$

Also

$$\int_0^\infty w(\lambda)(\lambda+m_1)^{-2}d\mu(\lambda) = -\mathcal{M}'(w,\mu)(m_1)$$

and then by (2.11) we get

$$\begin{aligned}(2.12)\quad &\frac{1}{2(m_2-m_1)^2}\\ &\times[(m_2-m_1)\mathcal{M}'(w,\mu)(m_1)+\mathcal{M}(w,\mu)(m_1)-\mathcal{M}(w,\mu)(m_2)]\\ &= \frac{1}{(m_2-m_1)^2}\\ &\times\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}\right.\\ &\times (m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\ &\times \left.((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\,dt.\end{aligned}$$

By utilising (2.8) and (2.10)-(2.12) we derive (2.7).

The case $m_2<m_1$ goes in a similar way and we also obtain (2.7).

Assume that $m_2=m_1>0$. Let $\epsilon>0$. Then $B+\epsilon\ge m+\epsilon>m$. By the first inequality for $m_2=m+\epsilon$ and $m_1=m$, we have

$$\begin{aligned}(2.13)\quad &\|D(\mathcal{M}(w,\mu))(A)(B+\epsilon-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B+\epsilon)\|\\ &\le \|B+\epsilon-A\|^2\frac{1}{\epsilon^2}[\mathcal{M}(w,\mu)(m)+\epsilon\mathcal{M}'(w,\mu)(m)-\mathcal{M}(w,\mu)(m+\epsilon)].\end{aligned}$$

By Taylor's expansion theorem with the Lagrange remainder we have

$$\mathcal{M}(w,\mu)(m)+\epsilon\mathcal{M}'(w,\mu)(m)-\mathcal{M}(w,\mu)(m+\epsilon) = -\frac{1}{2}\epsilon^2\mathcal{M}''(w,\mu)(\zeta_\epsilon)$$

with $m+\epsilon>\zeta_\epsilon>m$. Therefore

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon^2}\left[\mathcal{M}(w,\mu)(m)+\epsilon\mathcal{M}'(w,\mu)(m)-\mathcal{M}(w,\mu)(m+\epsilon)\right]=-\frac{1}{2}\mathcal{M}''(w,\mu)(m)$$

and by taking the limit $\epsilon\to 0+$ in (2.13) then we get

$$\begin{aligned}&\left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)-D(\mathcal{M}(w,\mu))(A)(B-A)\right\|\\&\le-\frac{1}{2}\|B-A\|^2\mathcal{M}''(w,\mu)(m)\end{aligned}$$

and the second part of (2.7) is proved. ☐

The case of operator monotone function is as follows:

Corollary 1. *Assume that $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$. If $A\ge m_1>0$, $B\ge m_2>0$, then*

$$\begin{aligned}&\|D(f)(A)(B-A)+f(A)-f(B)\|\\&\le\|B-A\|^2\\&\times\begin{cases}\frac{1}{(m_2-m_1)^2}\left[(m_2-m_1)f'(m_1)+f(m_1)-f(m_2)\right] & \text{if } m_1\ne m_2,\\[2ex] \left(-\frac{1}{2}f''(m)\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{2.14}$$

Proof. We have

$$\mathcal{M}(\ell,\mu)(t)=f(t)-a-bt,\ t>0,$$

where $a\in\mathbb{R}$, $b\ge 0$ and μ is a positive measure on $(0,\infty)$.

Therefore

$$\begin{aligned}&D(\mathcal{M}(w,\mu))(A)(B-A)+\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)(B)\\&=D(f-a-b\ell)(A)(B-A)+f(A)-a-bA-(f(B)-a-bB)\\&=D(f)(A)(B-A)-b(B-A)+f(A)-f(B)+bB-bA\\&=D(f)(A)(B-A)+f(A)-f(B),\end{aligned}$$

$$\begin{aligned}&(m_2-m_1)\mathcal{M}'(w,\mu)(m_1)+\mathcal{M}(w,\mu)(m_1)-\mathcal{M}(w,\mu)(m_2)\\&=(m_2-m_1)f'(m_1)+f(m_1)-f(m_2)\end{aligned}$$

and

$$\mathcal{M}''(\ell,\mu)(t)=f''(t),\ t>0.$$

By making use of (2.7) we deduce (2.14). ☐

We consider the representation obtained from (1.9) for the operator $T>0$ and the power $r\in(0,1]$,

$$T^r=\mathcal{M}(\tilde{w}_r)(T)$$

with the kernel $\tilde{w}_r(\lambda):=\frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$.

From (2.7) we get for $A\ge m_1>0$, $B\ge m_2>0$ and $r\in(0,1]$ that

$$\begin{aligned}&\left\|\int_0^\infty\lambda^{r-1}(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\lambda+A^r-B^r\right\|\\&\le\|B-A\|^2\begin{cases}\frac{1}{(m_2-m_1)^2}\left[(m_2-m_1)rm_1^{r-1}+m_1^r-m_2^r\right] & \text{if } m_1\ne m_2,\\[2ex] \frac{1}{2}r(1-r)m^{r-2} & \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{2.15}$$

We have the following error bounds for operator Jensen's gap related to the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{M}(w,\mu)):=\sum_{k=1}^{n}p_k\mathcal{M}(w,\mu)(A_k)-\mathcal{M}(w,\mu)\left(\sum_{k=1}^{n}p_kA_k\right).$$

Theorem 3. *Assume that $A_i\geq m>0$ for $i\in\{1,...,n\}$ and consider the probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$, then*

$$\begin{aligned}\|J(\mathbf{A},\mathbf{p},\mathcal{M}(w,\mu))\| &\leq -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\sum_{k=1}^{n}p_k\left\|A_k-\sum_{j=1}^{n}p_jA_j\right\|^2 \\ &\leq -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\sum_{k=1}^{n}\sum_{j=1}^{n}p_jp_k\|A_k-A_j\|^2 \\ &\leq -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\max_{k,j\in\{1,...,n\}}\|A_k-A_j\|^2.\end{aligned}\tag{2.16}$$

Proof. From (2.7) we get

$$\begin{aligned}&\left\|\mathcal{M}(w,\mu)(A_k)-\mathcal{M}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right.\\ &\left.-D(\mathcal{M}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\right\|\\ &\leq -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\left\|A_k-\sum_{j=1}^{n}p_jA_j\right\|^2\end{aligned}\tag{2.17}$$

for all $k\in\{1,...,n\}$.

If we multiply this inequality by $p_k\geq 0$ and sum over k from 1 to n, then we get

$$\begin{aligned}&\sum_{k=1}^{n}\left\|p_k\mathcal{M}(w,\mu)(A_k)-p_k\mathcal{M}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right.\\ &\left.-D(\mathcal{M}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(p_kA_k-\sum_{j=1}^{n}p_jA_j\right)\right\|\\ &\leq -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\sum_{k=1}^{n}p_k\left\|A_k-\sum_{j=1}^{n}p_jA_j\right\|^2.\end{aligned}\tag{2.18}$$

By making use of the triangle inequality for norms, we also have

$$\begin{aligned}(2.19)\quad & \sum_{k=1}^{n}\left\| p_k\mathcal{M}(w,\mu)(A_k) - p_k\mathcal{M}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right.\\ & \left. -D(\mathcal{M}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(p_kA_k-\sum_{j=1}^{n}p_jA_j\right)\right\|\\ & \geq \left\|\sum_{k=1}^{n}p_k\mathcal{M}(w,\mu)(A_k) - \sum_{k=1}^{n}p_k\mathcal{M}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right.\\ & \left. -D(\mathcal{M}(w,\mu))\left(\sum_{j=1}^{n}p_jA_j\right)\left(\sum_{k=1}^{n}p_kA_k-\sum_{j=1}^{n}p_jA_j\right)\right\|\\ & = \left\|\sum_{k=1}^{n}p_k\mathcal{M}(w,\mu)(A_k) - \mathcal{M}(w,\mu)\left(\sum_{j=1}^{n}p_jA_j\right)\right\|.\end{aligned}$$

By utilising (2.18) and (2.19) we deduce the first part of (2.16). The rest is obvious. □

Corollary 2. *Assume that $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$, then for $A_i \geq m > 0$ for $i\in\{1,...,n\}$ and the probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,*

$$\begin{aligned}(2.20)\quad \left\|\sum_{k=1}^{n}p_kf(A_k)-f\left(\sum_{k=1}^{n}p_kA_k\right)\right\| & \leq -\frac{1}{2}f''(m)\sum_{k=1}^{n}p_k\left\|A_k-\sum_{j=1}^{n}p_jA_j\right\|^2\\ & \leq -\frac{1}{2}f''(m)\sum_{k=1}^{n}\sum_{j=1}^{n}p_jp_k\|A_k-A_j\|^2\\ & \leq -\frac{1}{2}f''(m)\max_{k,j\in\{1,\ldots,n\}}\|A_k-A_j\|^2.\end{aligned}$$

Remark 1. *Assume that $A_i\geq m>0$ for $i\in\{1,...,n\}$ and consider the probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$, then we have the power inequalities for $r\in(0,1]$*

$$\begin{aligned}(2.21)\quad & \left\|\sum_{k=1}^{n}p_kA_k^r-\left(\sum_{k=1}^{n}p_kA_k\right)^r\right\|\\ & \leq \frac{1}{2}r(1-r)m^{r-2}\sum_{k=1}^{n}p_k\left\|A_k-\sum_{j=1}^{n}p_jA_j\right\|^2\\ & \leq \frac{1}{2}r(1-r)m^{r-2}\sum_{k=1}^{n}\sum_{j=1}^{n}p_jp_k\|A_k-A_j\|^2\\ & \leq \frac{1}{2}r(1-r)m^{r-2}\max_{k,j\in\{1,\ldots,n\}}\|A_k-A_j\|^2\end{aligned}$$

and the logarithmic inequalities

$$\left\|\sum_{k=1}^{n} p_k \ln A_k - \ln\left(\sum_{k=1}^{n} p_k A_k\right)\right\| \le \frac{1}{2m}\sum_{k=1}^{n} p_k \left\|A_k - \sum_{j=1}^{n} p_j A_j\right\|^2 \tag{2.22}$$
$$\le \frac{1}{2m}\sum_{k=1}^{n}\sum_{j=1}^{n} p_j p_k \left\|A_k - A_j\right\|^2$$
$$\le \frac{1}{2m}\max_{k,j\in\{1,\dots,n\}} \left\|A_k - A_j\right\|^2 .$$

3. Midpoint and Trapezoid Inequalities

We have the following midpoint norm inequality:

Theorem 4. *If $A,\ B \ge m > 0$ for some constant m, then*

$$\left\|\int_0^1 \mathcal{M}(w,\mu)\left((1-t)A + tB\right)dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \tag{3.1}$$
$$\le -\frac{1}{24}\mathcal{M}''(w,\mu)(m)\left\|B-A\right\|^2 .$$

Proof. From (2.7) we have for all $t \in [0,1]$ and $A,\ B \ge m > 0$,

$$\Bigg\|\mathcal{M}(w,\mu)\left((1-t)A + tB\right) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \tag{3.2}$$
$$-D\left(\mathcal{M}(w,\mu)\right)\left(\frac{A+B}{2}\right)\left((1-t)A + tB - \frac{A+B}{2}\right)\Bigg\|$$
$$\le -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\left\|(1-t)A + tB - \frac{A+B}{2}\right\|^2$$
$$= -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\left(t-\frac{1}{2}\right)^2\left\|B-A\right\|^2 .$$

If we integrate this inequality, we get

$$\int_0^1 \Bigg\|\mathcal{M}(w,\mu)\left((1-t)A + tB\right) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \tag{3.3}$$
$$-D\left(\mathcal{M}(w,\mu)\right)\left(\frac{A+B}{2}\right)\left((1-t)A + tB - \frac{A+B}{2}\right)\Bigg\| dt$$
$$\le -\frac{1}{2}\mathcal{M}''(w,\mu)(m)\left\|B-A\right\|^2\int_0^1\left(t-\frac{1}{2}\right)^2 dt$$
$$= -\frac{1}{24}\mathcal{M}''(w,\mu)(m)\left\|B-A\right\|^2 .$$

Using the properties of norm and integral, we also have

$$\begin{aligned}(3.4)\quad & \int_0^1 \left\| \mathcal{M}(w,\mu)((1-t)A+tB) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \right. \\ & \left. -D(\mathcal{M}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right) \right\| dt \\ & \geq \left\| \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \right. \\ & \left. -\left(\int_0^1 \left(t-\frac{1}{2}\right)dt\right) D(\mathcal{M}(w,\mu))\left(\frac{A+B}{2}\right)(B-A) \right\| \\ & = \left\| \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right) \right\|.\end{aligned}$$

By employing (3.3) and (3.4) we derive the desired result (3.1). □

Corollary 3. *Assume that $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$. If $A,\ B\geq m>0$ for some constant m, then*

$$(3.5)\qquad \left\| \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \right\| \leq -\frac{1}{24} f''(m)\|B-A\|^2 .$$

Remark 2. *If $A,\ B\geq m>0$ for some constant m, then for $r\in(0,1]$ we have the power inequality*

$$(3.6)\qquad \left\| \int_0^1 ((1-t)A+tB)^r\,dt - \left(\frac{A+B}{2}\right)^r \right\| \leq \frac{1}{24} r(1-r)m^{r-2}\|B-A\|^2 .$$

We also have the logarithmic inequality

$$(3.7)\qquad \left\| \int_0^1 \ln((1-t)A+tB)\,dt - \ln\left(\frac{A+B}{2}\right) \right\| \leq \frac{1}{24m}\|B-A\|^2 .$$

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B}:[0,1]\to\mathbb{R}$ defined by

$$f_{A,B}(t) := f((1-t)A+tB),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 4. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on $[0,1]$,*

$$(3.8)\qquad \frac{df_{A,B}(t)}{dt} = D(f)((1-t)A+tB)(B-A)$$

and

$$(3.9)\qquad \frac{d^2 f_{A,B}(t)}{dt^2} = D^2(f)((1-t)A+tB)(B-A,B-A)$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{f_{A,B}(t+h) - f(t)}{h} \\ & = \frac{f((1-(t+h))A + (t+h)B) - f((1-t)A + tB)}{h} \\ & = \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} \frac{df_{A,B}(t)}{dt} & = \lim_{h\to 0} \frac{f_{A,B}(t+h) - f(t)}{h} \\ & = \lim_{h\to 0} \left[\frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \right] \\ & = D(f)((1-t)A + tB)(B-A), \end{aligned}$$

which proves (3.8).

Similarly,

$$\begin{aligned} & \frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt} \right] \\ & = \frac{D(f)((1-(t+h))A + (t+h)B)(B-A) - D(f)((1-t)A + tB)(B-A)}{h} \\ & = \frac{D(f)((1-t)A + tB + h(B-A))(B-A) - D(f)((1-t)A + tB)(B-A)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} \frac{d^2 f_{A,B}(t)}{dt^2} & = \lim_{h\to 0} \left\{ \frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt} \right] \right\} \\ & = D^2(f)((1-t)A + tB)(B-A, B-A), \end{aligned}$$

which proves (3.9). □

For the transform $\mathcal{M}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\begin{aligned} \mathcal{M}(w,\mu)_{A,B}(t) & := \mathcal{M}(w,\mu)((1-t)A + tB) \\ & = \int_0^\infty w(\lambda)(\lambda + (1-t)A + tB)^{-1} d\mu(\lambda), \end{aligned}$$

where A, $B > 0$ and $t \in [0,1]$.

Corollary 4. *For all A, $B > 0$ and $t \in [0,1]$,*

$$\begin{aligned} \frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt} & = D(\mathcal{M}(w,\mu))((1-t)A + tB)(B-A) \\ & = \int_0^\infty \lambda w(\lambda)(\lambda + (1-t)A + tB)^{-1}(B-A) \\ & \times (\lambda + (1-t)A + tB)^{-1} d\mu(\lambda) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(3.11)\quad &\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}\\ &= D^2(\mathcal{M}(w,\mu))((1-t)A+tB)(B-A,B-A)\\ &= -2\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\ &\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}$$

We observe that if $f(t)=\mathcal{M}(w,\mu)(t)$, $t>0$, in Lemma 3, then by the representations from Lemma 1 and Lemma 2 we obtain the desired equalities (3.10) and (3.11).

We have the following identity for the trapezoid rule:

Lemma 5. *For all* A, $B>0$ *we have the identity*

$$\begin{aligned}(3.12)\quad &\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt-\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}\\ &=\int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt\\ &=\frac{1}{2}\left[t(1-t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}\Big|_0^1-\int_0^1(1-2t)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\right]\\ &=\int_0^1\left(t-\frac{1}{2}\right)\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}dt\\ &=\left(t-\frac{1}{2}\right)\mathcal{M}(w,\mu)_{A,B}(t)\Big|_0^1-\int_0^1\mathcal{M}(w,\mu)_{A,B}(t)\,dt\\ &=\frac{1}{2}\left[\mathcal{M}(w,\mu)_{A,B}(1)+\mathcal{M}(w,\mu)_{A,B}(0)\right]-\int_0^1\mathcal{M}(w,\mu)_{A,B}(t)\,dt,\end{aligned}$$

that gives the identity

$$\begin{aligned}(3.13)\quad &\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}-\int_0^1\mathcal{M}(w,\mu)((1-t)A+tB)\,dt\\ &=\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt.\end{aligned}$$

By (3.13) we have

$$\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{M}(w,\mu)_{A,B}(t)}{dt^2}dt \tag{3.14}$$
$$= -\int_0^1 t(1-t)\left[\int_0^\infty \lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.$$
$$\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.$$

By making use of (3.13) and (3.14) we deduce (3.12). □

We can state now the corresponding trapezoid norm inequality:

Theorem 5. *If A, $B \geq m > 0$ for some constant m, then*

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}-\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt\right\| \tag{3.15}$$
$$\leq -\frac{1}{12}\mathcal{M}''(w,\mu)(m)\|B-A\|^2.$$

Proof. By taking the norm in (3.12), we obtain

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}-\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt\right\| \tag{3.16}$$
$$\leq \|B-A\|^2$$
$$\times\int_0^1 t(1-t)\left(\int_0^\infty \lambda w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt.$$

Since A, $B \geq m > 0$, then for $\lambda \geq 0$ and $t \in [0,1]$,

$$\lambda+(1-t)A+tB \geq \lambda+m,$$

which implies that

$$(\lambda+(1-t)A+tB)^{-1} \leq (\lambda+m)^{-1}.$$

This implies that

$$\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 \leq (\lambda+m)^{-3}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

By multiplying this inequality by $t(1-t)\lambda w(\lambda) \geq 0$ and integrating we get

$$\int_0^1 t(1-t)\left(\int_0^\infty \lambda w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt \tag{3.17}$$
$$\leq\left(\int_0^1 t(1-t)\,dt\right)\left(\int_0^\infty \lambda w(\lambda)(\lambda+m)^{-3}d\mu(\lambda)\right)$$
$$=\frac{1}{6}\int_0^\infty \lambda w(\lambda)(\lambda+m)^{-3}d\mu(\lambda).$$

Taking the derivative over t twice in (1.15), we get

$$\mathcal{M}''(w,\mu)(t) := -2\int_0^\infty \frac{\lambda w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0,$$

that gives

$$\int_0^\infty w(\lambda)(\lambda+m)^{-3} d\mu(\lambda) = -\frac{1}{2}\mathcal{M}''(w,\mu)(m)$$

and by (3.16) and (3.17) we derive (3.15). □

Corollary 5. *Assume that $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$. If A, $B\geq m>0$, then*

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \leq -\frac{1}{12}f''(m)\|B-A\|^2. \tag{3.18}$$

The proof follows by (3.15) for

$$\mathcal{M}(\ell,\mu)(t) = \frac{f(t)}{t} - b,\ t>0,$$

where $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Remark 3. *If A, $B\geq m>0$ for some constant m, then for $r\in(0,1]$ we have the power inequality*

$$\left\|\frac{A^r+B^r}{2}-\int_0^1 ((1-t)A+tB)^r\,dt\right\| \leq \frac{1}{12}r(1-r)m^{r-2}\|B-A\|^2. \tag{3.19}$$

We also have the logarithmic inequality

$$\left\|\frac{\ln A+\ln B}{2}-\int_0^1 \ln((1-t)A+tB)\,dt\right\| \leq \frac{1}{12m}\|B-A\|^2. \tag{3.20}$$

References

[1] H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, *Commun. Math. Phys.* **81** (1981), 89-96.

[2] R. Bhatia, First and second order perturbation bounds for the operator absolute value, *Linear Algebra Appl.* **208/209** (1994), 367-376.

[3] R. Bhatia, Perturbation bounds for the operator absolute value. *Linear Algebra Appl.* **226/228** (1995), 639–645.

[4] R. Bhatia, D. Singh and K. B. Sinha, Differentiation of operator functions and perturbation bounds. *Comm. Math. Phys.* **191** (1998), no. 3, 603–611.

[5] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[6] R. Coleman, *Calculus on Normed Vector Spaces*, Springer, New York, 2012.

[7] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), to appear.

[8] Yu. B. Farforovskaya, Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric (in Russian), *Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom.* **4** (1967), 155-156.

[9] Yu. B. Farforovskaya, An estimate of the norm $\|f(B)-f(A)\|$ for self-adjoint operators A and B (in Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst.* **56** (1976), 143-162 .

[10] Yu. B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions. *Algebra i Analiz* **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.

[11] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[12] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[13] T. Furuta, Precise lower bound of $f(A)-f(B)$ for $A>B>0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[14] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[15] T. Kato, Continuity of the map $S \to |S|$ for linear operators, *Proc. Japan Acad.* **49** (1973), 143-162.

[16] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

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