LOWER AND UPPER BOUNDS IN TERMS OF SECOND DERIVATIVE FOR THE CONVEX INTEGRAL TRANSFORM WITH APPLICATIONS TO JENSEN'S GAP

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B - A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta \mathcal{C}''(w,\mu)(\beta) \\ &\leq \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) - D(\mathcal{C}(w,\mu))(A)(B - A) \\ &\leq \frac{1}{2}\Delta \mathcal{C}''(w,\mu)(\alpha), \end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as an operator functions and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function.

If $\beta \geq A_k \geq \alpha > 0$, $k \in \{1, ..., n\}$ and $0 < \delta \leq \left(A_k - \sum_{j=1}^n p_j A_j\right)^2 \leq \Delta$, $k \in \{1, ..., n\}$ for some constants α, β, δ, Δ, and $p \in [-1, 0] \cup [1, 2]$, then we have the power inequalities

$$0 \leq \frac{1}{2}\delta p(p-1)\beta^{p-2} \leq \sum_{k=1}^n p_k A_k^p - \left(\sum_{k=1}^n p_k A_k\right)^p \leq \frac{1}{2}\Delta p(p-1)\alpha^{p-2}$$

and the logarithmic inequalities

$$0 \leq \frac{\delta}{2\beta} \leq \sum_{k=1}^n p_k A_k \ln A_k - \sum_{k=1}^n p_k A_k \ln\left(\sum_{k=1}^n p_k A_k\right) \leq \frac{\Delta}{2\alpha}$$

and

$$0 < \frac{\delta}{2\beta^2} \leq \ln\left(\sum_{k=1}^n p_k A_k\right) - \sum_{k=1}^n p_k \ln(A_k) \leq \frac{\Delta}{2\alpha^2}.$$

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

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We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in\mathbb{R}$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

For some example of operator monotone functions see [3]-[5], [8], [9] and the references therein.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.11).

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)(t):=t^2\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{C}(w,\mu)(t)&:=\int_0^\infty w(\lambda)t^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)^2-2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)-2\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.13}$$

If $\int_0^\infty w(\lambda)d\mu(\lambda)<\infty$ and $\int_0^\infty w(\lambda)\lambda d\mu(\lambda)<\infty$, then we conclude that

$$\mathcal{C}(w,\mu)(t)=-\int_0^\infty w(\lambda)\lambda d\mu(\lambda)+t\int_0^\infty w(\lambda)d\mu(\lambda)+\mathcal{D}(\ell^2w,\mu)(t), \tag{1.14}$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq 0$$

where the *exponential integral* is given by

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty w(\lambda)\lambda d\lambda=\int_0^\infty \exp(-a\lambda)\lambda d\lambda=\frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda)d\lambda=\int_0^\infty \exp(-a\lambda)d\lambda=\frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t):=t^2\mathcal{D}(e_{-a},\mu)(t)=t^2E_1(at)\exp(at),\ t>0. \tag{1.15}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t)=\int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda}d\lambda$$

then by (1.14) we get

$$t^2E_1(at)\exp(at)=-\frac{1}{a^2}+\frac{t}{a}+\mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t)=t^2E_1(at)\exp(at)-\frac{t}{a}+\frac{1}{a^2},\ t>0,a>0.$$

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)d\lambda=\infty$ and the equality (1.14) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T):=\int_0^\infty w(\lambda)T^2(\lambda+T)^{-1}d\mu(\lambda), \tag{1.16}$$

provided the integral exist. By (1.13) we also have

$$\mathcal{C}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[T-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.17}$$

In this paper, we show among others that, if $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned}0&\leq\frac{1}{2}\delta\mathcal{C}''(w,\mu)(\beta)\\&\leq\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\\&\leq\frac{1}{2}\Delta\mathcal{C}''(w,\mu)(\alpha),\end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as an operator functions and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function.

If $\beta \geq A_k \geq \alpha > 0,\ k \in \{1,...,n\},\ 0 < \delta \leq \left(A_k - \sum_{j=1}^{n} p_j A_j\right)^2 \leq \Delta,\ k \in \{1,...,n\}$ for some constants $\alpha,\ \beta,\ \delta,\ \Delta$ and $p \in [-1,0] \cup [1,2]$, then we have the power inequalities

$$0 \leq \frac{1}{2}\delta p (p-1) \beta^{p-2} \leq \sum_{k=1}^{n} p_k A_k^p - \left(\sum_{k=1}^{n} p_k A_k\right)^p \leq \frac{1}{2}\Delta p (p-1) \alpha^{p-2}$$

and the logarithmic inequalities

$$0 \leq \frac{\delta}{2\beta} \leq \sum_{k=1}^{n} p_k A_k \ln A_k - \sum_{k=1}^{n} p_k A_k \ln \left(\sum_{k=1}^{n} p_k A_k\right) \leq \frac{\Delta}{2\alpha}$$

and

$$0 < \frac{\delta}{2\beta^2} \leq \ln \left(\sum_{k=1}^{n} p_k A_k\right) - \sum_{k=1}^{n} p_k \ln (A_k) \leq \frac{\Delta}{2\alpha^2}.$$

2. Representation Results

We have the following representation of the Fréchet derivative $D(\mathcal{C}(w,\mu))$:

Lemma 1. *For all $A > 0$,*

$$\begin{aligned} (2.1) \qquad & D(\mathcal{C}(w,\mu))(A)(V) \\ &= \int_0^\infty w(\lambda)(A+\lambda)^{-1}[AVA + \lambda(VA+AV)](A+\lambda)^{-1} d\mu(\lambda) \end{aligned}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. Let $A > 0$ and $V \in S(H)$. By the definition of $\mathcal{C}(w,\mu)$ and by (1.13) and we have for t in a small open interval around 0 that

$$\mathcal{C}(w,\mu)(A+tV) = \int_0^\infty w(\lambda)\left[A + tV - \lambda + \lambda^2 (A + tV + \lambda)^{-1}\right] d\mu(\lambda).$$

Then

$$\begin{aligned} & \mathcal{C}(w,\mu)(A+tV) - \mathcal{C}(w,\mu)(A) \\ &= \int_0^\infty w(\lambda)\left[A + tV - \lambda + \lambda^2 (A + tV + \lambda)^{-1}\right] d\mu(\lambda) \\ &- \int_0^\infty w(\lambda)\left[A - \lambda + \lambda^2 (A+\lambda)^{-1}\right] d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left\{tV + \lambda^2\left[(A+tV+\lambda)^{-1} - (A+\lambda)^{-1}\right]\right\} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left\{tV + \lambda^2\left[(A+tV+\lambda)^{-1}(-tV)(A+\lambda)^{-1}\right]\right\} d\mu(\lambda) \\ &= t\int_0^\infty w(\lambda)\left\{V - \lambda^2\left[(A+tV+\lambda)^{-1} V (A+\lambda)^{-1}\right]\right\} d\mu(\lambda). \end{aligned}$$

Therefore,

$$\begin{aligned}(2.2)\qquad & D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V\right)\\ & =\lim_{t\rightarrow 0}\frac{\mathcal{C}(w,\mu)\left(A+tV\right)-\mathcal{C}(w,\mu)\left(A\right)}{t}\\ & =\lim_{t\rightarrow 0}\int_{0}^{\infty}w\left(\lambda\right)\left\{V-\lambda^{2}\left[\left(A+tV+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}\right]\right\}d\mu\left(\lambda\right)\\ & =\int_{0}^{\infty}w\left(\lambda\right)\left\{V-\lambda^{2}\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right)\end{aligned}$$

for $A>0$ and $V\in S\left(H\right)$.

Define for $\lambda\geq 0$,

$$U_{\lambda}:=V-\lambda^{2}\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}.$$

If we multiply U_{λ} both sides by $A+\lambda$, then we get

$$\begin{aligned}\left(A+\lambda\right)U_{\lambda}\left(A+\lambda\right) & =\left(A+\lambda\right)V\left(A+\lambda\right)-\lambda^{2}V\\ & =\left(AV+\lambda V\right)\left(A+\lambda\right)-\lambda^{2}V\\ & =AVA+\lambda VA+\lambda AV+\lambda^{2}V-\lambda^{2}V\\ & =AVA+\lambda\left(VA+AV\right).\end{aligned}$$

If we multiply both sides by $\left(A+\lambda\right)^{-1}$ we get

$$U_{\lambda}=\left(A+\lambda\right)^{-1}\left[AVA+\lambda\left(VA+AV\right)\right]\left(A+\lambda\right)^{-1},$$

which, by (2.2), implies the representation (2.1). □

For the case of second Fréchet derivative $D^{2}\left(\mathcal{C}\left(w,\mu\right)\right)$, we have the representation:

Lemma 2. *For all* $A>0$*,*

$$\begin{aligned}(2.3)\qquad & D^{2}\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V,V\right)\\ & =2\int_{0}^{\infty}\lambda^{2}w\left(\lambda\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}\end{aligned}$$

for all $V\in S\left(H\right)$*.*

Proof. Let $A>0$ and $V\in S\left(H\right)$. We have by the definition of the Fréchet second derivative that

$$\begin{aligned}(2.4)\qquad & D^{2}\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V,V\right)\\ & =\lim_{t\rightarrow 0}\frac{D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V\right)}{t}.\end{aligned}$$

Observe, by (2.2), that we have for t in a small open interval around 0,

$$\begin{aligned}& D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)\\ & =\int_{0}^{\infty}w\left(\lambda\right)\left\{V-\lambda^{2}\left(A+tV+\lambda\right)^{-1}V\left(A+tV+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right).\end{aligned}$$

Therefore

$$
\begin{aligned}
(2.5)\qquad & D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V\right)\\
&=\int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left(A+tV+\lambda\right)^{-1}V\left(A+tV+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right)\\
&-\int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right)\\
&=\int_0^\infty \lambda^2 w\left(\lambda\right)\\
&\times\left[\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}-\left(A+tV+\lambda\right)^{-1}V\left(A+tV+\lambda\right)^{-1}\right].
\end{aligned}
$$

Define

$$
W_{t,\lambda}:=\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}-\left(A+tV+\lambda\right)^{-1}V\left(A+tV+\lambda\right)^{-1}.
$$

If we multiply both sides of $W_{t,\lambda}$ with $\lambda+A+tV$, the we get

$$
\begin{aligned}
&\left(\lambda+A+tV\right)W_{t,\lambda}\left(\lambda+A+tV\right)\\
&=\left(\lambda+A+tV\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}\left(\lambda+A+tV\right)-V\\
&=\left(1+tV\left(\lambda+A\right)^{-1}\right)V\left(1+t\left(\lambda+A\right)^{-1}V\right)-V\\
&=\left(V+tV\left(\lambda+A\right)^{-1}V\right)\left(1+t\left(\lambda+A\right)^{-1}V\right)-V\\
&=V+tV\left(\lambda+A\right)^{-1}V+tV\left(\lambda+A\right)^{-1}V\\
&+t^2V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V-V\\
&=2tV\left(\lambda+A\right)^{-1}V+t^2V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\\
&=t\left[2V\left(\lambda+A\right)^{-1}V+tV\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\right].
\end{aligned}
$$

If we multiply the equality by $\left(\lambda+A+tV\right)^{-1}$ both sides, we get for $t\neq 0$

$$
\begin{aligned}
(2.6)\qquad \frac{W_{t,\lambda}}{t}&=\left(\lambda+A+tV\right)^{-1}\left[2V\left(\lambda+A\right)^{-1}V+tV\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\right]\\
&\times\left(\lambda+A+tV\right)^{-1}.
\end{aligned}
$$

If we take the limit over $t\to 0$ in (2.6), then we get

$$
(2.7)\qquad \lim_{t\to 0}\left(\frac{W_{t,\lambda}}{t}\right)=2\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}.
$$

Then by (2.4), (2.5) and (2.7) we derive (2.3). □

We have the following representation for the transform $\mathcal{C}\left(w,\mu\right)$:

Theorem 3. *For all A, $B>0$,*

$$
\begin{aligned}
(2.8)\qquad & \mathcal{C}\left(w,\mu\right)\left(B\right)-\mathcal{C}\left(w,\mu\right)\left(A\right)-D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(B-A\right)\\
&=2\int_0^1\left(1-t\right)\left[\int_0^\infty \lambda^2 w\left(\lambda\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\right.\\
&\left.\times\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}d\mu\left(\lambda\right)\right]dt\\
&\geq 0.
\end{aligned}
$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E)=f(C)+D(f)(C)(E-C)+\int_0^1(1-t)D^2(f)((1-t)C+tE)(E-C,E-C)\,dt \tag{2.9}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and $C,\ E\in\mathcal{O}$.

If we write (2.9) for $\mathcal{C}(w,\mu)$ and $A,\ B>0$, we get

$$\mathcal{C}(w,\mu)(B)=\mathcal{C}(w,\mu)(A)+D(\mathcal{C}(w,\mu))(A)(B-A)+\int_0^1(1-t)D^2(\mathcal{C}(w,\mu))((1-t)A+tB)(B-A,B-A)\,dt. \tag{2.10}$$

By making use of (2.3) we obtain

$$\begin{aligned}&D^2(\mathcal{C}(w,\mu))((1-t)A+tB)(B-A,B-A)\\&=\left[\int_0^\infty\lambda^2w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt,\end{aligned}$$

which, by (2.10), produces the equality in (2.8).

Now, observe that

$$(\lambda+(1-t)A+tB)^{-1}\geq0$$

for all $\lambda\geq0$ and $t\in[0,1]$.

By multiplying this inequality both sides by $(B-A)$, we derive

$$(B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\geq0$$

for all $\lambda\geq0$ and $t\in[0,1]$.

Moreover, if we multiply this inequality both sides by $(\lambda+(1-t)A+tB)^{-1}$ we obtain

$$\begin{aligned}&(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\&(B-A)(\lambda+(1-t)A+tB)^{-1}\\&\geq0\end{aligned}$$

for all $\lambda\geq0$ and $t\in[0,1]$.

Finally, by multiplying with $(1-t)\lambda^2w(\lambda)\geq0$ and integrating we deduce the inequality part in (2.8). □

Corollary 1. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.9). Then for all* $A,\ B>0$ *we have*

$$\begin{aligned}&Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A)-b(B-A)^2\\&=2\int_0^1(1-t)\left[\int_0^\infty\lambda^3(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt\\&\geq0.\end{aligned} \tag{2.11}$$

Proof. From (1.9) we derive

$$tf(t) = at + bt^2 + t^2 \int_0^\infty \frac{\lambda}{t+\lambda} d\mu(\lambda) = at + bt^2 + \mathcal{C}(\ell,\mu)(t),$$

where $a \in \mathbb{R}$, $b \geq 0$, $\ell(\lambda) = \lambda$ and μ is a positive measure on $(0,\infty)$.

This gives that

$$\mathcal{C}(\ell,\mu)(t) = tf(t) - at - bt^2.$$

Observe that for A, $B > 0$

$$\begin{aligned} &D(\mathcal{C}(w,\mu))(A)(B-A) \\ &= D\left(\ell f - a\ell - b\ell^2\right)(A)(B-A) \\ &= D(\ell f)(A)(B-A) - aD(\ell)(A)(B-A) - bD\left(\ell^2\right)(A)(B-A). \end{aligned}$$

Since

$$\begin{aligned} D(\ell f)(A)(B-A) &= \ell(A) D(f)(A)(B-A) + f(A) D(\ell)(A)(B-A) \\ &= AD(f)(A)(B-A) + f(A)(B-A), \end{aligned}$$

and

$$D\left(\ell^2\right)(A)(B-A) = A(B-A) + (B-A)A = AB + BA - 2A^2,$$

hence

$$\begin{aligned} D(\ell f)(A)(B-A) = {} & AD(f)(A)(B-A) + f(A)(B-A) \\ & - a(B-A) - b\left(AB + BA - 2A^2\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) - D(\mathcal{C}(w,\mu))(A)(B-A) \\ &= Bf(B) - aB - bB^2 - Af(A) + aA + bA^2 \\ &- AD(f)(A)(B-A) - f(A)(B-A) + a(B-A) \\ &+ b\left(AB + BA - 2A^2\right) \\ &= Bf(B) - Af(A) - AD(f)(A)(B-A) - f(A)(B-A) \\ &+ b\left(AB + BA - A^2 - B^2\right) \\ &= Bf(B) - Af(A) - AD(f)(A)(B-A) - f(A)(B-A) - b(B-A)^2 \end{aligned}$$

and by (2.8) we derive (2.11). □

Corollary 2. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.11). Then for all A, $B > 0$,*

$$\begin{aligned} &f(B) - f(A) - D(f)(A)(B-A) - c(B-A)^2 \\ &= 2\int_0^1 (1-t)\left[\int_0^\infty \lambda^3 (\lambda + (1-t)A + tB)^{-1}(B-A)\right. \\ &\left.\times (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1} d\mu(\lambda)\right] dt \\ &\geq 0. \end{aligned} \tag{2.12}$$

Proof. From (1.11) we have

$$\mathcal{C}(\ell,\mu)(t)=f(t)-a-bt-ct^2,$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and μ a positive measure on $(0,\infty)$.

Therefore

$$\begin{aligned}&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\\&=f(B)-bB-cB^2-f(A)+bA+cA^2\\&-D(f)(A)(B-A)+bD(\ell)(A)(B-A)+cD\left(\ell^2\right)(A)(B-A)\\&=f(B)-bB-cB^2-f(A)+bA+cA^2\\&-D(f)(A)(B-A)+b(B-A)+c\left(AB+BA-2A^2\right)\\&=f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2\end{aligned}$$

and by (2.8) we derive the desired result (2.12). □

We have the following error bounds for *operator Jensen's gap* related to the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{C}(w,\mu)):=\sum_{k=1}^{n}p_k\mathcal{C}(w,\mu)(A_k)-\mathcal{C}(w,\mu)\left(\sum_{k=1}^{n}p_kA_k\right).$$

We have the following representation result for the Jensen's gap:

Theorem 4. *For the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,*

$$\begin{aligned}&J(\mathbf{A},\mathbf{p},\mathcal{C}(w,\mu)) \qquad (2.13)\\&=2\int_0^1(1-t)\sum_{k=1}^{n}p_k\left[\int_0^\infty\lambda^2w(\lambda)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}\right.\\&\times\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}\left(A_k-\sum_{j=1}^{n}p_jA_j\right)\\&\times\left.\left(\lambda+(1-t)\sum_{j=1}^{n}p_jA_j+tA_k\right)^{-1}d\mu(\lambda)\right]dt\\&\geq 0.\end{aligned}$$

In particular, this shows that $\mathcal{C}(w,\mu)$ is operator convex on $(0,\infty)$.

Proof. From (2.8) we have

$$\begin{aligned}
(2.14)\quad & \mathcal{C}(w,\mu)(A_k) - \mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \\
& - D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(A_k - \sum_{j=1}^{n} p_j A_j\right) \\
& = 2\int_0^1 (1-t)\left[\int_0^\infty \lambda^2 w(\lambda)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\right. \\
& \times \left(A_k - \sum_{j=1}^{n} p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\left(A_k - \sum_{j=1}^{n} p_j A_j\right) \\
& \times \left.\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1} d\mu(\lambda)\right] dt \\
& \geq 0,
\end{aligned}$$

for all $k \in \{1, ..., n\}$.

If we multiply (2.14) by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\begin{aligned}
(2.15)\quad & \sum_{k=1}^{n} p_k \mathcal{C}(w,\mu)(A_k) - \left(\sum_{k=1}^{n} p_k\right)\mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \\
& - \sum_{k=1}^{n} p_k D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(A_k - \sum_{j=1}^{n} p_j A_j\right) \\
& = 2\int_0^1 (1-t)\sum_{k=1}^{n} p_k\left[\int_0^\infty \lambda^2 w(\lambda)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\right. \\
& \times \left(A_k - \sum_{j=1}^{n} p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\left(A_k - \sum_{j=1}^{n} p_j A_j\right) \\
& \times \left.\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1} d\mu(\lambda)\right] dt \\
& \geq 0,
\end{aligned}$$

and since

$$\sum_{k=1}^{n} p_k D\left(\mathcal{C}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n} p_j A_j\right)\left(A_k - \sum_{j=1}^{n} p_j A_j\right)$$
$$= D\left(\mathcal{C}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n} p_j A_j\right)\left(\sum_{k=1}^{n} p_k A_k - \sum_{k=1}^{n} p_k \sum_{j=1}^{n} p_j A_j\right)$$
$$= D\left(\mathcal{C}\left(w,\mu\right)\right)\left(\sum_{j=1}^{n} p_j A_j\right)(0) = 0,$$

hence by (2.15) we obtain (2.13). □

Corollary 3. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). For the n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ and probability density n-tuple $\mathbf{p} = (p_1, ..., p_n)$,*

$$\begin{aligned}(2.16)\quad & \sum_{k=1}^{n} p_k A_k f\left(A_k\right) - f\left(\sum_{k=1}^{n} p_k A_k\right)\sum_{k=1}^{n} p_k A_k \\ & - b\left[\sum_{k=1}^{n} p_k A_k^2 - \left(\sum_{k=1}^{n} p_k A_k\right)^2\right] \\ & = 2\int_0^1 (1-t)\sum_{k=1}^{n} p_k\left[\int_0^{\infty} \lambda^3\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\right. \\ & \times\left(A_k - \sum_{j=1}^{n} p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\left(A_k - \sum_{j=1}^{n} p_j A_j\right) \\ & \times\left.\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1} d\mu\left(\lambda\right)\right] dt \\ & \geq 0.\end{aligned}$$

The proof follows by (2.13) observing that

$$\sum_{k=1}^{n} p_k\left(A_k - \sum_{j=1}^{n} p_j A_j\right)^2$$
$$= \sum_{k=1}^{n} p_k\left(A_k^2 - A_k\left(\sum_{j=1}^{n} p_j A_j\right) - \left(\sum_{j=1}^{n} p_j A_j\right)A_k + \left(\sum_{j=1}^{n} p_j A_j\right)^2\right)$$

$$= \sum_{k=1}^{n} p_k A_k^2 - \left(\sum_{k=1}^{n} p_k A_k\right)\left(\sum_{j=1}^{n} p_j A_j\right)$$
$$- \left(\sum_{j=1}^{n} p_j A_j\right)\left(\sum_{k=1}^{n} p_k A_k\right) + \left(\sum_{j=1}^{n} p_j A_j\right)^2$$
$$= \sum_{k=1}^{n} p_k A_k^2 - \left(\sum_{k=1}^{n} p_k A_k\right)^2.$$

Corollary 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.11). For the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,*

$$\sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) - b\left[\sum_{k=1}^{n} p_k A_k^2 - \left(\sum_{k=1}^{n} p_k A_k\right)^2\right] \tag{2.17}$$
$$= 2\int_0^1 (1-t)\sum_{k=1}^{n} p_k \left[\int_0^\infty \lambda^3 \left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\right.$$
$$\times \left(A_k - \sum_{j=1}^{n} p_j A_j\right)\left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1}\left(A_k - \sum_{j=1}^{n} p_j A_j\right)$$
$$\left.\times \left(\lambda + (1-t)\sum_{j=1}^{n} p_j A_j + tA_k\right)^{-1} d\mu(\lambda)\right] dt$$
$$\geq 0.$$

3. Upper and Lower Bounds

We have the following operator bounds:

Theorem 5. *Assume that $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta\mathcal{C}''(w,\mu)(\beta) \\ &\leq \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) - D(\mathcal{C}(w,\mu))(A)(B-A) \\ &\leq \frac{1}{2}\Delta\mathcal{C}''(w,\mu)(\alpha). \end{aligned} \tag{3.1}$$

Proof. For $\lambda \geq 0$ and $t \in [0,1]$ we have

$$\lambda + \alpha \leq \lambda + (1-t)A + tB \leq \lambda + \beta,$$

which implies that

$$(\lambda+\beta)^{-1} \leq (\lambda + (1-t)A + tB)^{-1} \leq (\lambda+\alpha)^{-1}. \tag{3.2}$$

If we multiply this both sides with $B-A$, then we obtain

$$(\lambda+\beta)^{-1}(B-A)^{2} \leq (B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \leq (\lambda+\alpha)^{-1}(B-A)^{2}. \tag{3.3}$$

Since $0<\delta\leq(B-A)^{2}\leq\Delta$, hence $(\lambda+\beta)^{-1}(B-A)^{2}\geq\delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^{2}\leq(\lambda+\alpha)^{-1}\Delta$, then by (3.3) we get

$$\delta(\lambda+\beta)^{-1} \leq (B-A)(\lambda+(1-t)A+tB)^{-1}(B-A) \leq (\lambda+\alpha)^{-1}\Delta, \tag{3.4}$$

for $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply this inequality both sides with $(\lambda+(1-t)A+tB)^{-1}$, we derive

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1}(\lambda+(1-t)A+tB)^{-2}\\
&\leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\
&\times(B-A)(\lambda+(1-t)A+tB)^{-1}\\
&\leq(\lambda+\alpha)^{-1}\Delta(\lambda+(1-t)A+tB)^{-2}
\end{aligned} \tag{3.5}$$

and by (3.2) we further obtain the bounds

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3}\\
&\leq(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\\
&\times(B-A)(\lambda+(1-t)A+tB)^{-1}\\
&\leq(\lambda+\alpha)^{-3}\Delta.
\end{aligned} \tag{3.6}$$

If we multiply with $2\lambda^{2}w(\lambda)(1-t)$ and integrate, then we get

$$\begin{aligned}
&2\delta\int_{0}^{\infty}\lambda^{2}w(\lambda)(\lambda+\beta)^{-3}\left(\int_{0}^{1}(1-t)\,dt\right)d\mu(\lambda)\\
&\leq 2\int_{0}^{\infty}\lambda^{2}w(\lambda)\left(\int_{0}^{1}(1-t)(\lambda+(1-t)A+tB)^{-1}\right.\\
&\times\left((B-A)(\lambda+(1-t)A+tB)^{-1}(B-A)\right)\\
&\left.\times(\lambda+(1-t)A+tB)^{-1}\,dt\right)d\mu(\lambda)\\
&\leq 2\Delta\int_{0}^{\infty}\lambda^{2}w(\lambda)(\lambda+\alpha)^{-3}\left(\int_{0}^{1}(1-t)\,dt\right)d\mu(\lambda),
\end{aligned}$$

which, by the equality (2.12), is equivalent to

$$\begin{aligned}
&\delta\int_{0}^{\infty}\lambda^{2}w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda)\\
&\leq\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\\
&\leq\Delta\int_{0}^{\infty}\lambda^{2}w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda).
\end{aligned} \tag{3.7}$$

If we take the derivative in (1.6) over t then we get

$$\mathcal{C}'(w,\mu)(t)=\int_{0}^{\infty}w(\lambda)\left[1-\lambda^{2}(t+\lambda)^{-2}\right]d\mu(\lambda),\ t>0,$$

and

$$\mathcal{C}''(w,\mu)(t) = 2\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+t)^3} d\mu(\lambda), \ t>0.$$

This gives

$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\alpha)^3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\alpha),$$
$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\beta)^3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\beta)$$

and by (3.7) we get (3.1). □

The case of operator monotone functions is as follows:

Corollary 5. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.9). If* $\beta \geq A$, $B\geq\alpha>0$ *and* $0<\delta\leq (B-A)^2 \leq \Delta$, *then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta\left[\beta f''(\beta)+2f'(\beta)\right] \\ &\leq \frac{1}{2}\delta\left[\beta f''(\beta)+2f'(\beta)\right] + b\left[(B-A)^2-\delta\right] \\ &\leq Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A) \\ &\leq \frac{1}{2}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+b\left[(B-A)^2-\Delta\right] \\ &\leq \frac{1}{2}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]. \end{aligned} \tag{3.8}$$

Proof. From (1.9) we derive

$$\mathcal{C}(\ell,\mu)(t) = tf(t) - at - bt^2,$$

where $a\in\mathbb{R}$, $b\geq 0$, $\ell(\lambda)=\lambda$ and μ is a positive measure on $(0,\infty)$.

This gives that

$$\mathcal{C}'(\ell,\mu)(t) = f(t)+tf'(t)-a-2bt$$

and

$$\mathcal{C}''(\ell,\mu)(t) = tf''(t)+2f'(t)-2b \geq 0.$$

By (3.1) we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta\left[\beta f''(\beta)+2f'(\beta)-2b\right] \\ &\leq Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A)-b(B-A)^2 \\ &\leq \frac{1}{2}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)-2b\right], \end{aligned}$$

namely

$$\begin{aligned} 0 &\leq \frac{1}{2}\delta\left[\beta f''(\beta)+2f'(\beta)\right] + b\left[(B-A)^2-\delta\right] \\ &\leq Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A) \\ &\leq \frac{1}{2}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+b\left[(B-A)^2-\Delta\right]. \end{aligned}$$

Since $b\left[(B-A)^2-\delta\right]\geq 0$ and $b\left[(B-A)^2-\Delta\right]\leq 0$, the last part of (3.8) is thus proved. □

The case of operator convex functions is as follows:

Corollary 6. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.11). If $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$, then*

$$\begin{aligned}0&\leq\frac{1}{2}\delta f''(\beta)\leq\frac{1}{2}\delta f''(\beta)+c\left[(B-A)^2-\delta\right]\\&\leq f(B)-f(A)-D(f)(A)(B-A)\\&\leq\frac{1}{2}\Delta f''(\alpha)+c\left[(B-A)^2-\Delta\right]\leq\frac{1}{2}\Delta f''(\alpha).\end{aligned}\tag{3.9}$$

Proof. From (1.11) we have

$$\mathcal{C}(\ell,\mu)(t)=f(t)-a-bt-ct^2,$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and μ a positive measure on $(0,\infty)$. This shows that $\mathcal{C}''(\ell,\mu)(t)=f''(t)-2c\geq 0$ and by (3.1) we get

$$\begin{aligned}0&\leq\frac{1}{2}\delta\left[f''(\beta)-2c\right]\\&\leq f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2\\&\leq\frac{1}{2}\Delta\left[f''(\alpha)-2c\right],\end{aligned}$$

namely

$$\begin{aligned}\frac{1}{2}\delta f''(\beta)+c\left[(B-A)^2-\delta\right]&\leq f(B)-f(A)-D(f)(A)(B-A)\\&\leq\frac{1}{2}\Delta f''(\alpha)+c\left[(B-A)^2-\Delta\right].\end{aligned}$$

Since $b\left[(B-A)^2-\delta\right]\geq 0$ and $b\left[(B-A)^2-\Delta\right]\leq 0$, the last part of (3.9) is thus proved. □

Proposition 1. *Assume that $\beta\geq A_k\geq\alpha>0$, $k\in\{1,...,n\}$ and $0<\delta\leq\left(A_k-\sum_{j=1}^n p_jA_j\right)^2\leq\Delta$, $k\in\{1,...,n\}$ for some constants α, β, δ, Δ, then*

$$0\leq\frac{1}{2}\delta\mathcal{C}''(w,\mu)(\beta)\leq J(\mathbf{A},\mathbf{p},\mathcal{C}(w,\mu))\leq\frac{1}{2}\Delta\mathcal{C}''(w,\mu)(\alpha).\tag{3.10}$$

Proof. From (3.1) we get

$$\begin{aligned}0&\leq\frac{1}{2}\delta\mathcal{C}''(w,\mu)(\beta)\leq\mathcal{C}(w,\mu)(A_k)-\mathcal{C}(w,\mu)\left(\sum_{j=1}^n p_jA_j\right)\\&\quad-D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^n p_jA_j\right)\left(A_k-\sum_{j=1}^n p_jA_j\right)\\&\leq\frac{1}{2}\Delta\mathcal{C}''(w,\mu)(\alpha),\end{aligned}\tag{3.11}$$

for $k\in\{1,...,n\}$.

If we multiply (3.11) by $p_k \geq 0$ and sum over k from 1 to n, we get the desired result. □

The case of operator monotone functions is as follows:

Corollary 7. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and* A_k, $k\in\{1,...,n\}$ *satisfy the assumption in Proposition 1, then*

$$
\begin{aligned}
(3.12)\qquad 0 &\leq \frac{1}{2}\delta\left[\beta f''(\beta)+2f'(\beta)\right] \leq \sum_{k=1}^{n} p_k A_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right)\sum_{k=1}^{n} p_k A_k \\
&\leq \frac{1}{2}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right].
\end{aligned}
$$

Remark 1. *If we write the inequality (3.12) for the operator monotone function* $f(t)=t^r$, $r\in(0,1]$, *then we have the power inequalities*

$$
(3.13)\quad 0 < \frac{1}{2}\delta r(r+1)\beta^{r-1} \leq \sum_{k=1}^{n} p_k A_k^{r+1} - \left(\sum_{k=1}^{n} p_k A_k\right)^{r+1} \leq \frac{1}{2}\Delta r(r+1)\alpha^{r-1}
$$

and the logarithmic inequalities

$$
(3.14)\qquad 0 < \frac{\delta}{2\beta} \leq \sum_{k=1}^{n} p_k A_k \ln A_k - \sum_{k=1}^{n} p_k A_k \ln\left(\sum_{k=1}^{n} p_k A_k\right) \leq \frac{\Delta}{2\alpha}.
$$

The case of operator convex functions is as follows:

Corollary 8. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *and* A_k, $k\in\{1,...,n\}$ *satisfy the assumption in Proposition 1, then*

$$
(3.15)\qquad 0 < \frac{1}{2}\delta f''(\beta) \leq \sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) \leq \frac{1}{2}\Delta f''(\alpha).
$$

Remark 2. *If we write the inequality (3.16) for the operator convex function* $f(t)=t^p$, $p\in[-1,0]\cup[1,2]$, *then we have the power inequalities*

$$
(3.16)\qquad 0 \leq \frac{1}{2}\delta p(p-1)\beta^{p-2} \leq \sum_{k=1}^{n} p_k A_k^{p} - \left(\sum_{k=1}^{n} p_k A_k\right)^{p} \leq \frac{1}{2}\Delta p(p-1)\alpha^{p-2}.
$$

In particular, we have

$$
(3.17)\qquad 0 \leq \delta\beta^{-3} \leq \sum_{k=1}^{n} p_k A_k^{-1} - \left(\sum_{k=1}^{n} p_k A_k\right)^{-1} \leq \Delta\alpha^{-3}.
$$

If we apply the inequality (3.15) for the operator convex function $f(t)=-\ln t$, $t>0$, *then we also obtain the logarithmic inequalities*

$$
(3.18)\qquad 0 < \frac{\delta}{2\beta^2} \leq \ln\left(\sum_{k=1}^{n} p_k A_k\right) - \sum_{k=1}^{n} p_k \ln(A_k) \leq \frac{\Delta}{2\alpha^2}.
$$

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