MIDPOINT AND TRAPEZOID OPERATOR INEQUALITIES IN TERMS OF SECOND DERIVATIVE FOR THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $\beta \geq A$, $B \geq \alpha > 0$, and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta \mathcal{C}''(w,\mu)(\beta) \\ &\leq \int_0^1 \mathcal{C}(w,\mu)((1-t)A + tB)\,dt - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{24}\Delta \mathcal{C}''(w,\mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{12}\delta \mathcal{C}''(w,\mu)(\beta) \\ &\leq \frac{\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A + tB)\,dt \\ &\leq \frac{1}{12}\Delta \mathcal{C}''(w,\mu)(\alpha), \end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as an operator functions and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function. Applications for operator monotone and operator convex functions are provided. Examples for power functions and logarithms are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot,\cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

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Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t = (t-1)\,\mathcal{D}(w_{\ln})(t), \ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.9}$$

where $a \in \mathbb{R}$, $b \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f *is operator monotone in* $[0, \infty)$, *then* $a = f(0)$ *in (1.9).*

For some example of operator monotone functions see [3]-[5], [8], [9] and the references therein.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function* $f : (0, \infty) \to \mathbb{R}$ *is operator convex in* $(0, \infty)$ *if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t+\lambda} d\mu(\lambda), \tag{1.11}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that (1.2) holds. If* f *is operator convex in* $[0, \infty)$, *then* $a = f(0)$ *and* $b = f'_+(0)$, *the right derivative, in (1.11).*

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w, \mu)(t) := t^2 \mathcal{D}(w, \mu)(t), \ t > 0. \tag{1.12}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{C}(w, \mu)(t) &:= \int_0^\infty w(\lambda) t^2 (t+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(t+\lambda-\lambda)^2 (t+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[(t+\lambda)^2 - 2\lambda(t+\lambda) + \lambda^2\right] (t+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[(t+\lambda) - 2\lambda + \lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[t - \lambda + \lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{1.13}$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$ and $\int_0^\infty w(\lambda) \lambda d\mu(\lambda) < \infty$, then we conclude that

$$\mathcal{C}(w, \mu)(t) = -\int_0^\infty w(\lambda) \lambda d\mu(\lambda) + t \int_0^\infty w(\lambda) d\mu(\lambda) + \mathcal{D}(\ell^2 w, \mu)(t), \tag{1.14}$$

where $\ell(t) = t$, $t > 0$.

Consider, for instance, the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \ t \geq 0$$

where the *exponential integral* is given by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

We also have

$$\int_0^\infty w(\lambda)\lambda d\lambda = \int_0^\infty \exp(-a\lambda)\lambda d\lambda = \frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t) := t^2\mathcal{D}(e_{-a},\mu)(t) = t^2 E_1(at)\exp(at), \ t > 0. \tag{1.15}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda} d\lambda$$

then by (1.14) we get

$$t^2 E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t) = t^2 E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2}, \ t > 0, a > 0.$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda) d\lambda = \infty$ and the equality (1.14) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2(\lambda+T)^{-1} d\mu(\lambda), \tag{1.16}$$

provided the integral exist. By (1.13) we also have

$$\mathcal{C}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[T - \lambda + \lambda^2(t+\lambda)^{-1}\right] d\mu(\lambda). \tag{1.17}$$

In this paper, we show among others that, if $\beta \geq A$, $B \geq \alpha > 0$ and $0 < \delta \leq (B-A)^2 \leq \Delta$ for some constants α, β, δ, Δ, then

$$\begin{aligned} 0 &\leq \frac{1}{24}\delta\mathcal{C}''(w,\mu)(\beta) \\ &\leq \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)dt - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{24}\Delta\mathcal{C}''(w,\mu)(\alpha) \end{aligned}$$

and

$$\begin{aligned} 0 &\le \frac{1}{12}\delta\mathcal{C}''(w,\mu)(\beta) \\ &\le \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \\ &\le \frac{1}{12}\Delta\mathcal{C}''(w,\mu)(\alpha), \end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as an operator functions and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function. Applications for operator monotone and operator convex functions are provided. Examples for power functions and logarithms are also given.

2. Some Preliminary Facts

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w,\mu))$:

Lemma 1. *For all $A>0$,*

$$\begin{aligned} &D(\mathcal{C}(w,\mu))(A)(V) \\ &= \int_0^\infty w(\lambda)(A+\lambda)^{-1}[AVA+\lambda(VA+AV)](A+\lambda)^{-1}\,d\mu(\lambda) \end{aligned} \tag{2.1}$$

for all $V\in S(H)$, the class of all selfadjoint operators on H.

Proof. Let $A>0$ and $V\in S(H)$. By the definition of $\mathcal{C}(w,\mu)$ and by (1.13) and we have for t in a small open interval around 0 that

$$\mathcal{C}(w,\mu)(A+tV) = \int_0^\infty w(\lambda)\left[A+tV-\lambda+\lambda^2(A+tV+\lambda)^{-1}\right]d\mu(\lambda).$$

Then

$$\begin{aligned} &\mathcal{C}(w,\mu)(A+tV)-\mathcal{C}(w,\mu)(A) \\ &= \int_0^\infty w(\lambda)\left[A+tV-\lambda+\lambda^2(A+tV+\lambda)^{-1}\right]d\mu(\lambda) \\ &- \int_0^\infty w(\lambda)\left[A-\lambda+\lambda^2(A+\lambda)^{-1}\right]d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(A+tV+\lambda)^{-1}-(A+\lambda)^{-1}\right]\right\}d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(A+tV+\lambda)^{-1}(-tV)(A+\lambda)^{-1}\right]\right\}d\mu(\lambda) \\ &= t\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(A+tV+\lambda)^{-1}V(A+\lambda)^{-1}\right]\right\}d\mu(\lambda). \end{aligned}$$

Therefore,

$$\begin{aligned} &D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V\right) \\ &= \lim_{t\to 0}\frac{\mathcal{C}(w,\mu)\left(A+tV\right)-\mathcal{C}(w,\mu)\left(A\right)}{t} \\ &= \lim_{t\to 0}\int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left[\left(A+tV+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}\right]\right\}d\mu\left(\lambda\right) \\ &= \int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right) \end{aligned} \tag{2.2}$$

for $A>0$ and $V\in S\left(H\right)$.

Define for $\lambda\geq 0$,

$$U_\lambda := V-\lambda^2\left(A+\lambda\right)^{-1}V\left(A+\lambda\right)^{-1}.$$

If we multiply U_λ both sides by $A+\lambda$, then we get

$$\begin{aligned} \left(A+\lambda\right)U_\lambda\left(A+\lambda\right) &= \left(A+\lambda\right)V\left(A+\lambda\right)-\lambda^2 V \\ &= \left(AV+\lambda V\right)\left(A+\lambda\right)-\lambda^2 V \\ &= AVA+\lambda VA+\lambda AV+\lambda^2 V-\lambda^2 V \\ &= AVA+\lambda\left(VA+AV\right). \end{aligned}$$

If we multiply both sides by $\left(A+\lambda\right)^{-1}$ we get

$$U_\lambda = \left(A+\lambda\right)^{-1}\left[AVA+\lambda\left(VA+AV\right)\right]\left(A+\lambda\right)^{-1},$$

which, by (2.2), implies the representation (2.1). □

For the case of second Fréchet derivative $D^2\left(\mathcal{C}\left(w,\mu\right)\right)$, we have the representation:

Lemma 2. *For all $A>0$,*

$$\begin{aligned} &D^2\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V,V\right) \\ &= 2\int_0^\infty \lambda^2 w\left(\lambda\right)\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1}V\left(\lambda+A\right)^{-1} \end{aligned} \tag{2.3}$$

for all $V\in S\left(H\right)$.

Proof. Let $A>0$ and $V\in S\left(H\right)$. We have by the definition of the Fréchet second derivative that

$$\begin{aligned} &D^2\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V,V\right) \\ &= \lim_{t\to 0}\frac{D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right)-D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A\right)\left(V\right)}{t}. \end{aligned} \tag{2.4}$$

Observe, by (2.2), that we have for t in a small open interval around 0,

$$\begin{aligned} &D\left(\mathcal{C}\left(w,\mu\right)\right)\left(A+tV\right)\left(V\right) \\ &= \int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left(A+tV+\lambda\right)^{-1}V\left(A+tV+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right). \end{aligned}$$

Therefore

$$
\begin{aligned}
&D(\mathcal{C}(w,\mu))(A+tV)(V)-D(\mathcal{C}(w,\mu))(A)(V) \\
&=\int_0^\infty w(\lambda)\left\{V-\lambda^2(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}\right\}d\mu(\lambda) \\
&-\int_0^\infty w(\lambda)\left\{V-\lambda^2(A+\lambda)^{-1}V(A+\lambda)^{-1}\right\}d\mu(\lambda) \\
&=\int_0^\infty \lambda^2 w(\lambda) \\
&\times\left[(A+\lambda)^{-1}V(A+\lambda)^{-1}-(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}\right].
\end{aligned}
\tag{2.5}
$$

Define

$$W_{t,\lambda}:=(A+\lambda)^{-1}V(A+\lambda)^{-1}-(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}.$$

If we multiply both sides of $W_{t,\lambda}$ with $\lambda+A+tV$, the we get

$$
\begin{aligned}
&(\lambda+A+tV)W_{t,\lambda}(\lambda+A+tV) \\
&=(\lambda+A+tV)(\lambda+A)^{-1}V(\lambda+A)^{-1}(\lambda+A+tV)-V \\
&=\left(1+tV(\lambda+A)^{-1}\right)V\left(1+t(\lambda+A)^{-1}V\right)-V \\
&=\left(V+tV(\lambda+A)^{-1}V\right)\left(1+t(\lambda+A)^{-1}V\right)-V \\
&=V+tV(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V \\
&+t^2V(\lambda+A)^{-1}V(\lambda+A)^{-1}V-V \\
&=2tV(\lambda+A)^{-1}V+t^2V(\lambda+A)^{-1}V(\lambda+A)^{-1}V \\
&=t\left[2V(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V(\lambda+A)^{-1}V\right].
\end{aligned}
$$

If we multiply the equality by $(\lambda+A+tV)^{-1}$ both sides, we get for $t\neq 0$

$$
\begin{aligned}
\frac{W_{t,\lambda}}{t}&=(\lambda+A+tV)^{-1}\left[2V(\lambda+A)^{-1}V+tV(\lambda+A)^{-1}V(\lambda+A)^{-1}V\right] \\
&\times(\lambda+A+tV)^{-1}.
\end{aligned}
\tag{2.6}
$$

If we take the limit over $t\to 0$ in (2.6), then we get

$$\lim_{t\to 0}\left(\frac{W_{t,\lambda}}{t}\right)=2(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1}. \tag{2.7}$$

Then by (2.4), (2.5) and (2.7) we derive (2.3). □

We have the following representation for the transform $\mathcal{C}(w,\mu)$:

Lemma 3. *For all* A, $B>0$,

$$
\begin{aligned}
&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A) \\
&=2\int_0^1(1-s)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-s)A+sB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds \\
&\geq 0.
\end{aligned}
\tag{2.8}
$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [2, p. 112],

$$f(E)=f(C)+D(f)(C)(E-C)+\int_0^1(1-s)D^2(f)((1-s)C+sE)(E-C,E-C)\,ds \tag{2.9}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and $C,\ E\in\mathcal{O}$.

If we write (2.9) for $\mathcal{C}(w,\mu)$ and $A,\ B>0$, we get

$$\mathcal{C}(w,\mu)(B)=\mathcal{C}(w,\mu)(A)+D(\mathcal{C}(w,\mu))(A)(B-A)+\int_0^1(1-s)D^2(\mathcal{C}(w,\mu))((1-s)A+sB)(B-A,B-A)\,ds. \tag{2.10}$$

By making use of (2.3) we obtain

$$\begin{aligned}&D^2(\mathcal{C}(w,\mu))((1-s)A+sB)(B-A,B-A)\\&=\left[\int_0^\infty\lambda^2 w(\lambda)(\lambda+(1-s)A+sB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds,\end{aligned}$$

which, by (2.10), produces the equality in (2.8).

Now, observe that

$$(\lambda+(1-s)A+sB)^{-1}\geq 0$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

By multiplying this inequality both sides by $(B-A)$, we derive

$$(B-A)(\lambda+(1-s)A+sB)^{-1}(B-A)\geq 0$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

Moreover, if we multiply this inequality both sides by $(\lambda+(1-s)A+sB)^{-1}$ we obtain

$$\begin{aligned}&(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}\\&(B-A)(\lambda+(1-s)A+sB)^{-1}\\&\geq 0\end{aligned}$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

Finally, by multiplying with $(1-s)\lambda^2 w(\lambda)\geq 0$ and integrating we deduce the inequality part in (2.8). □

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B}:[0,1]\to\mathbb{R}$ defined by

$$f_{A,B}(t):=f((1-t)A+tB),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 4. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on* $[0,1]$,

$$\frac{df_{A,B}(t)}{dt}=D(f)((1-t)A+tB)(B-A) \tag{2.11}$$

and

$$\frac{d^2 f_{A,B}(t)}{dt^2} = D^2(f)((1-t)A + tB)(B-A, B-A) \tag{2.12}$$

for $t \in [0,1]$, *where in* 0 *and* 1 *the derivatives are the right and left derivatives.*

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{f_{A,B}(t+h) - f(t)}{h} \\
&= \frac{f((1-(t+h))A + (t+h)B) - f((1-t)A + tB)}{h} \\
&= \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
\frac{df_{A,B}(t)}{dt} &= \lim_{h\to 0} \frac{f_{A,B}(t+h) - f(t)}{h} \\
&= \lim_{h\to 0}\left[\frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h}\right] \\
&= D(f)((1-t)A + tB)(B-A),
\end{aligned}$$

which proves (2.11).

Similarly,

$$\begin{aligned}
&\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt}\right] \\
&= \frac{D(f)((1-(t+h))A + (t+h)B)(B-A) - D(f)((1-t)A + tB)(B-A)}{h} \\
&= \frac{D(f)((1-t)A + tB + h(B-A))(B-A) - D(f)((1-t)A + tB)(B-A)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
\frac{d^2 f_{A,B}(t)}{dt^2} &= \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt}\right]\right\} \\
&= D^2(f)((1-t)A + tB)(B-A, B-A),
\end{aligned}$$

which proves (2.12). □

For the transform $\mathcal{C}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\mathcal{C}(w,\mu)_{A,B}(t) := \mathcal{C}(w,\mu)((1-t)A + tB)$$

where A, $B > 0$ and $t \in [0,1]$.

Corollary 1. *For all $A,\ B>0$ and $t\in[0,1]$,*

$$
\begin{aligned}
&\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt} \\
&=D(\mathcal{C}(w,\mu))((1-t)A+tB)(B-A) \\
&=\int_0^\infty w(\lambda)((1-t)A+tB+\lambda)^{-1} \\
&\times[((1-t)A+tB+\lambda)(B-A)((1-t)A+tB+\lambda) \\
&+\lambda((B-A)((1-t)A+tB+\lambda)+((1-t)A+tB+\lambda)(B-A))] \\
&\times((1-t)A+tB+\lambda)^{-1}d\mu(\lambda)
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
&\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2} \\
&=D^2(\mathcal{C}(w,\mu))((1-t)A+tB)(B-A,B-A) \\
&=2\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A) \\
&\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).
\end{aligned}
\tag{2.14}
$$

3. Main Results

We have the following identity for the midpoint rule:

Theorem 3. *For all $A,\ B>0$ we have the identity*

$$
\begin{aligned}
&\int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right. \\
&\times\left[\int_0^\infty \lambda^2 w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\left.\left.\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.
\end{aligned}
\tag{3.1}
$$

Proof. From (2.5) we have for $B=E>0$ and $A=C>0$ that

$$
\begin{aligned}
&\mathcal{C}(w,\mu)(E) \\
&=\mathcal{C}(w,\mu)(C)+D(\mathcal{C}(w,\mu))((1-t)C+tE)(E-C) \\
&+2\int_0^1(1-s)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-s)C+sE)^{-1}(E-C)\right. \\
&\left.\times(\lambda+(1-s)C+sE)^{-1}(E-C)(\lambda+(1-s)C+sE)^{-1}d\mu(\lambda)\right]ds,
\end{aligned}
$$

which implies for $E=(1-t)A+tB$, $t\in[0,1]$ and $C=\frac{A+B}{2}$, that

$$\begin{aligned}
&\mathcal{C}(w,\mu)((1-t)A+tB) \\
&=\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&+\left(t-\frac{1}{2}\right)D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)(B-A) \\
&+2\left(t-\frac{1}{2}\right)^2\int_0^1(1-s) \\
&\times\left[\int_0^\infty \lambda^2 w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds.
\end{aligned}\tag{3.2}$$

If we integrate (3.2) over $t\in[0,1]$, then we get

$$\begin{aligned}
&\int_0^1\mathcal{C}(w,\mu)((1-t)A+tB)\,dt \\
&=\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&-\int_0^1\left(t-\frac{1}{2}\right)dtD(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)(B-A) \\
&+2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right. \\
&\times\left[\int_0^\infty w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\
&\left.\left.\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt
\end{aligned}$$

and since $\int_0^1\left(t-\frac{1}{2}\right)dt=0$, hence the identity (3.1) is proved. □

Corollary 2. *Assume that $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$ for some constants α, β, δ, Δ, then*

$$\begin{aligned}
0&\leq\frac{1}{24}\delta\mathcal{C}''(w,\mu)(\beta) \\
&\leq\int_0^1\mathcal{C}(w,\mu)((1-t)A+tB)\,dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&\leq\frac{1}{24}\Delta\mathcal{C}''(w,\mu)(\alpha).
\end{aligned}\tag{3.3}$$

Proof. Since $\beta \geq A$, $B \geq \alpha > 0$, hence

$$\lambda + \alpha \leq \lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB) \leq \lambda + \beta,$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

This implies that

$$(\lambda+\beta)^{-1} \leq \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} \leq (\lambda+\alpha)^{-1} \tag{3.4}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply this both sides with $B-A$, then we obtain

$$\begin{aligned}
&(\lambda+\beta)^{-1}(B-A)^2 \\
&\leq (B-A)\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1}(B-A) \\
&\leq (\lambda+\alpha)^{-1}(B-A)^2
\end{aligned} \tag{3.5}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

Since $0 < \delta \leq (B-A)^2 \leq \Delta$, hence $(\lambda+\beta)^{-1}(B-A)^2 \geq \delta(\lambda+\beta)^{-1}$ and $(\lambda+\alpha)^{-1}(B-A)^2 \leq (\lambda+\alpha)^{-1}\Delta$, then by (3.5)

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1} \\
&\leq (B-A)\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1}(B-A) \\
&\leq \Delta(\lambda+\alpha)^{-1}
\end{aligned} \tag{3.6}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply both sides with $\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1}$ we derive

$$\begin{aligned}
&\delta(\lambda+\beta)^{-1}\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-2} \\
&\leq \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1}(B-A) \\
&\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1}(B-A) \\
&\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} \\
&\leq \Delta(\lambda+\alpha)^{-1}\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-2}
\end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

By utilising (3.4) we further obtain the bounds

$$\begin{aligned}
&\delta(\lambda+\beta)^{-3}\\
&\leq\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}\\
&\leq\Delta(\lambda+\alpha)^{-3}
\end{aligned}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply by $2\lambda^2 w(\lambda)\left(t-\frac{1}{2}\right)^2(1-s)\geq 0$ and integrate, then we get

$$\begin{aligned}
&2\delta\int_0^\infty\lambda^2 w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2 dt\int_0^1(1-s)\,ds \qquad (3.7)\\
&\leq 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right.\\
&\times\left[\int_0^\infty\lambda^2 w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right.\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt\\
&\leq 2\Delta\int_0^\infty\lambda^2 w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda)\int_0^1\left(t-\frac{1}{2}\right)^2 dt\int_0^1(1-s)\,ds
\end{aligned}$$

and by the identity (3.1) and the fact that

$$\int_0^1\left(t-\frac{1}{2}\right)^2 dt=\frac{1}{12}\text{ and }\int_0^1(1-s)\,ds=\frac{1}{2}$$

we obtain

$$\begin{aligned}
&\frac{1}{12}\delta\int_0^\infty\lambda^2 w(\lambda)(\lambda+\beta)^{-3}d\mu(\lambda) \qquad (3.8)\\
&\leq\int_0^1\mathcal{C}(w,\mu)((1-t)A+tB)\,dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\\
&\leq\frac{1}{12}\Delta\int_0^\infty\lambda^2 w(\lambda)(\lambda+\alpha)^{-3}d\mu(\lambda).
\end{aligned}$$

If we take the derivative in (1.13) over t, then we get

$$\mathcal{C}'(w,\mu)(t)=\int_0^\infty w(\lambda)\left[1-\lambda^2(t+\lambda)^{-2}\right]d\mu(\lambda),\ t>0,$$

and

$$\mathcal{C}''(w,\mu)(t)=2\int_0^\infty\frac{\lambda^2 w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0.$$

This gives

$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\alpha)^3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\alpha),$$
$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\beta)^3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\beta)$$

and by (3.2) we obtain (3.3). □

We have the following identity for the trapezoid rule:

Theorem 4. *For all A, $B>0$ we have the identity*

$$\begin{aligned}(3.9)\qquad & \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \\ &= \int_0^1 t(1-t)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt \\ &= \frac{1}{2}\left[t(1-t)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}\Big|_0^1 - \int_0^1 (1-2t)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}dt\right] \\ &= \int_0^1 \left(t-\frac{1}{2}\right)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}dt \\ &= \left(t-\frac{1}{2}\right)\mathcal{C}(w,\mu)_{A,B}(t)\Big|_0^1 - \int_0^1 \mathcal{C}(w,\mu)_{A,B}(t)\,dt \\ &= \frac{1}{2}\left[\mathcal{C}(w,\mu)_{A,B}(1)+\mathcal{C}(w,\mu)_{A,B}(0)\right] - \int_0^1 \mathcal{C}(w,\mu)_{A,B}(t)\,dt,\end{aligned}$$

that gives the identity

$$\begin{aligned}(3.10)\qquad & \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \\ &= \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt.\end{aligned}$$

By (2.10) we have

$$\begin{aligned}(3.11)\qquad & \frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt \\ &= \int_0^1 t(1-t)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}$$

By making use of (3.10) and (3.11). □

We have:

Corollary 3. *Assume that* $\beta \geq A$, $B \geq \alpha > 0$ *and* $0 < \delta \leq (B-A)^2 \leq \Delta$ *for some constants* α, β, δ, Δ, *then*

$$(3.12)\quad \begin{aligned} 0 &\leq \frac{1}{12}\delta \mathcal{C}''(w,\mu)(\beta) \\ &\leq \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \\ &\leq \frac{1}{12}\Delta \mathcal{C}''(w,\mu)(\alpha). \end{aligned}$$

Proof. As in the proof of Corollary 2 we have

$$(3.13)\quad \begin{aligned} &\delta(\lambda+\beta)^{-3} \\ &\leq (\lambda+(1-t)A+tB)^{-1}(B-A) \\ &\times (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1} \\ &\leq \Delta(\lambda+\alpha)^{-3} \end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we multiply by $t(1-t)\lambda^2 w(\lambda) \geq 0$ and integrate, then we get

$$(3.14)\quad \begin{aligned} &\delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty \lambda^2 w(\lambda)(\lambda+\beta)^{-3}\,d\mu(\lambda) \\ &\leq \int_0^1 t(1-t)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,d\mu(\lambda)\right]dt \\ &\leq \Delta\left(\int_0^1 t(1-t)\,dt\right)\int_0^\infty \lambda^2 w(\lambda)(\lambda+\alpha)^{-3}\,d\mu(\lambda). \end{aligned}$$

Since

$$\int_0^1 t(1-t)\,dt = \frac{1}{6},$$

$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\alpha)^3}\,d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\alpha)$$

and

$$\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+\beta)^3}\,d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(\beta),$$

then by (3.14) we derive (3.12). □

4. Examples for Operator Monotone Functions

The case of operator monotone function is as follows:

Proposition 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all A, $B>0$ we have the identity*

$$\begin{aligned}&Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A)-b(B-A)^2\\&=2\int_0^1(1-s)\left[\int_0^\infty\lambda^3(\lambda+(1-s)A+sB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds\\&\geq 0.\end{aligned}\tag{4.1}$$

Proof. From (1.9) we derive

$$sf(s)=as+bs^2+s^2\int_0^\infty\frac{\lambda}{s+\lambda}d\mu(\lambda)=as+bs^2+\mathcal{C}(\ell,\mu)(s),$$

where $a\in\mathbb{R}$, $b\geq 0$, $\ell(\lambda)=\lambda$ and μ is a positive measure on $(0,\infty)$.

This gives that

$$\mathcal{C}(\ell,\mu)(s)=sf(s)-as-bs^2.$$

Observe that for A, $B>0$

$$\begin{aligned}&D(\mathcal{C}(w,\mu))(A)(B-A)\\&=D\left(\ell f-a\ell-b\ell^2\right)(A)(B-A)\\&=D(\ell f)(A)(B-A)-aD(\ell)(A)(B-A)-bD\left(\ell^2\right)(A)(B-A).\end{aligned}$$

Since

$$\begin{aligned}D(\ell f)(A)(B-A)&=\ell(A)D(f)(A)(B-A)+f(A)D(\ell)(A)(B-A)\\&=AD(f)(A)(B-A)+f(A)(B-A),\end{aligned}$$

and

$$D\left(\ell^2\right)(A)(B-A)=A(B-A)+(B-A)A=AB+BA-2A^2,$$

hence

$$\begin{aligned}D(\ell f)(A)(B-A)&=AD(f)(A)(B-A)+f(A)(B-A)\\&\quad-a(B-A)-b\left(AB+BA-2A^2\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\\&=Bf(B)-aB-bB^2-Af(A)+aA+bA^2\\&-AD(f)(A)(B-A)-f(A)(B-A)+a(B-A)\\&+b\left(AB+BA-2A^2\right)\\&=Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A)\\&+b\left(AB+BA-A^2-B^2\right)\\&=Bf(B)-Af(A)-AD(f)(A)(B-A)-f(A)(B-A)-b(B-A)^2\end{aligned}$$

and by (2.8) we derive (4.1). □

The following mid-point identity for the operator monotone functions holds:

Proposition 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all $A,\ B>0$,*

$$\begin{aligned}
&\int_0^1 ((1-t)A+tB)\, f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)\\
&-\frac{b}{12}(B-A)^2\\
&=2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1 (1-s)\right.\\
&\times\left[\int_0^\infty \lambda^3\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right.\\
&\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\\
&\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt. \qquad (4.2)
\end{aligned}$$

Proof. From (1.9) we derive

$$\mathcal{C}(\ell,\mu)(s)=sf(s)-as-bs^2,$$

where $a\in\mathbb{R}$, $b\geq 0$, $\ell(\lambda)=\lambda$ and μ is a positive measure on $(0,\infty)$.

Observe that

$$\begin{aligned}
&\int_0^1 \mathcal{C}(\ell,\mu)((1-t)A+tB)\,dt\\
&=\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-a\int_0^1((1-t)A+tB)\,dt\\
&-b\int_0^1((1-t)A+tB)^2\,dt\\
&=\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-a\frac{A+B}{2}\\
&-b\left[\frac{1}{3}A^2+\frac{1}{6}(AB+BA)+\frac{1}{3}B^2\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{C}(\ell,\mu)\left(\frac{A+B}{2}\right)\\
&=\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)-a\left(\frac{A+B}{2}\right)-b\left(\frac{A+B}{2}\right)^2\\
&=\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)-a\left(\frac{A+B}{2}\right)-b\left(\frac{A^2+AB+BA+B^2}{4}\right).
\end{aligned}$$

Therefore

$$\int_0^1 \mathcal{C}(\ell,\mu)((1-t)A+tB)\,dt-\mathcal{C}(\ell,\mu)\left(\frac{A+B}{2}\right)$$
$$=\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)$$
$$-b\left[\frac{1}{3}A^2+\frac{1}{6}(AB+BA)+\frac{1}{3}B^2\right]+b\left(\frac{A^2+AB+BA+B^2}{4}\right)$$
$$=\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)$$
$$-\frac{b}{12}(B-A)^2$$

and by (3.1) we derive (4.2). □

Corollary 4. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.9). If* $\beta\geq A,\ B\geq\alpha>0$ *and* $0<\delta\leq(B-A)^2\leq\Delta$, *then*

$$0\leq\frac{1}{24}\delta\left[\beta f''(\beta)+2f'(\beta)\right] \tag{4.3}$$
$$\leq\frac{1}{24}\delta\left[\beta f''(\beta)+2f'(\beta)\right]+\frac{b}{12}\left[(B-A)^2-\delta\right]$$
$$\leq\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)$$
$$\leq\frac{1}{24}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+\frac{b}{12}\left[(B-A)^2-\Delta\right]$$
$$\leq\frac{1}{24}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right].$$

Proof. From (1.9) we derive

$$\mathcal{C}(\ell,\mu)(t)=tf(t)-at-bt^2,$$

where $a\in\mathbb{R}$, $b\geq0$, $\ell(\lambda)=\lambda$ and μ is a positive measure on $(0,\infty)$.

This gives that

$$\mathcal{C}'(\ell,\mu)(t)=f(t)+tf'(t)-a-2bt$$

and

$$\mathcal{C}''(\ell,\mu)(t)=tf''(t)+2f'(t)-2b\geq0.$$

By (3.3) we get

$$\frac{1}{24}\delta\left[\beta f''(\beta)+2f'(\beta)-2b\right]$$
$$\leq\int_0^1 ((1-t)A+tB)\,f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)f\left(\frac{A+B}{2}\right)$$
$$-\frac{b}{12}(B-A)^2$$
$$\leq\frac{1}{24}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)-2b\right],$$

namely

$$\begin{aligned}0 &\le \frac{1}{24}\delta\left[\beta f''(\beta)+2f'(\beta)\right]+\frac{b}{12}\left[(B-A)^2-\delta\right]\\&\le \int_0^1 ((1-t)A+tB)\, f(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right) f\left(\frac{A+B}{2}\right)\\&\le \frac{1}{24}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+\frac{b}{12}\left[(B-A)^2-\Delta\right].\end{aligned}$$

Since $b\left[(B-A)^2-\delta\right]\ge 0$ and $b\left[(B-A)^2-\Delta\right]\le 0$, the last part of (3.1) is thus proved. □

Remark 1. *If we write the inequality (4.3) for the power function $f(t)=t^r$, $r\in(0,1]$, then we have*

$$\begin{aligned}0 &\le \frac{1}{24}r(r+1)\beta^{r-1}\le \int_0^1 ((1-t)A+tB)^{r+1}\,dt-\left(\frac{A+B}{2}\right)^{r+1}\\&\le \frac{1}{24}\Delta r(r+1)\alpha^{r-1},\end{aligned}\tag{4.4}$$

if $\beta\ge A$, $B\ge\alpha>0$ and $0<\delta\le(B-A)^2\le\Delta$.

We also have the logarithmic inequality

$$\begin{aligned}0 &< \frac{\delta}{24\beta}\\&\le \int_0^1 ((1-t)A+tB)\ln(((1-t)A+tB))\,dt-\left(\frac{A+B}{2}\right)\ln\left(\frac{A+B}{2}\right)\\&\le \frac{\Delta}{24\alpha}.\end{aligned}\tag{4.5}$$

The following trapezoid identity for the operator monotone functions holds:

Proposition 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all A, $B>0$,*

$$\begin{aligned}&\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt\\&=\int_0^1 t(1-t)\left[\int_0^\infty \lambda^3(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.\end{aligned}\tag{4.6}$$

Proof. From (1.9) we derive

$$\mathcal{C}(\ell,\mu)(s)=sf(s)-as-bs^2,$$

where $a\in\mathbb{R}$, $b\ge 0$, $\ell(\lambda)=\lambda$ and μ is a positive measure on $(0,\infty)$.

We have

$$\begin{aligned}&\frac{\mathcal{C}(\ell,\mu)(A)+\mathcal{C}(\ell,\mu)(B)}{2}\\&=\frac{1}{2}\left(Af(A)-aA-bA^2+Bf(B)-aB-bB^2\right)\\&=\frac{Af(A)+Bf(B)}{2}-a\frac{A+B}{2}-\frac{b}{2}\left(A^2+B^2\right).\end{aligned}$$

Therefore

$$
\begin{aligned}
&\frac{\mathcal{C}(\ell,\mu)(A)+\mathcal{C}(\partial,\mu)(B)}{2}-\int_0^1 \mathcal{C}(\ell,\mu)((1-t)A+tB)\,dt\\
&=\frac{Af(A)+Bf(B)}{2}-a\frac{A+B}{2}-\frac{b}{2}\left(A^2+B^2\right)\\
&-\int_0^1((1-t)A+tB)\,f(((1-t)A+tB))\,dt+a\frac{A+B}{2}\\
&+b\left[\frac{1}{3}A^2+\frac{1}{6}(AB+BA)+\frac{1}{3}B^2\right]\\
&=\frac{Af(A)+Bf(B)}{2}-\int_0^1((1-t)A+tB)\,f(((1-t)A+tB))\,dt\\
&-\frac{b}{6}(B-A)^2.
\end{aligned}
$$

The proof follows by (3.9). □

Corollary 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). If $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$, then*

$$
\begin{aligned}
0&\leq\frac{1}{12}\delta\left[\beta f''(\beta)+2f'(\beta)\right]\\
&\leq\frac{1}{12}\delta\left[\beta f''(\beta)+2f'(\beta)\right]+\frac{b}{6}\left[(B-A)^2-\delta\right]\\
&\leq\frac{Af(A)+Bf(B)}{2}-\int_0^1((1-t)A+tB)\,f(((1-t)A+tB))\,dt\\
&\leq\frac{1}{12}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+\frac{b}{6}\left[(B-A)^2-\Delta\right]\\
&\leq\frac{1}{12}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right].
\end{aligned}\tag{4.7}
$$

Proof. By (3.12) we have

$$
\begin{aligned}
0&\leq\frac{1}{12}\delta\left[\beta f''(\beta)+2f'(\beta)-2b\right]\\
&\leq\frac{Af(A)+Bf(B)}{2}-\int_0^1((1-t)A+tB)\,f(((1-t)A+tB))\,dt\\
&-\frac{b}{6}(B-A)^2\\
&\leq\frac{1}{12}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)-2b\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
0&\leq\frac{1}{12}\delta\left[\beta f''(\beta)+2f'(\beta)\right]+\frac{b}{6}\left[(B-A)^2-\delta\right]\\
&\leq\frac{Af(A)+Bf(B)}{2}-\int_0^1((1-t)A+tB)\,f(((1-t)A+tB))\,dt\\
&\leq\frac{1}{12}\Delta\left[\alpha f''(\alpha)+2f'(\alpha)\right]+\frac{b}{6}\left[(B-A)^2-\Delta\right].
\end{aligned}
$$

Since $b\left[(B-A)^2-\delta\right]\geq 0$ and $b\left[(B-A)^2-\Delta\right]\leq 0$, the last part of (4.7) is thus proved. □

Remark 2. *If we write the inequality (4.3) for the power function $f(t)=t^r$, $r\in(0,1]$, then*

$$0\leq \frac{1}{12}r(r+1)\beta^{r-1}\leq \frac{A^{r+1}+B^{r+1}}{2}-\int_0^1((1-t)A+tB)^{r+1}dt \leq \frac{1}{12}\Delta r(r+1)\alpha^{r-1}, \tag{4.8}$$

if $\beta\geq A$, $B\geq\alpha>0$ and $0<\delta\leq(B-A)^2\leq\Delta$.

We also have the logarithmic inequality

$$0<\frac{\delta}{12\beta} \leq \frac{A\ln A+B\ln B}{2}-\int_0^1((1-t)A+tB)\ln(((1-t)A+tB))dt\leq\frac{\Delta}{12\alpha}. \tag{4.9}$$

5. Examples for Operator Convex Functions

The case of operator convex function is as follows:

Proposition 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.11). Then for all A, $B>0$,*

$$\begin{aligned} &f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2 \\ &=2\int_0^1(1-s)\left[\int_0^\infty\lambda^3(\lambda+(1-s)A+sB)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds \\ &\geq 0. \end{aligned} \tag{5.1}$$

Proof. From (1.11) we have

$$\mathcal{C}(\ell,\mu)(s)=f(s)-a-bs-cs^2,$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and μ a positive measure on $(0,\infty)$.

Therefore

$$\begin{aligned} &\mathcal{C}(\ell,\mu)(B)-\mathcal{C}(\ell,\mu)(A)-D(\mathcal{C}(\ell,\mu))(A)(B-A) \\ &=f(B)-bB-cB^2-f(A)+bA+cA^2 \\ &-D(f)(A)(B-A)+bD(\ell)(A)(B-A)+cD\left(\ell^2\right)(A)(B-A) \\ &=f(B)-bB-cB^2-f(A)+bA+cA^2 \\ &-D(f)(A)(B-A)+b(B-A)+c\left(AB+BA-2A^2\right) \\ &=f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2 \end{aligned}$$

and by (2.8) we derive the desired result (5.1). □

Proposition 5. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.11). Then for all* $A,\ B>0,$

$$\begin{aligned}(5.2)\qquad &\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) - \frac{c}{12}(B-A)^2 \\ &= 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1 (1-s)\right. \\ &\times\left[\int_0^\infty \lambda^3\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right. \\ &\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A) \\ &\times\left.\left.\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.\end{aligned}$$

Proof. From

$$\mathcal{C}(\ell,\mu)(t) = f(t) - a - bt - ct^2,\ t>0$$

where $a,\ b\in\mathbb{R}$, $c\geq 0$ and μ a positive measure on $(0,\infty)$.

We have

$$\begin{aligned}&\int_0^1 \mathcal{C}(\ell,\mu)((1-t)A+tB)\,dt - \mathcal{C}(\ell,\mu)\left(\frac{A+B}{2}\right) \\ &= \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) - \frac{c}{12}(B-A)^2\end{aligned}$$

and by (3.1) we derive (5.2). □

Corollary 6. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.11). If* $\beta\geq A,\ B\geq\alpha>0$ *and* $0<\delta\leq(B-A)^2\leq\Delta$, *then*

$$\begin{aligned}(5.3)\qquad 0 &\leq \frac{1}{24}\delta f''(\beta) \leq \frac{1}{24}\delta f''(\beta) + \frac{c}{12}\left[(B-A)^2-\delta\right] \\ &\leq \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{24}\Delta f''(\alpha) + \frac{c}{12}\left[(B-A)^2-\Delta\right] \leq \frac{1}{24}\Delta f''(\alpha).\end{aligned}$$

Remark 3. *If we write the inequality (5.3) for the operator convex function* $f(t)=t^p,\ p\in[-1,0]\cup[1,2]$, *then we have the power inequalities*

$$\begin{aligned}(5.4)\qquad 0 &< \frac{1}{24}\delta p(p-1)\beta^{p-2} \leq \int_0^1((1-t)A+tB)^p\,dt - \left(\frac{A+B}{2}\right)^p \\ &\leq \frac{1}{24}\Delta p(p-1)\alpha^{p-2}.\end{aligned}$$

In particular, \

$$(5.5)\qquad 0<\frac{1}{12}\delta\beta^{-3}\leq\int_0^1((1-t)A+tB)^{-1}\,dt-\left(\frac{A+B}{2}\right)^{-1}\leq\frac{1}{12}\Delta\alpha^{-3}.$$

If we write the inequality (5.3) for the operator convex function $f(t) = -\ln t$, $t > 0$, *then we get*

$$0 \le \frac{\delta}{24\beta^2} \le \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln\left((1-t)A + tB\right)dt \le \frac{\Delta}{24\alpha^2}. \tag{5.6}$$

For the trapezoid rule, we have:

Proposition 6. *Assume that the function* $f:(0,\infty) \to \mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.11). Then for all* A, $B > 0$,

$$\begin{aligned} &\frac{f(A)+f(B)}{2} - \int_0^1 f\left((1-t)A+tB\right)dt - \frac{c}{6}(B-A)^2 \\ &= \int_0^1 t(1-t)\left[\int_0^\infty \lambda^3 \left(\lambda + (1-t)A + tB\right)^{-1}(B-A)\right. \\ &\left.\times \left(\lambda + (1-t)A + tB\right)^{-1}(B-A)\left(\lambda + (1-t)A + tB\right)^{-1} d\mu(\lambda)\right] dt. \end{aligned} \tag{5.7}$$

The proof follows by the representation (4.6) applied for

$$\mathcal{C}(\ell,\mu)(t) = f(t) - a - bt - ct^2,\ t > 0$$

where a, $b \in \mathbb{R}$, $c \ge 0$ and μ a positive measure on $(0,\infty)$.

Corollary 7. *Assume that the function* $f:(0,\infty) \to \mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.11). If* $\beta \ge A$, $B \ge \alpha > 0$ *and* $0 < \delta \le (B-A)^2 \le \Delta$, *then*

$$\begin{aligned} 0 &\le \frac{1}{12}\delta f''(\beta) \le \frac{1}{12}\delta f''(\beta) + \frac{c}{6}\left[(B-A)^2 - \delta\right] \\ &\le \frac{f(A)+f(B)}{2} - \int_0^1 f\left((1-t)A+tB\right)dt \\ &\le \frac{1}{12}\Delta\alpha f''(\alpha) + \frac{c}{6}\left[(B-A)^2 - \Delta\right] \le \frac{1}{12}\Delta\alpha f''(\alpha). \end{aligned} \tag{5.8}$$

Remark 4. *If we write the inequality (5.3) for the operator convex function* $f(t) = t^p$, $p \in [-1,0] \cup [1,2]$, *then we have the power inequalities*

$$\begin{aligned} 0 &< \frac{1}{12}\delta p(p-1)\beta^{p-2} \le \frac{A^p + B^p}{2} - \int_0^1 \left((1-t)A+tB\right)^p dt \\ &\le \frac{1}{12}\Delta p(p-1)\alpha^{p-2}, \end{aligned} \tag{5.9}$$

if $\beta \ge A$, $B \ge \alpha > 0$ *and* $0 < \delta \le (B-A)^2 \le \Delta$.

In particular,

$$0 < \frac{1}{6}\delta\beta^{-3} \le \frac{A^{-1}+B^{-1}}{2} - \int_0^1 \left((1-t)A+tB\right)^{-1} dt \le \frac{1}{6}\Delta\alpha^{-3}. \tag{5.10}$$

If we write the inequality (5.3) for the operator convex function $f(t) = -\ln t$, $t > 0$, *we get*

$$0 < \frac{\delta}{12\beta^2} \le \int_0^1 \ln\left((1-t)A+tB\right)dt - \frac{\ln A + \ln B}{2} \le \frac{\Delta}{12\alpha^2}. \tag{5.11}$$

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