SOME COMPANIONS OF OSTROWSKI TYPE INEQUALITY FOR FIRST ORDER DIFFERENTIABLE $r-$CONVEX FUNCTIONS

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Abstract. We present some inequalities for first order differentiable $r-$convex functions which are companions of Ostrowski type inequality.

1. Introduction

The following inequality gives us upper bounds was established by Ostrowski (see [1]) for differentiable functions with bounded derivatives.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior of I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

holds for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible .

Ostrowski inequality gives us an inequality which contains the left hand side of Hermite-Hadamard inequality for $x = \frac{a+b}{2}$.

Before the definition of $r-$convex functions, let us recall the power mean $M_r(x, y; \lambda)$.([2]) The power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ x^\lambda y^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

Definition 1. [2] *A positive function f is $r-$convex on $[a, b]$ if*

$$f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

In [2], Gill et al. obtained the following result for $r-$convex functions.

Theorem 2. *Suppose f is a positive $r-$convex function on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(u)du \leq L_r(f(a), f(b)). \tag{1.1}$$

Key words and phrases. Ostrowski inequality, $r-$convex function, Hölder inequality.
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Here L_r is the generalized logarithmic mean of order r of positive numbers x, y is defined by

$$L_r(x,y)=\begin{cases} \frac{r}{r+1}.\frac{x^{r+1}-y^{r+1}}{x^r-y^r}, & r\neq 0,-1,\ x\neq y \\ \frac{x-y}{\ln x-\ln y}, & r=0,\ x\neq y \\ xy.\frac{\ln x-\ln y}{x-y}, & r=-1,\ x\neq y \\ x, & x=y. \end{cases}$$

There are some results for $r-$convex functions in the references [2]-[9].

In this paper, our main aim is to establish several inequalities for a companion of Ostrowski inequality for functions whose first derivatives absolute value are $r-$convex. In order to establish our results we need the following identity which is emboided in the following lemma:

Lemma 1. [10] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I°, where $a, b \in I$ with $a < b$, such that $f' \in L[a, b]$. Then the following equality holds*

$$\begin{aligned} & \frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \\ = & \frac{1}{b-a}\int_a^b k(x,t)f'(t)dt, \end{aligned}$$

$$k(x,t)=\begin{cases} t-a, & t\in[a,x] \\ t-\frac{a+b}{2}, & t\in(x,a+b-x] \\ t-b & t\in(a+b-x,b] \end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

2. Main Results

Theorem 3. *Let $f : [a, b] \to \mathbb{R}^+$ be an absolutely continuous function on $[a, b]$, $f' \in L[a, b]$. If $|f'|^q$ is $r-$convex on $[a, b]$, we obtain the following inequality for all $x \in \left[a, \frac{a+b}{2}\right]$ and $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$:*

$$\begin{aligned} & \left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq & \frac{1}{(b-a)}\left\{\frac{(x-a)^2}{(p+1)^{\frac{1}{p}}}\left[\left(L_r\left(|f'(a)|^q,|f'(x)|^q\right)\right)^{\frac{1}{q}}\right.\right. \\ & \left.+\left(L_r\left(|f'(a+b-x)|^q,|f'(b)|^q\right)\right)^{\frac{1}{q}}\right] \\ & \left.+\frac{(a+b-2x)^2}{2(p+1)^{\frac{1}{p}}}\left(L_r\left(|f'(x)|^q,|f'(a+b-x)|^q\right)\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.1}$$

Proof. We can write

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{(b-a)}\left[\int_a^x |t-a|\,|f'(t)|\,dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right||f'(t)|\,dt\right. \\ &\left.+\int_{a+b-x}^b |t-b|\,|f'(t)|\,dt\right] \end{aligned} \tag{2.2}$$

via Lemma 1 and property of modulus. If we use Hölder inequality in (2.2) we obtain

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{(b-a)}\left[\left(\int_a^x |t-a|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^x |f'(t)|^q\,dt\right)^{\frac{1}{q}}+\right. \\ &+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right)\left(\int_x^{a+b-x}|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\ &\left.+\left(\int_{a+b-x}^b |t-b|^p\,dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b |f'(t)|^q\,dt\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{2.3}$$

Since $|f'|^q$ is $r-$convex on $[a,b]$, we have

$$\frac{1}{(x-a)}\int_a^x |f'(t)|^q\,dt \leq L_r\left(|f'(a)|^q,|f'(x)|^q\right), \tag{2.4}$$

$$\frac{1}{a+b-x}\int_x^{a+b-x} |f'(t)|^q\,dt \leq L_r\left(|f'(a+b-x)|^q,|f'(b)|^q\right) \tag{2.5}$$

and

$$\frac{1}{x-a}\int_{a+b-x}^b |f'(t)|^q\,dt \leq L_r\left(|f'(x)|^q,|f'(a+b-x)|^q\right) \tag{2.6}$$

via the inequality in (1.1). If we use (2.4)-(2.6) in (2.2) and calculate the integrals in (2.2), we obtain the inequality in (2.1). □

Corollary 1. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Theorem 3, we have the following Ostrowski type inequality:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{1}{(b-a)}\left\{\frac{(x-a)^2}{(p+1)^{\frac{1}{p}}}\left[\left(L_r\left(|f'(a)|^q,|f'(x)|^q\right)\right)^{\frac{1}{q}}\right.\right.\\ &\left.+\left(L_r\left(|f'(a+b-x)|^q,|f'(b)|^q\right)\right)^{\frac{1}{q}}\right]\\ &\left.+\frac{(a+b-2x)^2}{2(p+1)^{\frac{1}{p}}}\left(L_r\left(|f'(x)|^q,|f'(a+b-x)|^q\right)\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Theorem 4. *Let $f:[a,b]\to\mathbb{R}^+$ be an absolutely continuous function on $[a,b]$, $f'\in L[a,b]$. If $|f'|$ is $r-$convex on $[a,b]$, we obtain the following inequality for all $x\in\left[a,\frac{a+b}{2}\right]$ and $r>1$:*

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{1}{(b-a)}\left[\left(\frac{r^2(x-a)^2}{(1+r)(1+2r)}\right)(|f'(a)|+|f'(b)|)\right.\\ &+\left(\frac{r(x-a)^2}{1+2r}+\frac{(a+b-2x)^2\left(2r^2+r2^{1+\frac{1}{r}}\right)}{2^{2+\frac{1}{r}}(1+r)(1+2r)}\right)\\ &\left.\times(|f'(x)|+|f'(a+b-x)|)\right].\end{aligned}\tag{2.7}$$

Proof. Using Lemma 1 and property of modulus, we can write the inequality (2.2). Since $|f'|$ is $r-$convex on $[a,b]$, we have

$$|f'(t)|\leq\left(\frac{x-t}{x-a}|f'(a)|^r+\frac{t-a}{x-a}|f'(x)|^r\right)^{\frac{1}{r}},\ t\in[a,x],\tag{2.8}$$

$$|f'(t)|\leq\left(\frac{a+b-x-t}{a+b-2x}|f'(x)|^r+\frac{t-x}{a+b-2x}|f'(a+b-x)|^r\right)^{\frac{1}{r}},\ t\in(x,a+b-x]\tag{2.9}$$

and

$$|f'(t)|\leq\left(\frac{b-t}{x-a}|f'(a+b-x)|^r+\frac{t-a-b+x}{x-a}|f'(b)|^r\right)^{\frac{1}{r}},\ t\in(a+b-x,b].\tag{2.10}$$

If we use (2.8)-(2.10) in (2.2), we have

$$\begin{aligned}
(2.11)\quad & \left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq & \frac{1}{b-a}\left[\int_a^x |t-a|\left(\frac{x-t}{x-a}\left|f'(a)\right|^r+\frac{t-a}{x-a}\left|f'(x)\right|^r\right)^{\frac{1}{r}}dt\right. \\
& +\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left(\frac{a+b-x-t}{a+b-2x}\left|f'(x)\right|^r\right. \\
& \left.+\frac{t-x}{a+b-2x}\left|f'(a+b-x)\right|^r\right)^{\frac{1}{r}}dt \\
& \left.+\int_{a+b-x}^b |t-b|\left(\frac{b-t}{x-a}\left|f'(a+b-x)\right|^r+\frac{t-a-b+x}{x-a}\left|f'(b)\right|^r\right)^{\frac{1}{r}}dt\right].
\end{aligned}$$

Using the fact that

$$\sum_{i=1}^n (x_i+y_i)^k \leq \sum_{i=1}^n x_i^k+\sum_{i=1}^n y_i^k \tag{2.12}$$

for $0<k<1, x_1,x_2,...,x_n \geq 0$ and $y_1,y_2,...,y_n \geq 0$, we have

$$\begin{aligned}
(2.13)\quad & \left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq & \frac{1}{b-a}\left[\int_a^x (t-a)\left(\left(\frac{x-t}{x-a}\right)^{\frac{1}{r}}\left|f'(a)\right|+\left(\frac{t-a}{x-a}\right)^{\frac{1}{r}}\left|f'(x)\right|\right)dt\right. \\
& +\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left(\left(\frac{a+b-x-t}{a+b-2x}\right)^{\frac{1}{r}}\left|f'(x)\right|\right. \\
& \left.+\left(\frac{t-x}{a+b-2x}\right)^{\frac{1}{r}}\left|f'(a+b-x)\right|\right)dt \\
& +\int_{a+b-x}^b (b-t)\left(\left(\frac{b-t}{x-a}\right)^{\frac{1}{r}}\left|f'(a+b-x)\right|\right. \\
& \left.\left.+\left(\frac{t-a-b+x}{x-a}\right)^{\frac{1}{r}}\left|f'(b)\right|\right)dt\right].
\end{aligned}$$

If we calculate the integrals in (2.13) we have

$$\begin{aligned}
(2.14)\quad \int_a^x (t-a)^{1+\frac{1}{r}}dt &= \int_{a+b-x}^b (b-t)^{1+\frac{1}{r}}dt \\
&= \frac{r\,(x-a)^{2+\frac{1}{r}}}{1+2r},
\end{aligned}$$

$$\int_a^x (t-a)(x-t)^{\frac{1}{r}}\,dt = \int_{a+b-x}^b (b-t)(t-a-b+x)^{\frac{1}{r}}\,dt = \frac{r^2(x-a)^{2+\frac{1}{r}}}{(1+r)(1+2r)} \tag{2.15}$$

and

$$\begin{aligned} &\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|(t-x)^{\frac{1}{r}}\,dt \\ =\ &\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|(a+b-x-t)^{\frac{1}{r}}\,dt \\ =\ &\frac{(a+b-2x)^{2+\frac{1}{r}}\left(2r^2+r2^{1+\frac{1}{r}}\right)}{2^{2+\frac{1}{r}}(1+r)(1+2r)}. \end{aligned} \tag{2.16}$$

If we use (2.14)-(2.16) in (2.13), we obtain the inequality in (2.7). □

Corollary 2. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Theorem 4 we have the following Ostrowski type inequality*

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ &\frac{1}{(b-a)}\left[\left(\frac{r^2(x-a)^2}{(1+r)(1+2r)}\right)(|f'(a)|+|f'(b)|)\right. \\ &+\left(\frac{r(x-a)^2}{1+2r}+\frac{(a+b-2x)^2\left(2r^2+r2^{1+\frac{1}{r}}\right)}{2^{2+\frac{1}{r}}(1+r)(1+2r)}\right) \\ &\left.\times(|f'(x)|+|f'(a+b-x)|)\right]. \end{aligned}$$

Theorem 5. *Under the assumptions of Theorem 3, for $r>1$ we have the following inequality:*

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ &\frac{1}{b-a}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\ &\times\left\{(x-a)^2\left[|f'(a)|^q+|f'(x)|^q\right]^{\frac{1}{q}}\right. \\ &+\frac{(a+b-2x)^2}{2}\left[|f'(x)|^q+|f'(a+b-x)|^q\right]^{\frac{1}{q}} \\ &\left.+(x-a)^2\left[|f'(a+b-x)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.17}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we obtain the inequality in (2.3). If we use $r-$convexity of $|f'|^q$, we can write

$$\int_a^x |f'(t)|^q\,dt \leq \frac{r(x-a)}{1+r}\left[|f'(a)|^q+|f'(b)|^q\right], \tag{2.18}$$

$$\int_{x}^{a+b-x} |f'(t)|^q \, dt \leq \frac{r(a+b-2x)}{1+r} \left[|f'(x)|^q + |f'(a+b-x)|^q \right] \tag{2.19}$$

and

$$\int_{a+b-x}^{b} |f'(t)|^q \, dt \leq \frac{r(x-a)}{1+r} \left[|f'(a+b-x)|^q + |f'(b)|^q \right] \tag{2.20}$$

via inequality in (2.12). If we use (2.18)-(2.20) in (2.2) and calculate the integrals in (2.2), we obtain the inequality in (2.17). □

Corollary 3. *If $f(x) = f(a+b-x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ in Theorem 5, we have the following Ostrowski type inequality*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ \leq & \frac{1}{b-a} \left(\frac{r}{r+1} \right)^{\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \times \left\{ (x-a)^2 \left[|f'(a)|^q + |f'(x)|^q \right]^{\frac{1}{q}} \right. \\ & + \frac{(a+b-2x)^2}{2} \left[|f'(x)|^q + |f'(a+b-x)|^q \right]^{\frac{1}{q}} \\ & \left. + (x-a)^2 \left[|f'(a+b-x)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Theorem 6. *Under the assumptions of Theorem 3, we have the following inequality:*

$$\begin{aligned} & \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ \leq & \frac{1}{b-a} \\ & \times \left\{ \frac{(x-a)^2}{2^{\frac{1}{p}}} \left[\frac{r^2}{(1+r)(1+2r)} |f'(a)|^q + \frac{r}{1+2r} |f'(x)|^q \right]^{\frac{1}{q}} \right. \\ & + \frac{(a+b-2x)^2}{4^{\frac{1}{p}}} \left[\frac{2r^2 + r2^{1+\frac{1}{r}}}{2^{2+\frac{1}{r}}(1+r)(1+2r)} \left(|f'(x)|^q + |f'(a+b-x)|^q \right) \right]^{\frac{1}{q}} \\ & \left. + \frac{(x-a)^2}{2^{\frac{1}{p}}} \left[\frac{r}{1+2r} |f'(a+b-x)|^q + \frac{r^2}{(1+r)(1+2r)} |f'(b)|^q \right] \right\}. \end{aligned} \tag{2.21}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we obtain

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{b-a}\left\{\left(\int_a^x |t-a|\,dt\right)^{\frac{1}{p}}\left(\int_a^x |t-a|\,|f''(t)|^q\,dt\right)^{\frac{1}{q}}\right. \\ &+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt\right)^{\frac{1}{p}}\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right||f''(t)|^q\,dt\right)^{\frac{1}{q}} \\ &\left.+\left(\int_{a+b-x}^b |t-b|\,dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b |t-b|\,|f''(t)|^q\,dt\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.22}$$

If we use $r-$convexity of $|f'|^q$, we can write

$$\int_a^x |t-a|\,|f'(t)|^q\,dt \leq (x-a)^2\left[\frac{r^2}{(1+r)(1+2r)}|f''(a)|^q+\frac{r}{1+2r}|f'(x)|^q\right], \tag{2.23}$$

$$\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right||f'(t)|^q\,dt \leq (a+b-2x)^2\,\frac{2r^2+r2^{1+\frac{1}{r}}}{2^{2+\frac{1}{r}}(1+r)(1+2r)}\left(|f'(x)|^q+|f'(a+b-x)|^q\right) \tag{2.24}$$

and

$$\int_{a+b-x}^b |t-b|\,|f'(t)|^q\,dt \leq (x-a)^2\,\frac{r}{1+2r}|f'(a+b-x)|^q+\frac{r^2}{(1+r)(1+2r)}|f'(b)|^q \tag{2.25}$$

via inequality in (2.12). If we use (2.23)-(2.25) in (2.22) and calculate the integrals in (2.22), we obtain the inequality in (2.21). □

Corollary 4. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Theorem 7, we have the following Ostrowski type inequality*

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{b-a} \\ &\times\left\{\frac{(x-a)^2}{2^{\frac{1}{p}}}\left[\frac{r^2}{(1+r)(1+2r)}|f'(a)|^q+\frac{r}{1+2r}|f'(x)|^q\right]^{\frac{1}{q}}\right. \\ &+\frac{(a+b-2x)^2}{4^{\frac{1}{p}}}\left[\frac{2r^2+r2^{1+\frac{1}{r}}}{2^{2+\frac{1}{r}}(1+r)(1+2r)}\left(|f'(x)|^q+|f'(a+b-x)|^q\right)\right]^{\frac{1}{q}} \\ &\left.+\frac{(x-a)^2}{2^{\frac{1}{p}}}\left[\frac{r}{1+2r}|f'(a+b-x)|^q+\frac{r^2}{(1+r)(1+2r)}|f'(b)|^q\right]\right\}. \end{aligned}$$

Theorem 7. *Under the assumptions of Theorem 3, we have the following inequality:*

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{(b-a)^{1-\frac{1}{q}}} \\ &\times\left(\frac{2\left(x-a\right)^{p+1}}{p+1}+\frac{(a+b-2x)^{p+1}}{2^p(p+1)}\right)^{\frac{1}{p}} \\ &\times\left(L_r\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right)^{\frac{1}{q}}. \end{aligned} \tag{2.26}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we can write

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{b-a}\left(\int_a^x \left|t-a\right|^p dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^p dt\right. \\ &\left.+\int_{a+b-x}^b \left|t-b\right|^p dt\right)^{\frac{1}{p}} \\ &\times\left(\int_a^b \left|f'(t)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.27}$$

Since $|f'|^q$ is $r-$convex on $[a,b]$ we can write

$$\frac{1}{b-a}\int_a^b \left|f'(t)\right|^q dt\leq L_r\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right) \tag{2.28}$$

via the inequality in (1.1). If we use (2.28) in (2.27) and calculate the integrals we obtain the inequality in (2.26). □

Corollary 5. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Theorem 7, we have the following Ostrowski type inequality*

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{1}{(b-a)^{1-\frac{1}{q}}} \\ &\times\left(\frac{2\left(x-a\right)^{p+1}}{p+1}+\frac{(a+b-2x)^{p+1}}{2^p(p+1)}\right)^{\frac{1}{p}} \\ &\times\left(L_r\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right)^{\frac{1}{q}}. \end{aligned}$$

References

[1] A. Ostrowski, Über die absolutabweichung einer differentiebaren function von ihren integralmittelwert. Commentarii Mathematici Helvetici, 10(1938) 226-227.

[2] P. M. Gill, C. E. M. Pearce and J. Pečarić, Hadamard's inequality for $r-$convex functions.Journal of Mathematical Analysis and Applications, 215 (1997) 461-470.

[3] M. W. N. Alomari, Several Inequalities of Hermite-Hadamard, Ostrowski and Simpson type for $s-$convex, quasi-convex and $r-$convex mappings and applications. Phd. Thesis, Faculty of Science and Technology, Universiti Kebangsaan, Malaysia 2011.

[4] F. Chen and X. Liu, Refinements on the Hermite-Hadamard Inequalities for$r-$Convex Functions. J. Appl. Math., vol. 2013, no. 1-2, p. 5, 2013, doi: 10.1155/2013/978493.

[5] S. Dragomir and C. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Its Applications. Victoria University: RGMIA Monograph, 2002.

[6] L. Han and L. Liu, Integral Inequalities of Hermite-Hadamard Type for $r-$Convex Functions. Appl. Math., no. 3, pp. 1967–1971, 2012, doi: 10.4236/am.2012.312270.

[7] N. Ngoc, N. Vinh, and P. Hien, Integral inequalities of Hadamard type for $r-$convex functions. Int. Math. Forum, vol. 4, no. 35, pp. 1723–1728, 2009.

[8] G. Yang and D. Hwang, Refinements of Hadamard inequality for $r-$convex functions. Indian J.Pure Appl.Math., vol. 32, no. 10, pp. 1571–1579, 2001.

[9] G. Zabandan, A. Bodagh, and A. Kilicman, The Hermite-Hadamard inequality for $r-$convex functions.Journal of Inequalities and Applications, vol. 2012, no. 1, p. 215, 2012.

[10] S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Bull. Korean Math. Soc. 42(2005), No:2, pp:213-230.

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