SECOND DERIVATIVE LIPSCHITZ TYPE INEQUALITIES FOR THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{C}(w, \mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. We show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned} & \|\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) - D(\mathcal{C}(w,\mu))(A)(B-A)\| \\ & \leq \|B-A\|^2 \\ & \times \begin{cases} \frac{\mathcal{C}(w,\mu)(m_2) - \mathcal{C}(w,\mu)(m_1) - (m_2-m_1)\mathcal{C}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1 \neq m_2, \\ \\ \frac{1}{2}\mathcal{C}''(w,\mu)(m) & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as a function of operator and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function. If $p \in [-1, 0] \cup [1, 2]$, then

$$\left\|\sum_{k=1}^n p_k A_k^p - \left(\sum_{k=1}^n p_k A_k\right)^p\right\| \leq \frac{1}{2} p(p-1) m^{p-2} \sum_{k=1}^n p_k \left\|A_k - \sum_{j=1}^n p_j A_j\right\|^2$$

provided that $A_k \geq m > 0$, $k \in \{1, ..., n\}$. We also have the midpoint inequality,

$$\left\|\int_0^1 ((1-t)A + tB)^p dt - \left(\frac{A+B}{2}\right)^p\right\| \leq \frac{1}{24} p(p-1) m^{p-2} \|B-A\|^2$$

and the trapezoid inequality,

$$\left\|\frac{A^p + B^p}{2} - \int_0^1 ((1-t)A + tB)^p dt\right\| \leq \frac{1}{12} p(p-1) m^{p-2} \|B-A\|^2,$$

provided that $A, B \geq m > 0$.

Some norm inequalities for the logarithm are given as well.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Midpoint inequality, Trapezoid inequality.

RGMIA Res. Rep. Coll. 23 (2020), Art. 110, 23 pp. Received 04/08/20

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [15], the following inequality holds

$$\||A| - |B|\| \le \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right)$$

for any $A, B \in \mathcal{B}(H)$ with $A \ne B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \le \sqrt{2} \|A - B\|_{HS}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \le a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right)$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \le f'(a) \|A - B\|$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \ge a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \ne 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [16], see for instance [5, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.9}$$

where $a\in\mathbb{R}$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.11).

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)(t):=t^2\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{C}(w,\mu)(t)&:=\int_0^\infty w(\lambda)t^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)^2-2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)-2\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.13}$$

If $\int_0^\infty w(\lambda)d\mu(\lambda)<\infty$ and $\int_0^\infty w(\lambda)\lambda d\mu(\lambda)<\infty$, then we conclude that

$$\mathcal{C}(w,\mu)(t)=-\int_0^\infty w(\lambda)\lambda d\mu(\lambda)+t\int_0^\infty w(\lambda)d\mu(\lambda)+\mathcal{D}(\ell^2w,\mu)(t), \tag{1.14}$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq0$ and $a>0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq0$$

where the *exponential integral* is given by

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty w(\lambda)\lambda d\lambda = \int_0^\infty \exp(-a\lambda)\lambda d\lambda = \frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t) := t^2\mathcal{D}(e_{-a},\mu)(t) = t^2 E_1(at)\exp(at), \ t>0. \tag{1.15}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda}d\lambda$$

then by (1.14) we get

$$t^2E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t) = t^2E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2}, \ t>0, a>0.$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)d\lambda = \infty$ and the equality (1.14) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda)T^2(\lambda+T)^{-1}d\mu(\lambda), \tag{1.16}$$

provided the integral exist. By (1.13) we also have

$$\mathcal{C}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[T-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.17}$$

In this paper, we show among others that, if $A\geq m_1>0$, $B\geq m_2>0$, then

$$\begin{aligned}&\|\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) - D(\mathcal{C}(w,\mu))(A)(B-A)\|\\ &\leq \|B-A\|^2\\ &\times\begin{cases}\frac{\mathcal{C}(w,\mu)(m_2)-\mathcal{C}(w,\mu)(m_1)-(m_2-m_1)\mathcal{C}'(w,\mu)(m_1)}{(m_2-m_1)^2} \text{ if } m_1\neq m_2,\\ \\ \frac{1}{2}\mathcal{C}''(w,\mu)(m) \text{ if } m_1=m_2=m,\end{cases}\end{aligned}$$

where $D(\mathcal{C}(w,\mu))$ is the Fréchet derivative of $\mathcal{C}(w,\mu)$ as a function of operator and $\mathcal{C}''(w,\mu)$ is the second derivative of $\mathcal{C}(w,\mu)$ as a real function. If $p\in[-1,0]\cup[1,2]$, then

$$\left\|\sum_{k=1}^n p_kA_k^p - \left(\sum_{k=1}^n p_kA_k\right)^p\right\| \leq \frac{1}{2}p(p-1)m^{p-2}\sum_{k=1}^n p_k\left\|A_k - \sum_{j=1}^n p_jA_j\right\|^2$$

provided that $A_k\geq m>0$, $k\in\{1,...,n\}$. We also have the midpoint inequality,

$$\left\|\int_0^1((1-t)A+tB)^p dt - \left(\frac{A+B}{2}\right)^p\right\| \leq \frac{1}{24}p(p-1)m^{p-2}\|B-A\|^2$$

and the trapezoid inequality,

$$\left\|\frac{A^p+B^p}{2}-\int_0^1((1-t)A+tB)^p\,dt\right\|\leq\frac{1}{12}p(p-1)m^{p-2}\|B-A\|^2,$$

provided that $A,\ B\geq m>0$.

Some norm inequalities for the logarithm are given as well.

2. General Lipschitz Inequalities

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w,\mu))$:

Lemma 1. *For all $A>0$,*

$$\begin{aligned}(2.1)\qquad & D(\mathcal{C}(w,\mu))(A)(V)\\ &=\int_0^\infty w(\lambda)(A+\lambda)^{-1}[AVA+\lambda(VA+AV)](A+\lambda)^{-1}\,d\mu(\lambda)\end{aligned}$$

for all $V\in S(H)$, the class of all selfadjoint operators on H.

Proof. Let $A>0$ and $V\in S(H)$. By the definition of $\mathcal{C}(w,\mu)$ and by (1.13) and we have for t in a small open interval around 0 that

$$\mathcal{C}(w,\mu)(A+tV)=\int_0^\infty w(\lambda)\left[A+tV-\lambda+\lambda^2(A+tV+\lambda)^{-1}\right]d\mu(\lambda).$$

Then

$$\begin{aligned}&\mathcal{C}(w,\mu)(A+tV)-\mathcal{C}(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)\left[A+tV-\lambda+\lambda^2(A+tV+\lambda)^{-1}\right]d\mu(\lambda)\\ &-\int_0^\infty w(\lambda)\left[A-\lambda+\lambda^2(A+\lambda)^{-1}\right]d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(A+tV+\lambda)^{-1}-(A+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(A+tV+\lambda)^{-1}(-tV)(A+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\ &=t\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(A+tV+\lambda)^{-1}V(A+\lambda)^{-1}\right]\right\}d\mu(\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}(2.2)\qquad & D(\mathcal{C}(w,\mu))(A)(V)\\ &=\lim_{t\to 0}\frac{\mathcal{C}(w,\mu)(A+tV)-\mathcal{C}(w,\mu)(A)}{t}\\ &=\lim_{t\to 0}\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(A+tV+\lambda)^{-1}V(A+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)\left\{V-\lambda^2(A+\lambda)^{-1}V(A+\lambda)^{-1}\right\}d\mu(\lambda)\end{aligned}$$

for $A>0$ and $V\in S(H)$.

Define for $\lambda\geq 0$,

$$U_\lambda:=V-\lambda^2(A+\lambda)^{-1}V(A+\lambda)^{-1}.$$

If we multiply U_λ both sides by $A+\lambda$, then we get

$$\begin{aligned}(A+\lambda)U_\lambda(A+\lambda) &= (A+\lambda)V(A+\lambda)-\lambda^2 V\\ &= (AV+\lambda V)(A+\lambda)-\lambda^2 V\\ &= AVA+\lambda VA+\lambda AV+\lambda^2 V-\lambda^2 V\\ &= AVA+\lambda(VA+AV).\end{aligned}$$

If we multiply both sides by $(A+\lambda)^{-1}$ we get

$$U_\lambda = (A+\lambda)^{-1}\left[AVA+\lambda(VA+AV)\right](A+\lambda)^{-1},$$

which, by (2.2), implies the representation (2.1). □

For the case of second Fréchet derivative $D^2(\mathcal{C}(w,\mu))$, we have the representation:

Lemma 2. *For all $A>0$,*

$$D^2(\mathcal{C}(w,\mu))(A)(V,V) = 2\int_0^\infty \lambda^2 w(\lambda)(\lambda+A)^{-1}V(\lambda+A)^{-1}V(\lambda+A)^{-1} \tag{2.3}$$

for all $V\in S(H)$.

Proof. Let $A>0$ and $V\in S(H)$. We have by the definition of the Fréchet second derivative that

$$D^2(\mathcal{C}(w,\mu))(A)(V,V) = \lim_{t\to 0}\frac{D(\mathcal{C}(w,\mu))(A+tV)(V)-D(\mathcal{C}(w,\mu))(A)(V)}{t}. \tag{2.4}$$

Observe, by (2.2), that we have for t in a small open interval around 0,

$$\begin{aligned}&D(\mathcal{C}(w,\mu))(A+tV)(V)\\ &= \int_0^\infty w(\lambda)\left\{V-\lambda^2(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}\right\}d\mu(\lambda).\end{aligned}$$

Therefore

$$\begin{aligned}&D(\mathcal{C}(w,\mu))(A+tV)(V)-D(\mathcal{C}(w,\mu))(A)(V)\\ &= \int_0^\infty w(\lambda)\left\{V-\lambda^2(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}\right\}d\mu(\lambda)\\ &- \int_0^\infty w(\lambda)\left\{V-\lambda^2(A+\lambda)^{-1}V(A+\lambda)^{-1}\right\}d\mu(\lambda)\\ &= \int_0^\infty \lambda^2 w(\lambda)\\ &\times\left[(A+\lambda)^{-1}V(A+\lambda)^{-1}-(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}\right].\end{aligned} \tag{2.5}$$

Define

$$W_{t,\lambda} := (A+\lambda)^{-1}V(A+\lambda)^{-1}-(A+tV+\lambda)^{-1}V(A+tV+\lambda)^{-1}.$$

If we multiply both sides of $W_{t,\lambda}$ with $\lambda + A + tV$, the we get

$$\begin{aligned}
&(\lambda + A + tV) W_{t,\lambda} (\lambda + A + tV) \\
&= (\lambda + A + tV) (\lambda + A)^{-1} V (\lambda + A)^{-1} (\lambda + A + tV) - V \\
&= \left(1 + tV (\lambda + A)^{-1}\right) V \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= \left(V + tV (\lambda + A)^{-1} V\right) \left(1 + t (\lambda + A)^{-1} V\right) - V \\
&= V + tV (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V \\
&+ t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V - V \\
&= 2tV (\lambda + A)^{-1} V + t^2 V (\lambda + A)^{-1} V (\lambda + A)^{-1} V \\
&= t \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right].
\end{aligned}$$

If we multiply the equality by $(\lambda + A + tV)^{-1}$ both sides, we get for $t \neq 0$,

$$\begin{aligned}
\frac{W_{t,\lambda}}{t} &= (\lambda + A + tV)^{-1} \left[2V (\lambda + A)^{-1} V + tV (\lambda + A)^{-1} V (\lambda + A)^{-1} V\right] \\
&\times (\lambda + A + tV)^{-1}.
\end{aligned} \tag{2.6}$$

If we take the limit over $t \to 0$ in (2.6), then we get

$$\lim_{t \to 0} \left(\frac{W_{t,\lambda}}{t}\right) = 2 (\lambda + A)^{-1} V (\lambda + A)^{-1} V (\lambda + A)^{-1}. \tag{2.7}$$

Then by (2.4), (2.5) and (2.7) we derive (2.3). □

We have the following representation for the transform $\mathcal{C}(w, \mu)$:

Lemma 3. *For all* A, $B > 0$,

$$\begin{aligned}
&\mathcal{C}(w, \mu)(B) - \mathcal{C}(w, \mu)(A) - D(\mathcal{C}(w, \mu))(A)(B - A) \\
&= 2 \int_0^1 (1 - s) \left[\int_0^\infty \lambda^2 w(\lambda) (\lambda + (1 - s) A + sB)^{-1} (B - A)\right. \\
&\left.\times (\lambda + (1 - s) A + sB)^{-1} (B - A) (\lambda + (1 - s) A + sB)^{-1} d\mu(\lambda)\right] ds \\
&\geq 0.
\end{aligned} \tag{2.8}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [6, p. 112],

$$\begin{aligned}
f(E) &= f(C) + D(f)(C)(E - C) \\
&+ \int_0^1 (1 - s) D^2(f)((1 - s) C + sE)(E - C, E - C) ds
\end{aligned} \tag{2.9}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $B(H)$ and C, $E \in \mathcal{O}$.

If we write (2.9) for $\mathcal{C}(w, \mu)$ and A, $B > 0$, then we get

$$\begin{aligned}
\mathcal{C}(w, \mu)(B) &= \mathcal{C}(w, \mu)(A) + D(\mathcal{C}(w, \mu))(A)(B - A) \\
&+ \int_0^1 (1 - s) D^2(\mathcal{C}(w, \mu))((1 - s) A + sB)(B - A, B - A) ds.
\end{aligned} \tag{2.10}$$

By making use of (2.3) we obtain

$$\begin{aligned}
&D^2 (\mathcal{C}(w,\mu))((1-s)A+sB)(B-A,B-A) \\
&= \left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-s)A+sB)^{-1}(B-A)\right. \\
&\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds,
\end{aligned}$$

which, by (2.10), produces the equality in (2.8).

Now, observe that

$$(\lambda+(1-s)A+sB)^{-1}\geq 0$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

By multiplying this inequality both sides by $(B-A)$, we derive

$$(B-A)(\lambda+(1-s)A+sB)^{-1}(B-A)\geq 0$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

Moreover, if we multiply this inequality both sides by $(\lambda+(1-s)A+sB)^{-1}$ we obtain

$$\begin{aligned}
&(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1} \\
&(B-A)(\lambda+(1-s)A+sB)^{-1} \\
&\geq 0
\end{aligned}$$

for all $\lambda\geq 0$ and $s\in[0,1]$.

Finally, by multiplying with $(1-s)\lambda^2 w(\lambda)\geq 0$ and integrating we deduce the inequality part in (2.8). □

We have the following Lipschitz type inequality:

Theorem 3. *Assume that* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\begin{aligned}
&\|\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\| \\
&\leq\|B-A\|^2 \\
&\times\begin{cases}\frac{\mathcal{C}(w,\mu)(m_2)-\mathcal{C}(w,\mu)(m_1)-(m_2-m_1)\mathcal{C}'(w,\mu)(m_1)}{(m_2-m_1)^2} & \text{if } m_1\neq m_2,\\ \\ \frac{1}{2}\mathcal{C}''(w,\mu)(m) & \text{if } m_1=m_2=m.\end{cases}
\end{aligned}\tag{2.11}$$

Proof. From (2.8) we get by taking the norm,

$$\begin{aligned}
&\|\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\| \\
&\leq 2\int_0^1(1-t)\left[\int_0^\infty\lambda^2 w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\
&\left.\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\right\|d\mu(\lambda)\right]dt \\
&\leq 2\|B-A\|^2 \\
&\times\int_0^1(1-t)\left(\int_0^\infty\lambda^2 w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt.
\end{aligned}\tag{2.12}$$

Assume that $m_2>m_1$. Then

$$(1-t)A+tB+\lambda\geq(1-t)m_1+tm_2+\lambda,$$

which implies that

$$((1-t)A+tB+\lambda)^{-1} \leq ((1-t)m_1+tm_2+\lambda)^{-1},$$

and

$$\left\|((1-t)A+tB+\lambda)^{-1}\right\|^3 \leq ((1-t)m_1+tm_2+\lambda)^{-3} \tag{2.13}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore, by integrating (2.13) we derive

$$\begin{aligned}
&\int_0^1 (1-t)\left(\int_0^\infty \lambda^2 w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt \quad (2.14)\\
&\leq \int_0^1 (1-t)\left(\int_0^\infty \lambda^2 w(\lambda)((1-t)m_1+tm_2+\lambda)^{-3} d\mu(\lambda)\right)dt\\
&= \frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\left(\int_0^\infty \lambda^2 w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}\right.\\
&\times (m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\
&\left.\times((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\,dt.
\end{aligned}$$

From (2.8) we have for $m_2>m_1$ that

$$\begin{aligned}
&\mathcal{C}(w,\mu)(m_2)-\mathcal{C}(w,\mu)(m_1)-\mathcal{C}'(w,\mu)(m_1)(m_2-m_1) \quad (2.15)\\
&=2\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right.\\
&\times((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\
&\left.\times\ ((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\Big]\,dt.
\end{aligned}$$

Then by (2.15) we get

$$\begin{aligned}
&\frac{1}{2(m_2-m_1)^2}\left[\mathcal{C}(w,\mu)(m_2)-\mathcal{C}(w,\mu)(m_1)-(m_2-m_1)\mathcal{C}'(w,\mu)(m_1)\right] \quad (2.16)\\
&=\frac{1}{(m_2-m_1)^2}\int_0^1 (1-t)\left(\int_0^\infty w(\lambda)((1-t)m_1+tm_2+\lambda)^{-1}\right.\\
&\times(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\\
&\left.((1-t)m_1+tm_2+\lambda)^{-1}\right)d\mu(\lambda)\,dt.
\end{aligned}$$

By utilising (2.12) and (2.14)-(2.16) we derive (2.11).

The case $m_2<m_1$ goes in a similar way and we also obtain (2.11).

Assume that $m_2=m_1>0$. Let $\epsilon>0$. Then $B+\epsilon\geq m+\epsilon>m$. By the first inequality for $m_2=m+\epsilon$ and $m_1=m$, we have

$$\begin{aligned}
&\|\mathcal{C}(w,\mu)(B+\epsilon)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B+\epsilon-A)\| \quad (2.17)\\
&\leq\|B+\epsilon-A\|^2\frac{1}{\epsilon^2}\left[\mathcal{C}(w,\mu)(m+\epsilon)-\mathcal{C}(w,\mu)(m)-\epsilon\mathcal{C}'(w,\mu)(m)\right].
\end{aligned}$$

By Taylor's expansion theorem with the Lagrange remainder we have

$$\mathcal{C}(w,\mu)(m+\epsilon)-\mathcal{C}(w,\mu)(m)-\epsilon\mathcal{C}'(w,\mu)(m)=\frac{1}{2}\epsilon^2\mathcal{C}''(w,\mu)(\zeta_\epsilon)$$

with $m+\epsilon>\zeta_\epsilon>m$. Therefore

$$\lim_{\epsilon\to 0+}\frac{1}{\epsilon^2}\left[\mathcal{C}(w,\mu)(m+\epsilon)-\mathcal{C}(w,\mu)(m)-\epsilon\mathcal{C}'(w,\mu)(m)\right]=\frac{1}{2}\mathcal{C}''(w,\mu)(m)$$

and by taking the limit $\epsilon\to 0+$ in (2.17) then we get

$$\begin{aligned}&\left\|\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)-D(\mathcal{C}(w,\mu))(A)(B-A)\right\|\\&\le\frac{1}{2}\left\|B-A\right\|^2\mathcal{C}''(w,\mu)(m)\end{aligned}$$

and the second part of (2.11) is proved. □

We have the following error bounds for *operator Jensen's gap* related to the n-tuple of positive operators $\mathbf{A}=(A_1,...,A_n)$ and probability density n-tuple $\mathbf{p}=(p_1,...,p_n)$,

$$J(\mathbf{A},\mathbf{p},\mathcal{C}(w,\mu)):=\sum_{k=1}^n p_k\mathcal{C}(w,\mu)(A_k)-\mathcal{C}(w,\mu)\left(\sum_{k=1}^n p_kA_k\right).$$

Theorem 4. *Assume that* $A_i\ge m>0$ *for* $i\in\{1,...,n\}$ *and consider the probability density* n*-tuple* $\mathbf{p}=(p_1,...,p_n)$, *then*

$$\begin{aligned}\left\|J(\mathbf{A},\mathbf{p},\mathcal{C}(w,\mu))\right\|&\le\frac{1}{2}\mathcal{C}''(w,\mu)(m)\sum_{k=1}^n p_k\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2\\&\le\frac{1}{2}\mathcal{C}''(w,\mu)(m)\sum_{k=1}^n\sum_{j=1}^n p_jp_k\left\|A_k-A_j\right\|^2\\&\le\frac{1}{2}\mathcal{C}''(w,\mu)(m)\max_{k,j\in\{1,...,n\}}\left\|A_k-A_j\right\|^2.\end{aligned}\tag{2.18}$$

Proof. From (2.11) we get

$$\begin{aligned}&\Bigg\|\mathcal{C}(w,\mu)(A_k)-\mathcal{C}(w,\mu)\left(\sum_{j=1}^n p_jA_j\right)\\&-D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^n p_jA_j\right)\left(A_k-\sum_{j=1}^n p_jA_j\right)\Bigg\|\\&\le\frac{1}{2}\mathcal{C}''(w,\mu)(m)\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2\end{aligned}\tag{2.19}$$

for all $k\in\{1,...,n\}$.

If we multiply this inequality by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\begin{aligned}
(2.20)\quad & \sum_{k=1}^{n} \left\| p_k \mathcal{C}(w,\mu)(A_k) - p_k \mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \right. \\
& \left. - D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(p_k A_k - \sum_{j=1}^{n} p_j A_j\right) \right\| \\
& \leq \frac{1}{2}\mathcal{C}''(w,\mu)(m)\sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 .
\end{aligned}$$

By making use of the triangle inequality for norms, we also have

$$\begin{aligned}
(2.21)\quad & \sum_{k=1}^{n} \left\| p_k \mathcal{C}(w,\mu)(A_k) - p_k \mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \right. \\
& \left. - D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(p_k A_k - \sum_{j=1}^{n} p_j A_j\right) \right\| \\
& \geq \left\| \sum_{k=1}^{n} p_k \mathcal{C}(w,\mu)(A_k) - \sum_{k=1}^{n} p_k \mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \right. \\
& \left. - D(\mathcal{C}(w,\mu))\left(\sum_{j=1}^{n} p_j A_j\right)\left(\sum_{k=1}^{n} p_k A_k - \sum_{j=1}^{n} p_j A_j\right) \right\| \\
& = \left\| \sum_{k=1}^{n} p_k \mathcal{C}(w,\mu)(A_k) - \mathcal{C}(w,\mu)\left(\sum_{j=1}^{n} p_j A_j\right) \right\| .
\end{aligned}$$

By utilising (2.20) and (2.21) we deduce the first part of (2.18). The rest is obvious. □

3. Midpoint and Trapezoid Inequalities

For a continuous function f on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $f_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$f_{A,B}(t) := f((1-t)A + tB),\ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 4. *Assume that the operator function generated by f is twice Fréchet differentiable in each $A>0$, then for $B>0$ we have that $f_{A,B}$ is twice differentiable on $[0,1]$,*

$$\frac{df_{A,B}(t)}{dt} = D(f)((1-t)A+tB)(B-A) \tag{3.1}$$

and

$$\frac{d^2 f_{A,B}(t)}{dt^2} = D^2(f)((1-t)A+tB)(B-A,B-A) \tag{3.2}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{f_{A,B}(t+h) - f(t)}{h} \\ & = \frac{f((1-(t+h))A + (t+h)B) - f((1-t)A + tB)}{h} \\ & = \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} \frac{df_{A,B}(t)}{dt} & = \lim_{h\to 0} \frac{f_{A,B}(t+h) - f(t)}{h} \\ & = \lim_{h\to 0} \left[\frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \right] \\ & = D(f)((1-t)A + tB)(B-A), \end{aligned}$$

which proves (3.1).

Similarly,

$$\begin{aligned} & \frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt} \right] \\ & = \frac{D(f)((1-(t+h))A + (t+h)B)(B-A) - D(f)((1-t)A + tB)(B-A)}{h} \\ & = \frac{D(f)((1-t)A + tB + h(B-A))(B-A) - D(f)((1-t)A + tB)(B-A)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} \frac{d^2 f_{A,B}(t)}{dt^2} & = \lim_{h\to 0} \left\{ \frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt} - \frac{df_{A,B}(t)}{dt} \right] \right\} \\ & = D^2(f)((1-t)A + tB)(B-A, B-A), \end{aligned}$$

which proves (3.2). □

For the transform $\mathcal{C}(w,\mu)(t)$ defined in the introduction, we consider the auxiliary function

$$\mathcal{C}(w,\mu)_{A,B}(t) := \mathcal{C}(w,\mu)((1-t)A + tB)$$

where A, $B > 0$ and $t \in [0,1]$.

Corollary 1. *For all A, $B > 0$ and $t \in [0,1]$,*

$$\begin{aligned} & \frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt} \\ & = D(\mathcal{C}(w,\mu))((1-t)A + tB)(B-A) \\ & = \int_0^\infty w(\lambda)((1-t)A + tB + \lambda)^{-1} \\ & \times [((1-t)A + tB + \lambda)(B-A)((1-t)A + tB + \lambda) \\ & + \lambda((B-A)((1-t)A + tB + \lambda) + ((1-t)A + tB + \lambda)(B-A))] \\ & \times ((1-t)A + tB + \lambda)^{-1} d\mu(\lambda) \end{aligned} \tag{3.3}$$

and

$$\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2} \tag{3.4}$$
$$= D^2(\mathcal{C}(w,\mu))((1-t)A+tB)(B-A,B-A)$$
$$= 2\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)$$
$$\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).$$

We have the following identity for the midpoint rule:

Lemma 5. *For all A, $B>0$ we have the identity*

$$\int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \tag{3.5}$$
$$= 2\int_0^1\left(t-\frac{1}{2}\right)^2\left\{\int_0^1(1-s)\right.$$
$$\times\left[\int_0^\infty \lambda^2 w(\lambda)\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)\right.$$
$$\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}(B-A)$$
$$\left.\left.\times\left(\lambda+(1-s)\frac{A+B}{2}+s((1-t)A+tB)\right)^{-1}d\mu(\lambda)\right]ds\right\}dt.$$

Proof. From (2.5) we have for $B=E>0$ and $A=C>0$ that

$$\mathcal{C}(w,\mu)(E)$$
$$=\mathcal{C}(w,\mu)(C)+D(\mathcal{C}(w,\mu))((1-t)C+tE)(E-C)$$
$$+2\int_0^1(1-s)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda+(1-s)C+sE)^{-1}(E-C)\right.$$
$$\left.\times(\lambda+(1-s)C+sE)^{-1}(E-C)(\lambda+(1-s)C+sE)^{-1}d\mu(\lambda)\right]ds,$$

which implies for $E=(1-t)A+tB$, $t\in[0,1]$ and $C=\frac{A+B}{2}$, that

$$\mathcal{C}(w,\mu)((1-t)A+tB) \tag{3.6}$$
$$=\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)$$
$$+\left(t-\frac{1}{2}\right)D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)(B-A)$$
$$+2\left(t-\frac{1}{2}\right)^2\int_0^1(1-s)$$

$$\times \left[\int_0^\infty \lambda^2 w(\lambda) \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A)\right.$$
$$\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A)$$
$$\left.\left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} d\mu(\lambda)\right] ds.$$

If we integrate (3.6) over $t \in [0,1]$, then we get

$$\int_0^1 \mathcal{C}(w,\mu)((1-t)A + tB)\,dt$$
$$= \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)$$
$$- \int_0^1 \left(t - \frac{1}{2}\right) dt D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)(B-A)$$
$$+ 2\int_0^1 \left(t - \frac{1}{2}\right)^2 \left\{\int_0^1 (1-s)\right.$$
$$\times \left[\int_0^\infty w(\lambda) \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A)\right.$$
$$\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A)$$
$$\left.\left.\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} d\mu(\lambda)\right] ds\right\} dt$$

and since $\int_0^1 \left(t - \frac{1}{2}\right) dt = 0$, hence the identity (3.5) is proved. □

We have the following identity for the trapezoid rule:

Lemma 6. *For all A, $B > 0$ we have the identity*

$$\frac{\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A + tB)\,dt \tag{3.7}$$
$$= \int_0^1 t(1-t)\left[\int_0^\infty \lambda^2 w(\lambda)(\lambda + (1-t)A + tB)^{-1}(B-A)\right.$$
$$\left.\times (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1} d\mu(\lambda)\right] dt.$$

Proof. Using integration by parts for the Bochner integral, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt\\
&=\frac{1}{2}\left[t(1-t)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}\bigg|_0^1-\int_0^1(1-2t)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}dt\right]\\
&=\int_0^1\left(t-\frac{1}{2}\right)\frac{d\mathcal{C}(w,\mu)_{A,B}(t)}{dt}dt\\
&=\left(t-\frac{1}{2}\right)\mathcal{C}(w,\mu)_{A,B}(t)\bigg|_0^1-\int_0^1\mathcal{C}(w,\mu)_{A,B}(t)\,dt\\
&=\frac{1}{2}\left[\mathcal{C}(w,\mu)_{A,B}(1)+\mathcal{C}(w,\mu)_{A,B}(0)\right]-\int_0^1\mathcal{C}(w,\mu)_{A,B}(t)\,dt,
\end{aligned}$$

that gives the identity

$$\begin{aligned}
&\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\int_0^1\mathcal{C}(w,\mu)((1-t)A+tB)\,dt\\
&=\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt.
\end{aligned}\tag{3.8}$$

By (2.10) we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 t(1-t)\frac{d^2\mathcal{C}(w,\mu)_{A,B}(t)}{dt^2}dt\\
&=\int_0^1 t(1-t)\left[\int_0^\infty\lambda^2 w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\
&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt.
\end{aligned}\tag{3.9}$$

By making use of (3.8) and (3.9). □

We have the following midpoint norm inequality:

Theorem 5. *If A, $B\geq m>0$ for some constant m, then*

$$\begin{aligned}
&\left\|\int_0^1\mathcal{C}(w,\mu)((1-t)A+tB)\,dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right\|\\
&\leq\frac{1}{24}\mathcal{C}''(w,\mu)(m)\|B-A\|^2.
\end{aligned}\tag{3.10}$$

Proof. From (3.5) we have for all $t\in[0,1]$ and A, $B\geq m>0$,

$$\begin{aligned}
&\left\|\mathcal{C}(w,\mu)((1-t)A+tB)-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right.\\
&\left.-D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\|\\
&\leq\frac{1}{2}\mathcal{C}''(w,\mu)(m)\left\|(1-t)A+tB-\frac{A+B}{2}\right\|^2\\
&=\frac{1}{2}\mathcal{C}''(w,\mu)(m)\left(t-\frac{1}{2}\right)^2\|B-A\|^2.
\end{aligned}\tag{3.11}$$

If we integrate this inequality, we get

$$\begin{aligned}
&\int_0^1 \left\| \mathcal{C}(w,\mu)((1-t)A+tB) - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \right. \\
&\left. -D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\| dt \\
&\leq \frac{1}{2}\mathcal{C}''(w,\mu)(m)\|B-A\|^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 dt \\
&= \frac{1}{24}\mathcal{C}''(w,\mu)(m)\|B-A\|^2 .
\end{aligned} \tag{3.12}$$

Using the properties of norm and integral, we also have

$$\begin{aligned}
&\int_0^1 \left\| \mathcal{C}(w,\mu)((1-t)A+tB) - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \right. \\
&\left. -D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)\left((1-t)A+tB-\frac{A+B}{2}\right)\right\| dt \\
&\geq \left\| \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \right. \\
&\left. -\left(\int_0^1 \left(t-\frac{1}{2}\right)dt\right) D(\mathcal{C}(w,\mu))\left(\frac{A+B}{2}\right)(B-A)\right\| \\
&= \left\| \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right\| .
\end{aligned} \tag{3.13}$$

By employing (3.12) and (3.13) we derive the desired result (3.10). □

Theorem 6. *If A, $B \geq m > 0$ for some constant m, then*

$$\begin{aligned}
&\left\| \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \right\| \\
&\leq \frac{1}{12}\mathcal{C}''(w,\mu)(m)\|B-A\|^2 .
\end{aligned} \tag{3.14}$$

Proof. By taking the norm in (2.15), we obtain

$$\begin{aligned}
&\left\| \frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2} - \int_0^1 \mathcal{C}(w,\mu)((1-t)A+tB)\,dt \right\| \\
&\leq \|B-A\|^2 \\
&\times \int_0^1 t(1-t)\left(\int_0^\infty \lambda^2 w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right) dt.
\end{aligned} \tag{3.15}$$

Since A, $B \geq m > 0$, then for $\lambda \geq 0$ and $t \in [0,1]$,

$$\lambda + (1-t)A + tB \geq \lambda + m,$$

which implies that

$$(\lambda + (1-t)A + tB)^{-1} \leq (\lambda + m)^{-1} .$$

This implies that

$$\left\|(\lambda + (1-t)A + tB)^{-1}\right\|^3 \leq (\lambda + m)^{-3}$$

for $\lambda \geq 0$ and $t \in [0,1]$.

By multiplying this inequality by $t(1-t)w(\lambda)\geq 0$ and integrating we get

$$\begin{aligned}(3.16)\quad &\int_0^1 t(1-t)\left(\int_0^\infty \lambda^2 w(\lambda)\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^3 d\mu(\lambda)\right)dt\\ &\leq\left(\int_0^1 t(1-t)\,dt\right)\left(\int_0^\infty \lambda^2 w(\lambda)(\lambda+m)^{-3}d\mu(\lambda)\right)\\ &=\frac{1}{6}\int_0^\infty \lambda^2 w(\lambda)(\lambda+m)^{-3}d\mu(\lambda).\end{aligned}$$

Taking the derivative over t twice in (1.13), we get

$$\mathcal{C}''(w,\mu)(t):=2\int_0^\infty \frac{\lambda^2 w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\ t>0,$$

that gives

$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+m)^{-3}d\mu(\lambda)=\frac{1}{2}\mathcal{C}''(w,\mu)(m)$$

and by (3.15) and (3.16) we derive (3.14). □

4. Examples for Operator Convex Functions

Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda),$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and μ a positive measure on $(0,\infty)$. Then

$$(4.1)\qquad \mathcal{C}(\ell,\mu)(t)=f(t)-a-bt-ct^2,\ t>0$$

where $\ell(\lambda)=\lambda$, $\lambda\geq 0$.

We have for A, $B>0$ that

$$\begin{aligned}&\mathcal{C}(\ell,\mu)(B)-\mathcal{C}(\ell,\mu)(A)-D(\mathcal{C}(\ell,\mu))(A)(B-A)\\ &=f(B)-bB-cB^2-f(A)+bA+cA^2\\ &-D(f)(A)(B-A)+bD(\ell)(A)(B-A)+cD\left(\ell^2\right)(A)(B-A)\\ &=f(B)-bB-cB^2-f(A)+bA+cA^2\\ &-D(f)(A)(B-A)+b(B-A)+c\left(AB+BA-2A^2\right)\\ &=f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2.\end{aligned}$$

From (2.8) we then have the equality

$$\begin{aligned}(4.2)\quad &f(B)-f(A)-D(f)(A)(B-A)-c(B-A)^2\\ &=2\int_0^1(1-s)\left[\int_0^\infty \lambda^3(\lambda+(1-s)A+sB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-s)A+sB)^{-1}(B-A)(\lambda+(1-s)A+sB)^{-1}d\mu(\lambda)\right]ds\\ &\geq 0\end{aligned}$$

for all A, $B>0$.

If $A,\ B \geq m > 0$, then by (2.11)

$$\left\| f(B) - f(A) - D(f)(A)(B-A) - c(B-A)^2 \right\| \leq \frac{1}{2}\left[f''(m) - 2c\right]\|B-A\|^2 . \tag{4.3}$$

By the properties of norm,

$$\|f(B) - f(A) - D(f)(A)(B-A)\| - c\left\|(B-A)^2\right\| \leq \left\| f(B) - f(A) - D(f)(A)(B-A) - c(B-A)^2 \right\|$$

and by (4.3) we get

$$\|f(B) - f(A) - D(f)(A)(B-A)\| - c\left\|(B-A)^2\right\| \leq \frac{1}{2}\left[f''(m) - 2c\right]\|B-A\|^2 , \tag{4.4}$$

namely

$$\|f(B) - f(A) - D(f)(A)(B-A)\| \leq \frac{1}{2}f''(m)\|B-A\|^2 - c\left(\|B-A\|^2 - \left\|(B-A)^2\right\|\right)$$

and since $c\left(\|B-A\|^2 - \left\|(B-A)^2\right\|\right) \geq 0$, then we get the inequality that does not contain c,

$$\|f(B) - f(A) - D(f)(A)(B-A)\| \leq \frac{1}{2}f''(m)\|B-A\|^2 \tag{4.5}$$

for $A,\ B \geq m > 0$.

We have the operator Jensen's gap related to the n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ and probability density n-tuple $\mathbf{p} = (p_1, ..., p_n)$,

$$\begin{aligned} & J(\mathbf{A}, \mathbf{p}, \mathcal{C}(\ell, \mu)) \\ & = \sum_{k=1}^{n} p_k\left[f(A_k) - a - bA_k - cA_k^2\right] \\ & - f\left(\sum_{k=1}^{n} p_k A_k\right) + a + b\sum_{k=1}^{n} p_k A_k + c\left(\sum_{k=1}^{n} p_k A_k\right)^2 \\ & = \sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) - c\left[\sum_{k=1}^{n} p_k A_k^2 - \left(\sum_{k=1}^{n} p_k A_k\right)^2\right] \\ & = \sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) - c\sum_{k=1}^{n} p_k\left(A_k - \sum_{j=1}^{n} p_j A_j\right)^2 . \end{aligned} \tag{4.6}$$

From (2.18) we get for $A_k \geq m > 0$, $k \in \{1, ..., n\}$

$$\left\| \sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) - c \sum_{k=1}^{n} p_k \left(A_k - \sum_{j=1}^{n} p_j A_j\right)^2 \right\| \leq \frac{1}{2}\left[f''(m) - 2c\right] \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 \tag{4.7}$$

and the weaker version but without $c \geq 0$,

$$\left\| \sum_{k=1}^{n} p_k f(A_k) - f\left(\sum_{k=1}^{n} p_k A_k\right) \right\| \leq \frac{1}{2} f''(m) \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 \tag{4.8}$$

We have

$$\int_0^1 \mathcal{C}(\ell, \mu)((1-t)A + tB)\, dt - \mathcal{C}(\ell, \mu)\left(\frac{A+B}{2}\right) = \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) - \frac{c}{12}(B-A)^2.$$

From (3.5) we get

$$\begin{aligned} &\int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) - \frac{c}{12}(B-A)^2 \\ &= 2\int_0^1 \left(t - \frac{1}{2}\right)^2 \left\{ \int_0^1 (1-s) \right. \\ &\times \left[\int_0^\infty \lambda^3 \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A) \right. \\ &\times \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} (B-A) \\ &\times \left. \left. \left(\lambda + (1-s)\frac{A+B}{2} + s((1-t)A + tB)\right)^{-1} d\mu(\lambda) \right] ds \right\} dt \end{aligned} \tag{4.9}$$

and by (3.10) we obtain

$$\left\| \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) - \frac{c}{12}(B-A)^2 \right\| \leq \frac{1}{24}\left[f''(m) - 2c\right] \|B-A\|^2 \tag{4.10}$$

and

$$\left\| \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right) \right\| \leq \frac{1}{24} f''(m) \|B-A\|^2. \tag{4.11}$$

From the identity (3.7) we derive

$$\begin{aligned}(4.12)\qquad & \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt-\frac{c}{6}(B-A)^2\\ &=\int_0^1 t(1-t)\left[\int_0^\infty \lambda^3(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda)\right]dt,\end{aligned}$$

while by (3.14),

$$\begin{aligned}(4.13)\qquad & \left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt-\frac{c}{6}(B-A)^2\right\|\\ &\leq\frac{1}{12}\left[f''(m)-2c\right]\|B-A\|^2\end{aligned}$$

and

$$(4.14)\qquad \left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|\leq\frac{1}{12}f''(m)\|B-A\|^2.$$

Consider the operator convex function $f(t)=t^p$, $p\in[-1,0]\cup[1,2]$.

From (4.8) we obtain

$$\left\|\sum_{k=1}^n p_kA_k^p-\left(\sum_{k=1}^n p_kA_k\right)^p\right\|\leq\frac{1}{2}p(p-1)m^{p-2}\sum_{k=1}^n p_k\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2$$

provided that $A_k\geq m>0$, $k\in\{1,...,n\}$.

We have the midpoint inequality

$$\left\|\int_0^1((1-t)A+tB)^p\,dt-\left(\frac{A+B}{2}\right)^p\right\|\leq\frac{1}{24}p(p-1)m^{p-2}\|B-A\|^2$$

and the trapezoid inequality

$$\left\|\frac{A^p+B^p}{2}-\int_0^1((1-t)A+tB)^p\,dt\right\|\leq\frac{1}{12}p(p-1)m^{p-2}\|B-A\|^2,$$

provided that $A,\ B\geq m>0$.

Consider the operator convex function $f(t)=-\ln t$, $t>0$, then by (4.8) we get

$$\left\|\ln\left(\sum_{k=1}^n p_kA_k\right)-\sum_{k=1}^n p_k\ln A_k\right\|\leq\frac{1}{2m^2}\sum_{k=1}^n p_k\left\|A_k-\sum_{j=1}^n p_jA_j\right\|^2.$$

We have the midpoint inequality

$$\left\|\ln\left(\frac{A+B}{2}\right)-\int_0^1\ln((1-t)A+tB)\,dt\right\|\leq\frac{1}{24m^2}\|B-A\|^2$$

and the trapezoid inequality

$$\left\|\int_0^1\ln((1-t)A+tB)\,dt-\frac{\ln A+\ln B}{2}\right\|\leq\frac{1}{12m^2}\|B-A\|^2,$$

provided that $A,\ B\geq m>0$.

Consider the operator convex function $f(t) = t \ln t$, $t > 0$, then by (4.8) we get

$$\left\| \sum_{k=1}^{n} p_k A_k \ln A_k - \sum_{k=1}^{n} p_k A_k \ln \left(\sum_{k=1}^{n} p_k A_k \right) \right\| \leq \frac{1}{2m} \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^2 .$$

We have the midpoint inequality

$$\left\| \int_0^1 ((1-t) A + tB) \ln ((1-t) A + tB) \, dt - \left(\frac{A+B}{2} \right) \ln \left(\frac{A+B}{2} \right) \right\| \leq \frac{1}{24m} \|B - A\|^2$$

and the trapezoid inequality

$$\left\| \frac{A \ln A + B \ln B}{2} - \int_0^1 ((1-t) A + tB) \ln ((1-t) A + tB) \, dt \right\| \leq \frac{1}{12m} \|B - A\|^2 ,$$

provided that $A, B \geq m > 0$.

References

[1] H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, *Commun. Math. Phys.* **81** (1981), 89-96.

[2] R. Bhatia, First and second order perturbation bounds for the operator absolute value, *Linear Algebra Appl.* **208/209** (1994), 367-376.

[3] R. Bhatia, Perturbation bounds for the operator absolute value. *Linear Algebra Appl.* **226/228** (1995), 639–645.

[4] R. Bhatia, D. Singh and K. B. Sinha, Differentiation of operator functions and perturbation bounds. *Comm. Math. Phys.* **191** (1998), no. 3, 603–611.

[5] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[6] R. Coleman, *Calculus on Normed Vector Spaces*, Springer, New York, 2012.

[7] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), to appear.

[8] Yu. B. Farforovskaya, Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric (in Russian), *Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom.* **4** (1967), 155-156.

[9] Yu. B. Farforovskaya, An estimate of the norm $\|f(B) - f(A)\|$ for self-adjoint operators A and B (in Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst.* **56** (1976), 143-162 .

[10] Yu. B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions. *Algebra i Analiz* **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.

[11] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[12] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[13] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[14] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[15] T. Kato, Continuity of the map $S \to |S|$ for linear operators, *Proc. Japan Acad.* **49** (1973), 143-162.

[16] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.