REVERSES OF MINKOWSKI'S, HÖLDER'S AND HARDY'S TYPE INEQUALITIES USING ψ-FRACTIONAL INTEGRAL OPERATORS

BHAGWAT R. YEWALE AND DEEPAK B. PACHPATTE

Abstract. In present paper, we establish new reverses of Minkowski, Hölder and Hardy type inequalities by using ψ-Riemann-Liouville fractional integral operator, which is the classical Riemann-Liouville fractional integral of any function with respect to another function.

1. Introduction

In [4], Bougoffa proved following reverse integral Minkowski inequality:

$$\left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}} + \left(\int_a^b g^p(t)dt\right)^{\frac{1}{p}} \leq c\left(\int_a^b \left(f(t)+g(t)\right)^p dt\right)^{\frac{1}{p}}, \tag{1.1}$$

where f and g are positive functions on [a, b] such that $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for all t $\in$ [a, b] and $c = \frac{M(m+2)+1}{(m+1)(M+1)}$. In recent years, researchers proved many improvements and generalizations of integral inequalities. In particular, in [20], Sulaiman established some variant's of integral inequalities concerning reverses of Minkowski, Hölder and Hardy inequalities and then improvements of these inequalities appeared in the literature [17, 18]. We can found certain extensions and generalizations of the Minkowski, Hölder and Hardy type inequalities in the literature [3, 8, 12, 13, 22].

Fractional inequalities have been investigated by many authors due to their importance in the field of Mathematics. Fractional inequalities play a major role to study the nature of solution of fractional boundary value problems such as uniqueness, boundedness, stability of solution, continuous dependance etc. In [5, 6, 7, 14, 15, 16], authors established reverses of Minkowski, Hölder and Hardy type inequalities for different kinds of fractional integral operators.

In [21], Wu et al. presented generalized Hardy like inequalities and Consequently, Khameli et al. [9], established fractional form of these inequalities by using Riemann-Liouville fractional integral operator as follows:

2010 *Mathematics Subject Classification.* 26A33, 26D10.

Key words and phrases. Minkowski inequality, Hardy inequality, Hölder Inequality, ψ-Riemann-Liouville fractional integral.

RGMIA Res. Rep. Coll. 23 (2020), Art. 111, 14. pp. Received 08/08/20

(i) For $p > 1$, $q > 0$ and $\alpha > 0$, we have

$$\int_a^b \frac{\left(I_a^\alpha f(x)\right)^p}{g^q(x)}dx \leq \frac{1}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)} \left[(b-a)^{\alpha p-\alpha+1} I_a^\alpha\left(\frac{f^p(b)}{g^q(b)}\right) - I_a^\alpha\left(\frac{f^p(b)}{g^q(b)}(b-a)^{\alpha p-\alpha+1}\right)\right], \tag{1.2}$$

where f and g are positive on [a, b]$\subseteq [0,\infty)$ with g is non decreasing.
(ii) For $0 < p < 1$, $q > 0$ and $\alpha > 0$, we have

$$\int_a^b \frac{\left(I_a^\alpha f(x)\right)^p}{g^q(x)}dx \geq \frac{g^{-q}(b)}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)} \left[\frac{(-1)^{\alpha p-\alpha+1}}{\Gamma(\alpha)}\Gamma(\alpha p+1) I_b^{\alpha p+1} f^p(a) - (b-a)^{\alpha p-\alpha+1} I_b^\alpha f^p(a)\right], \tag{1.3}$$

where f is non negative and g is positive on [a, b]$\subseteq [0,\infty)$ with g is non decreasing.

In [1], Benaissa proved generalized integral inequalities related to the reverses of Minkowski and Hardy type integral inequalities. Furthemore, reverses Hölder's like integral inequalities proved by Benaissa and Budak in [2], among these the Hardy type integral inequality is given as follows:
(i) For $p \geq 1$, we have

$$p\int_a^b \frac{\left(\int_a^s f(t)dt\right)^p}{g(s)}ds \leq (b-a)^p \int_a^b \frac{f^p(s)}{g(s)}ds - \int_a^b \frac{(s-a)^p}{g(s)} f^p(s)ds \tag{1.4}$$

(ii) For $0 < p < 1$,

$$p\int_a^b \frac{\left(\int_a^s f(t)dt\right)^p}{g(s)}ds \geq \frac{(b-a)^p}{g(b)} \int_a^b f^p(s)ds - \frac{1}{g(b)}\int_a^b (s-a)^p f^p(s)ds, \tag{1.5}$$

where f, g are positive functions defined on [a, b] and g is non decreasing.

Motivated by above literature in this paper we established the reverses of Minkowski, Hölder and Hardy type integral inequalities proved in [1, 2] by employing ψ fractional integral operator. Remaining paper is organized as follows: In next sections we give some preliminaries, definitions and lemmas. In section 3, we established Hardy type integral inequalities by using ψ fractional integral operator. Reverses of Minkowski and Hölder inequalities for ψ fractional integral are given in section 4.

2. Preliminaries

In this section, we give some preliminaries, basic definitions of fractional integral operators and lemmas which will be used to carry out our main results.

Definition 1. [10] The Riemann-Liouville fractional integral operator of the integrable function ϕ on [a, b] of order $\beta > 0$ is defined as

$$\mathfrak{J}^{\beta}_{a+}\phi(t) = \frac{1}{\Gamma(\beta)}\int_a^t (t-x)^{\beta-1}\phi(x)dx, \quad \text{for all } t > a,$$

where Γ is the Gamma function.

Definition 2. [10] The Hadamard fractional integral of the integrable function ϕ on [a, b] of order $\beta > 0$ is defined as

$${}^{H}\mathfrak{J}^{\beta}_{a+}\phi(t) = \frac{1}{\Gamma(\beta)}\int_a^t \left(log\frac{t}{x}\right)\frac{\phi(x)}{x}dx, \quad (a < t < b),$$

where Γ is the Gamma function.

Definition 3. [10] Let ϕ be an integrable function defined on [a, b] and $\psi \in C^1[a, b]$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$. Then ψ-Riemann-Liouville fractional integral of the function ϕ with respect to the function ψ of order $\beta > 0$ is defined by

$$\mathfrak{J}^{\beta,\psi}_{a+}\phi(t) = \frac{1}{\Gamma(\beta)}\int_a^t \psi'(x)(\psi(t)-\psi(x))^{\beta-1}\phi(x)dx, \quad \text{for all } t > a,$$

where Γ is the Gamma function.

Lemma 2.1. (Hölder inequality)[11, 21]. Let $p \geq 1, \frac{1}{p} + \frac{1}{q} = 1$. Let f and g be non-negative integrable functions defined on [a, b], then

$$\int_a^b f(s)g(s)ds \leq \Big(\int_a^b f^p(s)ds\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(s)ds\Big)^{\frac{1}{q}}. \quad \Box \tag{2.1}$$

Lemma 2.2. (Reverse Hölder inequality)[21]. Let $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$. Let f and g be positive integrable functions defined on [a, b], then

$$\int_a^b f(s)g(s)ds \geq \Big(\int_a^b f^p(s)ds\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(s)ds\Big)^{\frac{1}{q}}. \quad \Box \qquad (2.1)$$

Remark 2.1. For different values of $\psi(x)$ we get various types of fractional integrals and consequently the obtained inequalities in this paper reduces to these integral operators:
1. If we put $\psi(x) = x$, then ψ-Reimann Liouville fractional integral reduces to the Riemann-Liouville fractional integral.
2. If we put $\psi(x) = lnx$, then ψ-Reimann Liouville fractional integral reduces to the Hadamard fractional integral.

3. If we put $\psi(x) = x^{\sigma}$, then ψ-Reimann Liouville fractional integral reduces to the Erdelyi-Kober fractional integral.

3. Hardy Type Inequalities Using ψ fractional integral

In this section, we prove Hardy type inequalities using ψ fractional integral. Our proofs based on the applications of the well known Fubini's theorem.

Theorem 3.1. Let $\beta > 0, p \geq 0$. Let $f,\ g$ be two positive functions defined on $[a, b] \subseteq [0, \infty)$ such that g is non-decreasing and ψ is defined as in Definition 3, then following inequalities hold:

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt \leq \frac{1}{(\beta p - \beta + 1)\Gamma^{p-1}(\beta+1)} \Big[(\psi(b) - \psi(a))^{\beta p - \beta + 1} \mathfrak{J}_{a+}^{\beta,\psi} \frac{f^p(b)}{g(b)} - \mathfrak{J}_{a+}^{\beta,\psi} \frac{f^p(b)}{g(b)} (\psi(b) - \psi(a))^{\beta p - \beta + 1} \Big]. \tag{3.1}$$

Proof. For $p \geq 1$, we have

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt = \int_a^b g^{-1}(t) \Big(\frac{1}{\Gamma(\beta)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s) ds \Big)^p dt. \tag{3.2}$$

By using Hölder inequality, we get

$$\begin{aligned}
\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt \leq & \int_a^b g^{-1}(t) \Big[\Big(\frac{1}{\Gamma(\beta)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f^p(s) ds \Big)^{\frac{1}{p}} \\
& \Big(\frac{1}{\Gamma(\beta)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} ds \Big)^{\frac{p-1}{p}} \Big]^p dt \\
= & \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta)} \int_a^b g^{-1}(t) \Big(\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f^p(s) ds \Big) \\
& \Big(\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} ds \Big)^{p-1} dt \\
= & \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)} \int_a^b g^{-1}(t)(\psi(t) - \psi(a))^{\beta(p-1)} \\
& \Big(\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f^p(s) ds \Big) dt.
\end{aligned}$$

Since g is non decreasing and by changing the order of integration, we obtain

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt \leq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)} \int_a^b g^{-1}(s)\psi'(s)(\psi(b) - \psi(s))^{\beta-1} f^p(s) \Big(\int_s^b (\psi(t) - \psi(s))^{p(\beta-1)} dt \Big) ds.$$

It follows that

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)}dt \leq \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)\Gamma(\beta)}\Big[\int_a^b g^{-1}(s)\psi'(s)(\psi(b)-\psi(s))^{\beta-1} (\psi(b)-\psi(s))^{\beta p-\beta+1}f^p(s)ds - \int_a^b g^{-1}(t)\psi'(s)(\psi(b)-\psi(s))^{\beta-1} (\psi(s)-\psi(a))^{\beta p-\beta+1}f^p(s)ds\Big],$$

which is equivalent to

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)}dt \leq \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Big[(\psi(b)-\psi(a))^{\beta p-\beta+1}\mathfrak{J}_{a+}^{\beta,\psi}\frac{f^p(b)}{g(b)} -\mathfrak{J}_{a+}^{\beta,\psi}\frac{f^p(b)}{g(b)}(\psi(b)-\psi(a))^{\beta p-\beta+1}\Big]. \quad \square$$

Corollary 3.1. By choosing different values of $\psi(s)$ we can establish corresponding fractional integral inequalities such as
(i) Choosing $\psi(s) = s$, we get Hardy type inequality (3.1) for the Riemann-Liouville fractional integral.
(ii) Choosing $\psi(s) = lns$, we get Hardy type inequality (3.1) for the Hadamard fractional integral.
(iii) Choosing $\psi(s) = s^\sigma$, we get Hardy type inequality (3.1) for the Erdelyi-Kober fractional integral. □

Remark 3.1. For $\psi(s) = s$ and $\beta = 1$, the inequality (3.1), reduces to the inequality (1.4). □

Theorem 3.2. Let $\beta > 0, 0 < p < 1$ and f, g be two positive functions defined on $[a, b] \subseteq [0,\infty)$ such that g is non-decreasing and ψ is defined as in Definition 3, then following inequalities hold:

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)}dt \geq \frac{g^{-1}(b)}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Big[\frac{(-1)^{\beta p-\beta+1}\Gamma(\beta p+1)}{\Gamma(\beta)}\mathfrak{J}_b^{\beta p+1,\psi}f^p(a) -(\psi(b)-\psi(a))^{\beta p-\beta+1}\mathfrak{J}_b^{\beta,\psi}f^p(a)\Big]. \tag{3.3}$$

Proof. For $0 < p < 1$, we have

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)}dt = \int_a^b g^{-1}(t)\Big(\frac{1}{\Gamma(\beta)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}f(s)ds\Big)^p dt \tag{3.4}$$

By using reverse Hölder inequality, we get

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)}dt \geq \int_a^b g^{-1}(t)\Big[\Big(\frac{1}{\Gamma(\beta)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}f^p(s)ds\Big)^{\frac{1}{p}}$$

$$\left(\frac{1}{\Gamma(\beta)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\right)^{\frac{p-1}{p}}\Bigg]^p dt$$
$$=\frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta)}\int_a^b g^{-1}(t)\Big(\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}f^p(s)ds\Big)$$
$$\Big(\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\Big)^{p-1}dt$$
$$=\frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_a^b g^{-1}(t)(\psi(t)-\psi(a))^{\beta(p-1)}$$
$$\Big(\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}f^p(s)ds\Big)dt.$$

Since g is non decreasing and by changing the order of integration, we have

$$\int_a^b \frac{[\mathfrak{I}_{a+}^{\beta,\psi}f(t)]^p}{g(t)}dt \geq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_a^b g^{-1}(b)\psi'(s)(\psi(a)-\psi(s))^{\beta-1}f^p(s)$$
$$\Big(\int_s^b (\psi(t)-\psi(s))^{p(\beta-1)}dt\Big)ds.$$

Therefore

$$\int_a^b \frac{[\mathfrak{I}_{a+}^{\beta,\psi}f(t)]^p}{g(t)}dt \geq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_b^a g^{-1}(b)\psi'(s)(\psi(a)-\psi(s))^{\beta-1}f^p(s)$$
$$\Big(\int_b^s (\psi(t)-\psi(s))^{p(\beta-1)}dt\Big)ds.$$

From above we get

$$\int_a^b \frac{[\mathfrak{I}_{a+}^{\beta,\psi}f(t)]^p}{g(t)}dt \geq \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)\Gamma(\beta)}\Bigg[\int_b^a g^{-1}(b)\psi'(s)(\psi(a)-\psi(s))^{\beta-1}$$
$$(\psi(s)-\psi(a))^{\beta p-\beta+1}f^p(s)ds-\int_b^a g^{-1}(b)\psi'(s)(\psi(a)-\psi(s))^{\beta-1}$$
$$(\psi(b)-\psi(a))^{\beta p-\beta+1}f^p(s)ds\Bigg].$$

It follows that

$$\int_a^b \frac{[\mathfrak{I}_{a+}^{\beta,\psi}f(t)]^p}{g(t)}dt \geq \frac{g^{-1}(b)}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Bigg[\frac{1}{\Gamma(\beta)}\int_b^a \psi'(s)(\psi(a)-\psi(s))^{\beta-1}$$
$$(\psi(s)-\psi(a))^{\beta p-\beta+1}f^p(s)ds-\frac{1}{\Gamma(\beta)}(\psi(b)-\psi(a))^{\beta p-\beta+1}$$
$$\int_b^a \psi'(s)(\psi(a)-\psi(s))^{\beta-1}f^p(s)ds\Bigg].$$

Therefore

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt \geq \frac{g^{-1}(b)}{(\beta p - \beta + 1)\Gamma^{p-1}(\beta+1)} \left[\frac{(-1)^{\beta p - \beta + 1}}{\Gamma(\beta)} \int_b^a \psi'(s)(\psi(a)-\psi(s))^{\beta p} f^p(s) ds - \frac{1}{\Gamma(\beta)}(\psi(b)-\psi(a))^{\beta p - \beta + 1} \int_b^a \psi'(s)(\psi(a)-\psi(s))^{\beta - 1} f^p(s) ds\right],$$

which is equivalent to

$$\int_a^b \frac{[\mathfrak{J}_{a+}^{\beta,\psi} f(t)]^p}{g(t)} dt \geq \frac{g^{-1}(b)}{(\beta p - \beta + 1)\Gamma^{p-1}(\beta+1)} \left[\frac{(-1)^{\beta p - \beta + 1}\Gamma(\beta p + 1)}{\Gamma(\beta)} \mathfrak{J}_{b-}^{\beta p + 1,\psi} f^p(a) - (\psi(b)-\psi(a))^{\beta p - \beta + 1} \mathfrak{J}_{b-}^{\beta,\psi} f^p(a)\right]. \quad \square$$

Corollary 3.2. By choosing different values of $\psi(s)$ we can establish corresponding fractional integral inequalities such as
(i) Choosing $\psi(s) = s$, we get Hardy type inequality (3.3) for the Riemann-Liouville fractional integral.
(ii) Choosing $\psi(s) = lns$, we get Hardy type inequality (3.3) for the Hadamard fractional integral.
(iii) Choosing $\psi(s) = s^\sigma$, we get Hardy type inequality (3.3) for the Erdelyi-Kober fractional integral. $\square$

Remark 3.2. For $\psi(s) = s$ and $\beta = 1$, the inequality (3.3), reduces to the inequality (1.5). $\square$

4. Reverse Minkowski and Hölder type Inequality using ψ fractional integral

In this section, we prove reverse Minkowski type and Hölder type inequalities for ψ-Riemann-Liouville fractional integral.

Our next result concerning reverses of Minkowski by using ψ fractional integral operator.

Theorem 4.1. Let $\beta > 0$, $p \geq 1$. Let f and g be two positive functions defined on $[a,t]$, for all $t > a \geq 0$ such that $\mathfrak{J}_{a+}^{\beta,\psi} f^p(t) < \infty, \mathfrak{J}_{a+}^{\beta,\psi} g^p(t) < \infty$ and ψ is defined as in Definition 3. If $0 < c < l \leq \frac{kf(s)}{g(s)} \leq Ł$, for $k > 0$, $s \in [a,t]$, then following inequalities hold:

$$\frac{Ł + k}{k(Ł - c)} \left[\mathfrak{J}_{a+}^{\beta,\psi}(kf - cg)^p(t)\right]^{\frac{1}{p}} \leq \left[\mathfrak{J}_{a+}^{\beta,\psi} f^p(t)\right]^{\frac{1}{p}} + \left[\mathfrak{J}_{a+}^{\beta} g^p(t)\right]^{\frac{1}{p}}$$

$$\leq \frac{l+k}{k(l-c)}\big[\mathfrak{J}^{\beta,\psi}_{a+}(kf-cg)^p(t)\big]^{\frac{1}{p}}. \tag{4.1}$$

Proof. Since for $k>0, s\in[a,\ t], t>0$, we have

$$0<c<l\leq \frac{kf(s)}{g(s)}\leq Ł, \tag{4.2}$$

then

$$-\frac{1}{l}\leq -\frac{g(s)}{kf(s)}\leq -\frac{1}{Ł},$$

which implies

$$\frac{1}{c}-\frac{1}{l}\leq \frac{1}{c}-\frac{g(s)}{kf(s)}\leq \frac{1}{c}-\frac{1}{Ł}.$$

Therefore

$$\frac{l-c}{cl}\leq \frac{kf(s)-cg(s)}{ckf(s)}\leq \frac{Ł-c}{cŁ}.$$

From above we get

$$\frac{Ł}{Ł-c}\leq \frac{kf(s)}{kf(s)-cg(s)}\leq \frac{l}{l-c}.$$

It follows that

$$\frac{Ł}{k(Ł-c)}\big(kf(s)-cg(s)\big)\leq f(s)\leq \frac{l}{k(l-c)}\big(kf(s)-cg(s)\big). \tag{4.3}$$

Taking p^{th} power of (4.3), we get

$$\Big[\frac{Ł}{k(Ł-c)}\Big]^p\big(kf(s)-cg(s)\big)^p\leq f^p(s)\leq \Big[\frac{l}{k(l-c)}\Big]^p\big(kf(s)-cg(s)\big)^p. \tag{4.4}$$

Multiplying (4.4) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s\in(a,t)$ and integrating with respect to s from a to t, we get

$$\Big[\frac{Ł}{k(Ł-c)}\Big]\big[\mathfrak{J}^{\beta,\psi}_{a+}(kf-cg)^p(t)\big]^{\frac{1}{p}}\leq\big[\mathfrak{J}^{\beta,\psi}_{a+}\rho^p(t)\big]^{\frac{1}{p}}$$
$$\leq\Big[\frac{l}{k(l-c)}\Big]\big[\mathfrak{J}^{\beta,\psi}_{a+}(kf-cg)^p(t)\big]^{\frac{1}{p}}. \tag{4.5}$$

Again from the condition (4.2), we have

$$l-c\leq \frac{kf(s)-cg(s)}{g(s)}\leq Ł-c.$$

Therefore

$$\frac{1}{Ł-c}\leq \frac{f(s)}{kf(s)-cg(s)}\leq \frac{1}{l-c}.$$

It follows that

$$\frac{(kf(s)-cg(s))^p}{(Ł-c)^p}\leq g^p(s)\leq \frac{(kf(s)-cg(s))^p}{(l-c)^p}. \tag{4.6}$$

Multiplying by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ to (4.6) and integrating with respect to s from a to t, we get

$$\frac{1}{Ł-c}\big[\mathfrak{I}_{a+}^{\beta,\psi}(kf-cg)^p(t)\big]^{\frac{1}{p}} \leq \big[\mathfrak{I}_{a+}^{\beta,\psi}g^p(t)\big]^{\frac{1}{p}} \leq \frac{1}{l-c}\big[\mathfrak{I}_{a+}^{\beta,\psi}(kf-cg)^p(t)\big]^{\frac{1}{p}}. \tag{4.7}$$

Adding inequalities (4.5) and (4.7), we get required inequality (4.1). □

Corollary 4.1. By choosing different values of $\psi(s)$ we can establish corresponding fractional integral inequalities such as
(i) Choosing $\psi(s) = s$, we get Minkowski type inequality (4.1) for the Riemann-Liouville fractional integral.
(ii) Choosing $\psi(s) = lns$, we get Minkowski type inequality (4.1) for the Hadamard fractional integral.
(iii) Choosing $\psi(s) = s^\sigma$, we get Minkowski type inequality (4.1) for the Erdelyi-Kober fractional integral. □

Remark 4.1. For $\psi(s) = s$ and $\beta = 1$, the inequality (4.1), reduces to the inequality (2.2) of Theorem(2.1) in [1]. □

In next theorem we prove reverse Hölder type inequalities and give corollaries and remarks for these results.
Theorem 4.2. Let $\beta > 0$, $p > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $m > 0$, $n > 0$, $t > a \geq 0$. Let f and g be two positive integrable functions on $[a,t]$ and ψ is defined as in Definition 3. Let w be a weight function on [a, t] . If $0 < l \leq \frac{f^m(s)}{g^n(s)} \leq Ł$, for $s \in [a,t]$, then following inequalities hold:

$$\big(\mathfrak{I}_{a+}^{\beta,\psi}f^m(t)w(t)\big)^{\frac{1}{p}}\big(\mathfrak{I}_{a+}^{\beta,\psi}g^n(t)w(t)\big)^{\frac{1}{q}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{pq}}\big(\mathfrak{I}_{a+}^{\beta,\psi}f^{\frac{m}{p}}(t)g^{\frac{n}{q}}(t)w(t)\big). \tag{4.8}$$

Proof. Since for $m,n > 0$ and $s \in [a,t]$, we have

$$0 < l \leq \frac{f^m(s)}{g^n(s)} \leq Ł. \tag{4.9}$$

Then

$$\left(\frac{1}{l}\right)^{\frac{1}{q}} \geq \frac{g^{\frac{n}{q}}(s)}{f^{\frac{m}{q}}(s)} \geq \left(\frac{1}{Ł}\right)^{\frac{1}{q}}. \tag{4.10}$$

Multiplying (4.10) by $f^m(s)$, we get

$$\left(\frac{1}{l}\right)^{\frac{1}{q}}f^m(s) \geq \frac{g^{\frac{n}{q}}(s)f^m(s)}{f^{\frac{m}{q}}(s)} \geq \left(\frac{1}{Ł}\right)^{\frac{1}{q}}f^m(s). \tag{4.11}$$

From (4.11), we have

$$\left(\frac{1}{l}\right)^{\frac{1}{q}} f^m(s) \geq g^{\frac{n}{q}}(s) f^{\frac{m}{p}}(s) \geq \left(\frac{1}{Ł}\right)^{\frac{1}{q}} f^m(s).$$

Which gives

$$l^{\frac{1}{q}} f^{\frac{m}{p}} g^{\frac{n}{q}} \leq f^m(s) \leq Ł^{\frac{1}{q}} f^{\frac{m}{p}} g^{\frac{n}{q}}. \tag{4.12}$$

Multiplying right hand side of (4.12) by *w(s)*, we obtain

$$f^m(s) w(s) \leq Ł^{\frac{1}{q}} f^{\frac{m}{p}} g^{\frac{n}{q}} w(s). \tag{4.13}$$

Multiplying (4.13) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ and integrating with respect to s from a to t, we get

$$\left[\mathfrak{J}_{a+}^{\beta,\psi} f^m(t) w(t)\right]^{\frac{1}{p}} \leq Ł^{\frac{1}{pq}} \left[\mathfrak{J}_{a+}^{\beta,\psi} f^{\frac{m}{p}}(t) f^{\frac{m}{p}}(t) g^{\frac{n}{q}}(t) w(t)\right]^{\frac{1}{p}}. \tag{4.14}$$

Now from (4.9), we have

$$l^{\frac{1}{p}} \leq \frac{f^{\frac{m}{p}}(s)}{g^{\frac{n}{p}}(s)} \leq Ł^{\frac{1}{p}} \tag{4.15}$$

Multiplying (4.15) by $g^n(s)$, we get

$$l^{\frac{1}{p}}(s) g^n(s) \leq \frac{f^{\frac{m}{p}}(s) g^n(s)}{g^{\frac{n}{p}}(s)} \leq Ł^{\frac{1}{p}} g^n(s).$$

Which gives

$$l^{\frac{1}{p}} g^n \leq f^{\frac{m}{p}}(s) g^{\frac{n}{q}}(s) \leq Ł^{\frac{1}{p}} g^n(s). \tag{4.16}$$

It follows that

$$\left(\frac{1}{Ł}\right)^{\frac{1}{p}} f^{\frac{m}{p}}(s) g^{\frac{n}{q}}(s) \leq g^n(s) \leq \left(\frac{1}{l}\right)^{\frac{1}{p}} f^{\frac{m}{p}}(s) g^{\frac{n}{q}}(s). \tag{4.17}$$

Multiplying right hand side inequality of (4.17) by *w(s)*, we obtain

$$g^n(s) w(s) \leq \left(\frac{1}{l}\right)^{\frac{1}{p}} f^{\frac{m}{p}}(s) g^{\frac{n}{q}}(s) w(s). \tag{4.18}$$

Multiplying (4.18) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ and integrating with respect to s from a to t, we get

$$\left[\mathfrak{J}_{a+}^{\beta,\psi} g^n(t) w(t)\right]^{\frac{1}{q}} \leq \left(\frac{1}{l}\right)^{\frac{1}{pq}} \left[\mathfrak{J}_{a+}^{\beta,\psi} f^{\frac{m}{p}}(t) g^{\frac{n}{q}}(t) w(t)\right]^{\frac{1}{q}}. \tag{4.19}$$

Taking multiplication between (4.14) and (4.19), we get required inequality (4.8).
□

Corollary 4.2. By choosing different values of $\psi(s)$ we can establish corresponding fractional integral inequalities such as
(i) Choosing $\psi(s) = s$, we get Hölder type inequality (4.8) for the Riemann-Liouville fractional integral.
(ii) Choosing $\psi(s) = lns$, we get Hölder type inequality (4.8) for the Hadamard fractional integral.
(iii) Choosing $\psi(s) = s^\sigma$, we get Hölder type inequality (4.8) for the Erdelyi-Kober fractional integral. □

Remark 4.2. For $\psi(s) = s$ and $\beta = 1$, the inequality (4.8), reduces to the inequality (6) of Theorem(2.1) in [2]. □

Corollary 4.3. Let $\beta > 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be two positive integrable functions on $[a, t]$ and ψ is defined as in Definition 3. If $0 < l \leq \frac{f^{p-1}(s)}{g(s)} \leq$ Ł, for $s \in [a, t]$, where $t > a \geq 0$,then following inequalities hold:

$$\left[\mathfrak{I}_{a+}^{\beta,\psi} f^p(t)\right]^{\frac{1}{p}} \left[\mathfrak{I}_{a+}^{\beta,\psi} g^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{\text{Ł}}{l}\right)^{\frac{1}{p}} \mathfrak{I}_{a+}^{\beta,\psi} f(t)g(t). \quad \square$$

Corollary 4.4. Let $\beta > 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be two positive integrable functions on [a, t], for all $t > a \geq 0$, and ψ is defined as in Definition 3. If $0 < l \leq \frac{f(s)}{g^{q-1}(s)} \leq$ Ł, for $s \in [a, t]$, then following inequalities hold:

$$\left[\mathfrak{I}_{a+}^{\beta,\psi} f^p(t)\right]^{\frac{1}{p}} \left[\mathfrak{I}_{a+}^{\beta,\psi} g^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{\text{Ł}}{l}\right)^{\frac{1}{q}} \mathfrak{I}_{a+}^{\beta,\psi} f(t)g(t). \quad \square$$

Remark 4.3. For $\psi(s) = s$ and $\beta = 1$, the inequalities in corollary (4.1) and (4.2) reduces to the inequalities in corollary (2.4) and (2.5) in [2] respectively. □

Theorem 4.3. Let $\beta > 0$, $m > 0, k > 0, p > 0, q > 0, \mu > 0, \nu > 0$. Let *f*, *g* be non-negative integrable functions on [a, t], for all $t > a \geq 0$ and ψ is defined as in Definition 3. If $0 < k < l \leq \frac{mf(s)}{g(s)} \leq$ Ł, for all $s \in [a, t]$ where $t \geq a > 0$, then following inequalities hold:

$$\begin{aligned}\left[\mathfrak{I}_{a+}^{\beta,\psi} f^p(t)\right]^{\frac{1}{p}} \left[\mathfrak{I}_{a+}^{\beta,\psi} g^q(t)\right]^{\frac{1}{q}} \leq \quad & \left(\frac{\text{Ł}}{m}\right)\left(\frac{m}{l}\right)^{\frac{2\mu}{\mu+\nu}} (l+k)^{\frac{\mu-\nu}{\mu+\nu}} (\text{Ł}+k)^{\frac{\nu-\mu}{\mu+\nu}} \\ & \left[\mathfrak{I}_{a+}^{\beta,\psi} \left(f^\mu(t) g^\nu(t)\right)^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}} \left[\mathfrak{I}_{a+}^{\beta,\psi} \left(f^\mu(t) g^\nu(t)\right)^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \quad (4.20)\end{aligned}$$

Proof. Since for $m > 0$, $s \in [a, t]$, $t > a \geq 0$ we have

$$0 < k < l \leq \frac{mf(s)}{g(s)} \leq \text{Ł}, \tag{4.21}$$

then

$$l+k \leq \frac{mf(s)+kg(s)}{g(s)} \leq Ł+k. \tag{4.22}$$

From above we have

$$(l+k)^q \leq \Big(\frac{mf(s)+kg(s)}{g(s)}\Big)^q \leq (Ł+k)^q. \tag{4.23}$$

Multiplying left hand side inequality of (4.23) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ and integrating with respect to s from a to t, we get

$$(l+k)\big[\mathfrak{J}_{a+}^{\beta,\psi} g^q(t)\big]^{\frac{1}{q}} \leq \big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^q(t)\big]^{\frac{1}{q}}. \tag{4.24}$$

Also, from (4.21) we have

$$\frac{Ł+k}{Ł} \leq \frac{mf(s)+kg(s)}{mf(s)} \leq \frac{l+k}{l} \tag{4.25}$$

Multiplying the left hand side inequality of (4.25) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ and integrating with respect to s from a to t, we get

$$m\bigg(\frac{Ł+k}{Ł}\bigg)\Big[\mathfrak{J}_{a+}^{\beta,\psi} f^p(t)\Big]^{\frac{1}{p}} \leq \Big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^p(t)\Big]^{\frac{1}{p}}. \tag{4.26}$$

Taking multiplication between the inequalities (4.24) and (4.26), we obtain

$$\begin{aligned} &\bigg(\frac{m}{Ł}\bigg)(Ł+k)(l+k)\big[\mathfrak{J}_{a+}^{\beta,\psi} f^p(t)\big]^{\frac{1}{p}}\big[\mathfrak{J}_{a+}^{\beta,\psi} g^q(t)\big]^{\frac{1}{q}} \\ \leq \quad &\big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^p(t)\big]^{\frac{1}{p}}\big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^q(t)\big]^{\frac{1}{q}}. \end{aligned} \tag{4.27}$$

From the right hand side inequalities of (4.22) and (4.25), we get

$$(mf(s)+kg(s))^\nu \leq (Ł+k)^\nu g^\nu(s) \tag{4.28}$$

and

$$(mf(s)+kg(s))^\mu \leq \Big(\frac{m}{l}(l+k)\Big)^\mu f^\mu(s). \tag{4.29}$$

Adding inequalities (4.28) and (4.29), we obtain

$$mf(s)+kg(s) \leq (\frac{m}{l})^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\Big(f^\mu(s)g^\nu(s)\Big)^{\frac{1}{\mu+\nu}}. \tag{4.30}$$

Multiplying (4.30) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (a,t)$ and integrating with respect to s from a to t, we get

$$\big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^p(t)\big]^{\frac{1}{p}} \leq (\frac{m}{l})^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\Big[\mathfrak{J}_{a+}^{\beta,\psi}(f^\mu(t)g^\nu(t))^{\frac{p}{\mu+\nu}}\Big]^{\frac{1}{p}}. \tag{4.31}$$

Similarly, from (4.30), we have

$$\big[\mathfrak{J}_{a+}^{\beta,\psi}(mf+kg)^q(t)\big]^{\frac{1}{q}} \leq (\frac{m}{l})^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\Big[\mathfrak{J}_{a+}^{\beta,\psi}(f^\mu(t)g^\nu(t))^{\frac{q}{\mu+\nu}}\Big]^{\frac{1}{q}}. \tag{4.32}$$

Multiplying inequalities (4.31) and (4.32), we get

$$\begin{aligned}&\left[\mathfrak{I}_{a+}^{\beta,\psi}(mf+kg)^{p}(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{a+}^{\beta,\psi}(mf+kg)^{q}(t)\right]^{\frac{1}{q}}\\ \leq\ &(\frac{m}{l})^{\frac{2\mu}{\mu+\nu}}(l+k)^{\frac{2\mu}{\mu+\nu}}(Ł+k)^{\frac{2\nu}{\mu+\nu}}\left[\mathfrak{I}_{a+}^{\beta,\psi}(f^{\mu}(t)g^{\nu}(t))^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}\\ &\left[\mathfrak{I}_{a+}^{\beta,\psi}(f^{\mu}(t)g^{\nu}(t))^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \qquad (4.33)\end{aligned}$$

From the inequalities (4.26) and (4.33), we obtain

$$\begin{aligned}&\frac{m}{Ł}(Ł+k)(l+k)\left[\mathfrak{I}_{a+}^{\beta,\psi}f^{p}(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{a+}^{\beta,\psi}g^{q}(t)\right]^{\frac{1}{q}}\\ \leq\ &(\frac{m}{l})^{\frac{2\mu}{\mu+\nu}}(l+k)^{\frac{2\mu}{\mu+\nu}}(Ł+k)^{\frac{2\nu}{\mu+\nu}}\left[\mathfrak{I}_{a+}^{\beta,\psi}(f^{\mu}(t)g^{\nu}(t))^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}\\ &\left[\mathfrak{I}_{a+}^{\beta,\psi}(f^{\mu}(t)g^{\nu}(t))^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \qquad (4.34)\end{aligned}$$

From inequality (4.34), we get required inequality (4.20). □

Corollary 4.5. By choosing different values of $\psi(s)$ we can establish corresponding fractional integral inequalities such as
(i) Choosing $\psi(s) = s$, we get Hölder type inequality (4.20) for the Riemann-Liouville fractional integral.
(ii) Choosing $\psi(s) = lns$, we get Hölder type inequality (4.20) for the Hadamard fractional integral.
(iii) Choosing $\psi(s) = s^{\sigma}$, we get Hölder type inequality (4.20) for the Erdelyi-Kober fractional integral. □

Remark 4.4. For $\psi(s) = s$ and $\beta = 1$, the inequality (4.20), reduces to the inequality (13) of Theorem(2.6) in [2]. □

References

[1] B. Benaissa, More on reverses of Minkowski's inequalities and Hardy's integral inequalities, *Asian Eur. J. Math.*, 13(1), 2018, 1-7.
[2] B. Benaissa, H. Budak, More on reverses of Hölder integral inequality, *Korean J. Math.*, 28(1), 2020, 9-15.
[3] B. Benaissa, On the reverse of Minkowski's integral inequality, *Kragujevac J. Math.*, 46(3), 2020, 407-416.
[4] L. Bougoffa, On Minkowski and Hardy integral inequalities, *J. Inequal. Pure and Appl. Math.*, 7(2), art.60, 2006.
[5] Z. Dahmani, On Minkowski and Hermite-Hadmard integral inequalities via fractional integration, Ann. Funct. Anal., 1(1), 2010, 51-58.
[6] J. E. H. Hernandez, M. J. V. Cortez, On a Hardy's inequality for a fractional integral operator, *An. Univ. Craiova Ser. Inform.*, 45(2), 2018, 232-242.
[7] S. Iqbal, J. Pecaric, M. Samraiz, Z. Tomovski, Hardy-type inequalities for generalized fractional integral operators *Tbil. Math. J.*, 10(1), 2017, 75-90.
[8] A. Kashuri, R. Liko Some inequalities similar to Hardy's inequality, *PJMMS*, 17(1), 2016, 1-6.

[9] A. Khameli, Z. Dahmani, K. Freha, M. Z. Sarikaya New Riemann-Liouville generalizations for some inequalities of Hardy type, *Malaya J. Mat.*, 4(2), 2016, 277-283.

[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, *Elsevier, Amsterdam*, 204, 2006.

[11] D. S. Mitrinovic, J. Pecaric, A. M. Fink, Classical and New inequalities in analysis, *Kluwer Acad. Publish., Dordrecht*, 61, 1993.

[12] B. G. Pachpatte, On Hardy type integral inequalities for a functions of two variables, *Demonstratio Math.*, XXVIII(2), 1995.

[13] J. Pecaric, K. Smoljak, Improvement of an extension of a Hölder-type inequality, *Anal. Math.*, 38, 2012, 135-146.

[14] G. Rahman, A. Khan, T. Abdeljawad, K. S. Nisar, The Minkowski inequalities via generalized proportional fractional integral operators, *Adv. Differ.Equ.*, 287(2019), 2019.

[15] J. E. Restrepo, V. L. Chinchane, P. Agarwal, Weighted reverse fractional inequalities of Minkowski's and Hölder's type, *TWMS J. Pure Appl. Math.*, 10(2), 2019, 188-198.

[16] J. Sousa, E. Olivera, The Minkowski's inequality by means of a generalized fractional integral, *AIMS Mathematics*, 3(1), 2018, 131-147.

[17] B. Sroysang, A generalization of some integral inequalities similar to Hardy's Inequality, *Math. Aeterna*, 3(7), 2013, 593-596.

[18] B. Sroysang, More on reverses of Minkowski's integral inequality, *Math. Aeterna*, 3(7), 2013, 597-600.

[19] B. Sroysang, More on some Hardy type intgral inequalities, *J. Math. Inequal.*, 8(3), 2014, 497-501.

[20] W. T. Sulaiman, Reverses of Minkowski's, Hölder's and Hardy's integral inequalities, *Int. J. Mod. Math. Sci.*, 1(1), 2012, 14-24.

[21] S. Wu, B. Sroysang, S. Li A further generaliztion of certain integral inequalities similar to Hardy's inequality *J. Nonlinear Sci. Appl.*, 9, 2016, 1093-1102.

[22] Y. Yilmaz, M. K. Ozdemir, I. Solak A generaliztion of Hölder and Minkowski Inequalities, *J. Inequal. Pure and Appl. Math.*, 7(5), art.193, 2006.

Bhagwat R. Yewale
Dept. of Mathematics, Dr. B. A. M. University, Aurangabad, Maharashtra 431004, India
E-mail address: `yewale.bhagwat@gmail.com`

Deepak B. Pachpatte
Dept. of Mathematics, Dr. B. A. M. University, Aurangabad, Maharashtra 431004, India
E-mail address: `pachpatte@gmail.com`