REVERSE MINKOWSKI INEQUALITY VIA EXTENDED GENERALIZED FRACTIONAL INTEGRAL OPERATOR

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Abstract. In this paper, we establish some reverse integral Minkowski inequalities and Holder type inequalities by using Extended generalized fractional integral operator. Examples are provided to illustrate our main results.

1. Introduction

In [5], Bougoffa obtained following reverse integral Minkowski inequality for $k \geq 1$:

$$\left(\int_a^b \zeta^k(t)dt\right)^{\frac{1}{k}} + \left(\int_a^b \eta^k(t)dt\right)^{\frac{1}{k}} \leq c\left(\int_a^b \left(\zeta(t)+\eta(t)\right)^k dt\right)^{\frac{1}{k}}, \tag{1.1}$$

where ζ and η are positive functions on [a, b] such that $0 < m \leq \frac{\zeta(t)}{\eta(t)} \leq M$, for all t $\in$ [a, b] and $c = \frac{M(m+2)+1}{(m+1)(M+1)}$. Different versions and generalizations of the reverse integral Minkowski inequality and Holder inequality have been established which serve as powerful tool in the field mathematical analysis (see [3, 4, 25, 26]).

Fractional Calculus is a generalization of Ordinary Calculus(Classical Calculus), which allow us to take integration and derivative of fractional order. Fractional Calculus has become popular area in Mathematics due to it's extensive applications in various areas of science and engineering. The contemporary development of Fractional Calculus can be found in the literature [10, 14]. In the middle of Fractional Calculus, the fractional operators (i.e. fractional integrals and fractional derivatives) have deep significance and made effective impact in the field of research. Recently, many generalizations of fractional operators are given and used to extends various inequalities in fractional form (see [1, 12, 13, 18, 20, 21, 22, 29]).

In [7], Dahmani presented following fractional version of inequality (1.1) by employing Riemann-Liouville fractional integral:

Let $\alpha > 0$, $p \geq 1$. Let ζ and η be two positive functions defined on $[0, \infty)$ such that for all $t > 0, I^\alpha \zeta^p(t) < \infty, I^\alpha \eta^p(t) < \infty$. If $0 < l \leq \frac{\zeta(x)}{\eta(x)} \leq Ł$, for $x \in [0, t]$, then

$$[I^\alpha \zeta^p(t)]^{\frac{1}{p}} + [I^\alpha \eta^p(t)]^{\frac{1}{p}} \leq c[I^\alpha (\zeta+\eta)^p(t)]^{\frac{1}{p}}, \tag{1.2}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

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Moreover, reverse Minkowski inequalities and it's related inequalities for various fractional integral can be seen in the literature [6, 9, 17, 18, 19, 23, 27, 28]. Inspired by above cited work, in this paper we establish reverse Minkowski integral type inequalities, Holder type inequalities and it's related inequalities by using extended generalized fractional integral operator. Also, we give some examples to illustrate our main results.

2. Preliminaries

In this section, we recall some essential notations, basic definitions and theorems which will be used in the subsequent of this paper:
Let $[a,\ b]$ be a finite interval of $\mathbb{R}$. Denote $L_1([a,\ b])$ be the space of all Lebesgue measurable functions with

$$\|f\|_1 = \int_a^b |f(t)|dt < \infty, \quad \text{for } t \in [a,\ b].$$

Now we state some definitions of fractional integral operators which contains Mittag-Leffler fuctions as kernel:

Definition 1. [15] Let $\vartheta, \alpha, \gamma, \omega, z \in \mathbb{C}$ such that $Re(\vartheta) > 0, Re(\alpha) > 0$. Let $t \in (a, b)$ and $f \in L_1([a,\ b])$, then Prabhakar fractional integral is defined as

$$\epsilon(\vartheta, \alpha, \gamma, \omega)f(t) = \int_a^t (t-x)^{\alpha-1} E^{\gamma}_{\vartheta,\alpha}(\omega(t-x)^{\vartheta})f(x)dx, \tag{2.1}$$

where $E^{\gamma}_{\vartheta,\alpha}(z)$ is a Mittag-Leffler function

$$E^{\gamma}_{\vartheta,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\vartheta n+\alpha)} \frac{z^n}{n!}.$$

Definition 2. [24] Let $\vartheta, \alpha, \kappa, \gamma, \omega, z \in \mathbb{C}$ such that $Re(\vartheta) > \max\{0, Re(\kappa)-1\}$, $\min\{Re(\alpha), Re(\kappa)\} > 0$. Let $t \in [a, b]$ and $f \in L_1([a,\ b])$, then Srivastva-Tomovski fractional integral is defined as

$$(\epsilon^{\omega,\gamma,\kappa}_{a^+,\vartheta,\alpha} f)(t) = \int_a^t (t-x)^{\alpha-1} E^{\gamma,\kappa}_{\vartheta,\alpha}(\omega(t-x)^{\vartheta})f(x)dx, \quad (t > a). \tag{2.2}$$

where $E^{\gamma,\kappa}_{\vartheta,\alpha}(z)$ is a Mittag-Leffler function

$$E^{\gamma,\kappa}_{\vartheta,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\kappa}}{\Gamma(\vartheta n+\alpha)} \frac{z^n}{n!}.$$

Definition 3. [2] Let $\vartheta,\ \alpha,\ \iota,\ \gamma,\ \varsigma, z \in \mathbb{C}$, $Re(\vartheta), Re(\alpha), Re(\iota) > 0$, $Re(\varsigma) > Re(\gamma) > 0$ with $p \geq 0,\ \lambda > 0$ and $0 < \kappa \leq \lambda + Re(\vartheta)$. Then the extended generalized Mittag-Leffler function is defined as

$$E^{\gamma,\lambda,\kappa,\varsigma}_{\vartheta,\alpha,\iota}(z;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+n\kappa, \varsigma-\gamma)}{\beta(\gamma, \varsigma-\gamma)} \frac{(\varsigma)_{n\kappa}}{\Gamma(\vartheta n+\alpha)} \frac{z^n}{(\iota)_{n\lambda}}, \tag{2.3}$$

where β_p is an extension of the beta function

$$\beta_p(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}e^{-\frac{p}{t(1-t)}}dt, \quad (Re(r),Re(s),Re(p)>0).$$

Definition 4. [2] Let $\omega,\ \vartheta,\ \alpha,\ \iota,\ \gamma,\ \varsigma\in\mathbb{C}$, $Re(\vartheta),Re(\alpha),Re(\iota)>0$, $Re(\varsigma)>Re(\gamma)>0$ with $p\geq 0,\ \lambda>0$ and $0<\kappa\leq\lambda+Re(\vartheta)$. Let $f\in L_1[a,\ b]$ and $x\in[a,\ b]$. Then the generalized fractional integral operator is defined as

$$(J^{\omega,\gamma,\lambda,\kappa,\varsigma}_{a^+,\vartheta,\alpha,\iota}f)(x;p)=\int_a^x(x-z)^{\alpha-1}E^{\gamma,\lambda,\kappa,\varsigma}_{\vartheta,\alpha,\iota}(\omega(x-z)^{\vartheta};p)f(z)dz. \tag{2.4}$$

We use following notations:

$$\boldsymbol{E}_\alpha(z;p)=E^{\gamma,\lambda,\kappa,\varsigma}_{\vartheta,\alpha,\iota}(z;p).$$

$$(\mathfrak{J}_\alpha f)(x;p)=(J^{\omega,\gamma,\lambda,\kappa,\varsigma}_{a^+,\vartheta,\alpha,\iota}f)(x;p).$$

Theorem 2.1. [2] Let $\omega,\ \vartheta,\ \alpha,\ \iota,\ \gamma,\ \varsigma\in\mathbb{C}$, $Re(\vartheta),Re(\alpha),Re(\iota)>0$, $Re(\varsigma)>Re(\gamma)>0$ with $p\geq 0,\ \lambda>0$ and $0<\kappa\leq\lambda+Re(\vartheta)$. Let $f\in L_1[a,\ b]$ and $x\in[a,\ b]$, then the generalized fractional integral operator $(J^{\omega,\gamma,\lambda,\kappa,\varsigma}_{a,\vartheta,\alpha,\iota})$ is bounded on $L_1[a,b]$, that is

$$\|(\mathfrak{J}_\alpha\ f)\|\leq C\|f\|_1$$

where C is constant given by

$$C=(b-a)^{Re(\alpha)}\sum_{n=0}^{\infty}\frac{|\beta_p(\gamma+n\kappa,\varsigma-\gamma)|}{|\beta(\gamma,\varsigma-\gamma)|}\frac{|(\varsigma)_{n\kappa}|}{(nRe(\vartheta)+Re(\alpha))|\Gamma(\vartheta n+\alpha)|}\frac{|\omega(b-a)^{Re(\vartheta)}|^n}{|(\iota)_{n\lambda}|}. \quad \square$$

By choosing different values of parameters in (2.4), we get corresponding fractional integral operators for more details see remark (2.2) in [2].

3. Reverse Minkowski fractional integral inequality using extended generalized fractional integral

In this section, we prove reverse Minkowski integral inequalities by using extended generalized fractional integral operator.

Theorem 3.1. Let ζ and η be two positive functions defined on $[0,\infty)$ such that for all $t>a\geq 0$, $k\geq 1$, $\alpha>0$, $(\mathfrak{J}_\alpha\zeta^k)(t;p)<\infty$, $(\mathfrak{J}_\alpha\eta^k)(t;p)<\infty$. If $0<l\leq\frac{\zeta(s)}{\eta(s)}\leq Ł$, $s\in[a,t]$, then following inequalities hold:

$$\Big[(\mathfrak{J}_\alpha\zeta^k)(t;p)\Big]^{\frac{1}{k}}+\Big[(\mathfrak{J}_\alpha\eta^k)(t;p)\Big]^{\frac{1}{k}}\leq\frac{1+Ł(l+2)}{(l+1)(Ł+1)}\Big[\mathfrak{J}_\alpha(\zeta+\eta)^k(t;p)\Big]^{\frac{1}{k}}. \tag{3.1}$$

Proof. Since $\frac{\zeta(s)}{\eta(s)}\leq Ł$, $s\in[0,t]$, $t>0$, we have

$$\zeta(s)\leq Ł\eta(s). \tag{3.2}$$

Adding $Ł\zeta(s)$ on both sides in (3.2), we get

$$\zeta(s) + Ł\zeta(s) \leq Ł\eta(s) + Ł\zeta(s).$$

From above we get

$$(Ł + 1)^k \zeta^k(s) \leq Ł^k (\zeta + \eta)^k(s). \tag{3.3}$$

Multiplying both the sides of (3.3) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)$, $s \in$ (a, t) and integrating resulting inequality with respect to s from a to t, we obtain

$$\begin{aligned} &(Ł + 1)^k \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)\zeta^k(s)ds \\ \leq\ & Ł^k \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)(\zeta+\eta)^k(s)ds. \end{aligned}$$

From above we have

$$(Ł + 1)^k (\mathfrak{J}_\alpha \zeta^k)(t; p) \leq Ł^k (\mathfrak{J}_\alpha (\zeta + \eta)^k)(t; p). \tag{3.4}$$

Taking $(\frac{1}{k})^{th}$ power on both sides of (3.4), we get

$$\big[(\mathfrak{J}_\alpha \zeta^k)(t; p)\big]^{\frac{1}{k}} \leq \frac{Ł}{Ł + 1}\big[(\mathfrak{J}_\alpha (\zeta + \eta)^k)(t; p)\big]^{\frac{1}{k}}. \tag{3.5}$$

Also from $l \leq \frac{\zeta(s)}{\eta(s)}$, we have

$$\eta(s) \leq \frac{1}{l}\zeta(s). \tag{3.6}$$

Adding $\frac{1}{l}\eta(s)$ on both sides of (3.6), we obtain

$$\eta(s) + \frac{1}{l}\eta(s) \leq \frac{1}{l}\zeta(s) + \frac{1}{l}\eta(s).$$

Which gives

$$\Big(1 + \frac{1}{l}\Big)^k \eta^k(s) \leq \Big(\frac{1}{l}\Big)^k (\zeta + \eta)^k(s). \tag{3.7}$$

Multiplying both the sides of (3.7) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)$, $s \in$ (a, t) and integrating obtained inequality with respect to s from a to t, we get

$$\begin{aligned} &\Big(1 + \frac{1}{l}\Big)^k \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)\eta^k(s)ds \\ \leq\ & \Big(\frac{1}{l}\Big)^k \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta; p)(\zeta+\eta)^k ds. \end{aligned}$$

It follows that

$$(\mathfrak{J}_\alpha \eta^k)(t; p) \leq \frac{1}{(l+1)^k}(\mathfrak{J}_\alpha (\zeta + \eta)^k)(t; p). \tag{3.8}$$

Taking $(\frac{1}{k})^{th}$ power on both sides of (3.8), we get

$$\big[(\mathfrak{J}_\alpha \eta^k)(t; p)\big]^{\frac{1}{k}} \leq \frac{1}{l + 1}\big[(\mathfrak{J}_\alpha (\zeta + \eta)^k)(t; p)\big]^{\frac{1}{k}}. \tag{3.9}$$

Adding inequalities (3.5) and (3.9), we get inequality (3.1). □

Theorem 3.2. Let ζ and η be two positive functions defined on $[0,\infty)$ such that for all $t > a \geq 0$, $k \geq 1$, $\alpha > 0$, $(\mathfrak{J}_\alpha \zeta^k)(t;p) < \infty$, $(\mathfrak{J}_\alpha \eta^k)(t;p) < \infty$. If $0 < l \leq \frac{\zeta(s)}{\eta(s)} \leq Ł$, $s \in [a,t]$, then following inequalities hold:

$$\Big[\frac{(Ł+1)(l+1)}{Ł} - 2\Big]\Big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\Big]^{\frac{1}{k}}\Big[(\mathfrak{J}_\alpha \eta^k)(t;p)\Big]^{\frac{1}{k}} \leq \Big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\Big]^{\frac{2}{k}} + \Big[(\mathfrak{J}_\alpha \eta^k)(t;p)\Big]^{\frac{2}{k}}. \tag{3.10}$$

Proof. Multiplying the inequalities (3.5) and (3.9), we get

$$\frac{(Ł+1)(l+1)}{Ł}\Big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\Big]^{\frac{1}{k}}\Big[(\mathfrak{J}_\alpha \eta^k)(t;p)\Big]^{\frac{1}{k}} \leq \Big[\big((\mathfrak{J}_\alpha (\zeta+\eta)^k)(t;p)\big)^{\frac{1}{k}}\Big]^2. \tag{3.11}$$

Applying the Minkowski inequality [8] to the right hand side of (3.11), we have

$$\Big[\big((\mathfrak{J}_\alpha (\zeta+\eta)^k)(t;p)\big)^{\frac{1}{k}}\Big]^2 \leq \Big[\big((\mathfrak{J}_\alpha \zeta^k)(t;p)\big)^{\frac{1}{k}} + \big((\mathfrak{J}_\alpha \eta^k)(t;p)\big)^{\frac{1}{k}}\Big]^2. \tag{3.12}$$

It follows that

$$\begin{aligned}\big[(\mathfrak{J}_\alpha (\zeta+\eta)^k)(t;p)\big]^{\frac{2}{k}} \leq& \big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{2}{k}} + \big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{2}{k}} \\ &+ 2\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{1}{k}}\big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{1}{k}}.\end{aligned} \tag{3.13}$$

From (3.11) and (3.13), we obtain

$$\begin{aligned}&\Big[\frac{(Ł+1)(l+1)}{Ł}\Big]\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{1}{k}}\big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{1}{k}} \\ \leq\ & \big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{2}{k}} + \big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{2}{k}} + 2\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{1}{k}} \\ & \big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{1}{k}}.\end{aligned} \tag{3.14}$$

From (3.14) we get required inequality (3.10). □

Theorem 3.3. Let $\alpha > 0$, $k \geq 1$. Let ζ and η be two positive functions defined on $[0,\infty)$ such that for all $t > a \geq 0$, $(\mathfrak{J}_\alpha \zeta^k)(t;p) < \infty$, $(\mathfrak{J}_\alpha \eta^k)(t;p) < \infty$. If $0 < \delta < l \leq \frac{z\zeta(s)}{\eta(s)} \leq Ł$, for $z > 0$, $s \in [a,t]$, then following inequalities hold:

$$\begin{aligned}\frac{Ł+z}{z(Ł-\delta)}\Big[(\mathfrak{J}_\alpha (z\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}} \leq& \Big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\Big]^{\frac{1}{k}} + \Big[(\mathfrak{J}_\alpha \eta^k)(t;p)\Big]^{\frac{1}{k}} \\ \leq& \frac{l+z}{z(l-\delta)}\Big[(\mathfrak{J}_\alpha (z\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}}.\end{aligned} \tag{3.15}$$

Proof. From the given condition $\delta < l \leq \frac{z\zeta(s)}{\eta(s)} \leq Ł$, we get

$$l - \delta \leq \frac{z\zeta(s)}{\eta(s)} - \delta \leq Ł - \delta.$$

It follows that

$$\frac{1}{(Ł-\delta)^k}[z\zeta(s)-\delta\eta(s)]^k \le \eta^k(s) \le \frac{1}{(l-\delta)^k}[z\zeta(s)-\delta\eta(s)]^k. \tag{3.16}$$

Multiplying by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s\in$ (a, t) to (3.16) and integrating with respect to s from a to t, we get

$$\begin{aligned}\frac{1}{(Ł-\delta)}\big[(\mathfrak{J}_\alpha(z\zeta-\delta\eta)^k)(t;p)\big]^{\frac{1}{k}} \le& \big[(\mathfrak{J}_\alpha\eta^k)(t;p)\big]^{\frac{1}{k}}\\ \le&\frac{1}{(l-\delta)}\big[(\mathfrak{J}_\alpha(z\zeta-\delta\eta)^k)(t;p)\big]^{\frac{1}{k}}.\end{aligned} \tag{3.17}$$

Again, from the condition $l\le\frac{z\zeta(s)}{\eta(s)}\le Ł$, $s\in[a,t]$, we get

$$-\frac{1}{l}\le-\frac{\eta(s)}{z\zeta(s)}\le-\frac{1}{Ł},$$

then

$$\frac{1}{\delta}-\frac{1}{l}\le\frac{1}{\delta}-\frac{\eta(s)}{z\zeta(s)}\le\frac{1}{\delta}-\frac{1}{Ł},$$

which gives

$$\frac{l-\delta}{\delta l}\le\frac{z\zeta(s)-\delta\eta(s)}{\delta z\zeta(s)}\le\frac{Ł-\delta}{\delta Ł}.$$

It follows that

$$\frac{Ł}{z(Ł-\delta)}\big[z\zeta(s)-\delta\eta(s)\big]\le\zeta(s)\le\frac{l}{z(l-\delta)}\big[z\zeta(s)-\delta\eta(s)\big].$$

From above we get

$$\left[\frac{Ł}{z(Ł-\delta)}\right]^k\Big[z\zeta(s)-\delta\eta(s)\Big]^k\le\Big[\zeta(s)\Big]^k\le\left[\frac{l}{z(l-\delta)}\right]^k\Big[z\zeta(s)-\delta\eta(s)\Big]^k. \tag{3.18}$$

Multiplying by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s\in$ (a, t) to (3.18) and integrating with respect to s from a to t, we get

$$\begin{aligned}\frac{Ł}{z(Ł-\delta)}\Big[(\mathfrak{J}_\alpha(z\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}} \le& \Big[(\mathfrak{J}_\alpha\zeta^k)(t;p)\Big]^{\frac{1}{k}}\\ \le&\frac{l}{z(l-\delta)}\Big[(\mathfrak{J}_\alpha(z\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}}. \quad\square\end{aligned}$$

Remark 3.1. For $p=\omega=0$ and $\alpha=1$ in theorem (3.3), we get inequality (2.2)*(see theorem (2.1))* in [3]. $\square$

4. Reverse Holder's inequality using extended generalized fractional integral operator

In this section we prove some reverse Holder's like fractional integral inequalities using extended generalized fractional integral operator.

Theorem 4.1. Let $\alpha > 0$, $k > 1$, $\frac{1}{k} + \frac{1}{j} = 1$ and ζ, η be two positive functions defined on $[0, \infty)$ such that $(\mathfrak{J}_\alpha \zeta)(t;p) < \infty$ and $(\mathfrak{J}_\alpha \eta)(t;p) < \infty$. If $0 < l \leq \frac{\zeta(s)}{\eta(s)} \leq$ Ł, for $s \in [a, t]$, $t > a \geq 0$, then following inequalities hold:

$$\left[(\mathfrak{J}_\alpha \zeta)(t;p)\right]^{\frac{1}{k}} \left[(\mathfrak{J}_\alpha \eta)(t;p)\right]^{\frac{1}{j}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{kj}} \left[(\mathfrak{J}_\alpha \zeta^{\frac{1}{k}} \eta^{\frac{1}{j}})(t;p)\right]. \tag{4.1}$$

Proof. Since $\frac{\zeta(s)}{\eta(s)} \leq Ł, s \in [a, t], t > 0$, we have

$$\zeta(s) \leq Ł\eta(s). \tag{4.2}$$

Taking $\left(\frac{1}{j}\right)^{th}$ power on both sides of (4.2), we get

$$\zeta^{\frac{1}{j}}(s) \leq Ł^{\frac{1}{j}} \eta^{\frac{1}{j}}(s). \tag{4.3}$$

Multiplying both sides of (4.3) by $\zeta^{\frac{1}{k}}(s)$, we have

$$\zeta(s)Ł^{-\frac{1}{j}} \leq \eta^{\frac{1}{j}}(s)\zeta^{\frac{1}{k}}(s). \tag{4.4}$$

Multiplying both the sides of (4.4) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) and integrating with respect to s from a to t, we get

$$\begin{aligned} & Ł^{-\frac{1}{j}} \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)\zeta(s)ds \\ \leq & \int_a^t (t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)\eta^{\frac{1}{j}}(s)\zeta^{\frac{1}{k}}(s)ds. \end{aligned}$$

It follows that

$$Ł^{-\frac{1}{kj}} \left[(\mathfrak{J}_\alpha \zeta)(t;p)\right]^{\frac{1}{k}} \leq \left[(\mathfrak{J}_\alpha \zeta^{\frac{1}{k}} \eta^{\frac{1}{j}})(t;p)\right]^{\frac{1}{k}}. \tag{4.5}$$

On the other hand, we have

$$l \leq \frac{\zeta(s)}{\eta(s)} \Rightarrow l^{\frac{1}{k}} \eta^{\frac{1}{k}}(s) \leq \zeta^{\frac{1}{k}}(s). \tag{4.6}$$

Multiplying by $\eta^{\frac{1}{j}}(s)$ both sides of (4.6), we get

$$l^{\frac{1}{k}} \eta(s) \leq \zeta^{\frac{1}{k}}(s)\eta^{\frac{1}{j}}(s). \tag{4.7}$$

Multiplying both the sides of (4.7) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) and integrating with respect to s from a to t, we get

$$l^{\frac{1}{kj}} \left[(\mathfrak{J}_\alpha \eta)(t;p)\right]^{\frac{1}{j}} \leq \left[(\mathfrak{J}_\alpha \zeta^{\frac{1}{k}} \eta^{\frac{1}{j}})(t;p)\right]^{\frac{1}{j}}. \tag{4.8}$$

Multiplying (4.5) and (4.8), we get required inequality (4.1). □

Corollary 4.1. Let $\alpha > 0$, $k \geq 1$, $\frac{1}{k} + \frac{1}{j} = 1$ and ζ, η be two positive functions defined on $[0, \infty)$ such that $(\mathfrak{J}_\alpha \zeta^k)(t;p) < \infty$ and $(\mathfrak{J}_\alpha \eta^j)(t;p) < \infty$. If $0 < l \leq \frac{(\zeta(s))^k}{(\eta(s))^j} \leq$ Ł, for $s \in [a, t]$, $t > a \geq 0$, then following inequalities hold:

$$\left[(\mathfrak{J}_\alpha \zeta^k)(t;p)\right]^{\frac{1}{k}} \left[(\mathfrak{J}_\alpha \eta^j)(t;p)\right]^{\frac{1}{j}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{kj}} \left[(\mathfrak{J}_\alpha \zeta\eta)(t;p)\right]. \quad \square \tag{4.9}$$

5. Certain Fractional integral inequalities

Theorem 5.1. Let $\alpha > 0$, $k \geq 1$, $\frac{1}{k} + \frac{1}{j} = 1$. Let ζ, η be two positive functions defined on $[0, \infty)$. If $0 < l \leq \frac{\zeta(s)}{\eta(s)} \leq Ł$, for $l, Ł \in \mathbb{R}$, and $s \in [a, t]$, $t > a \geq 0$, then following inequalities hold:

$$\big[(\mathfrak{J}_\alpha \zeta\eta)(t;p)\big] \leq c_1\big[(\mathfrak{J}_\alpha(\zeta^k + \eta^k))(t;p)\big] + c_2\big[(\mathfrak{J}_\alpha(\zeta^j + \eta^j))(t;p)\big], \tag{5.1}$$

$$\text{where} \quad c_1 = \frac{2^{k-1}Ł^k}{k(Ł+1)^k} \text{ and } c_2 = \frac{2^{j-1}}{j(l+1)^j}.$$

Proof. Since $\frac{\zeta(s)}{\eta(s)} \leq Ł$, $s \in [a, t]$, $t > 0$, we have

$$(Ł+1)^k \zeta^k(s) \leq Ł^k(\zeta + \eta)^k(s). \tag{5.2}$$

Multiplying both the sides of (5.2) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) and integrating with respect to s from a to t, we get

$$(Ł+1)^k(\mathfrak{J}_\alpha \zeta^k)(t;p) \leq Ł^k(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p).$$

It follows that

$$\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big] \leq \frac{Ł^k}{(Ł+1)^k}\big[(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p)\big]. \tag{5.3}$$

Now by using the condition $l \leq \frac{\zeta(s)}{\eta(s)}$, $s \in [a, t]$, $t > 0$, we have

$$\eta^j(s) \leq \frac{1}{(l+1)^j}(\zeta+\eta)^j(s). \tag{5.4}$$

Multiplying by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) to (5.4) and integrating with respect to s from a to t, we get

$$(\mathfrak{J}_\alpha \eta^j)(t;p) \leq \frac{1}{(l+1)^j}(\mathfrak{J}_\alpha(\zeta+\eta)^j)(t;p). \tag{5.5}$$

Now by using Young's inequality, we have

$$\zeta(s)\eta(s) \leq \frac{\zeta^k(s)}{k} + \frac{\eta^j(s)}{j}. \tag{5.6}$$

Multiplying by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) to (5.6) and integrating with respect to s from a to t, we get

$$(\mathfrak{J}_\alpha \zeta\eta)(t;p) \leq \frac{1}{k}\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big] + \frac{1}{j}\big[(\mathfrak{J}_\alpha \eta^j)(t;p)\big]. \tag{5.7}$$

From (5.3), (5.5) and (5.7), we obtain

$$(\mathfrak{J}_\alpha \zeta\eta)(t;p) \leq \frac{Ł^k}{k(Ł+1)^k}\big[(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p)\big] + \frac{1}{j(l+1)^j}(\mathfrak{J}_\alpha(\zeta+\eta)^j)(t;p). \tag{5.8}$$

Now using elementary inequality, $(m+n)^v \leq 2^{v-1}(m^v + n^v), v > 1, m, n \geq 0$, we get

$$(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p) \leq 2^{k-1}(\mathfrak{J}_\alpha(\zeta^k + \eta^k))(t;p) \tag{5.9}$$

and

$$(\mathfrak{J}_\alpha(\zeta+\eta)^j)(t;p) \leq 2^{j-1}(\mathfrak{J}_\alpha(\zeta^j+\eta^j)(t;p). \tag{5.10}$$

From (5.8), (5.9) and (5.10), we get required inequality (5.1). □

Theorem 5.2. Let $\alpha > 0$, $k \geq 1$ and ζ and η be two positive functions in $[0, \infty)$ such that for all $t \geq 0$, $(\mathfrak{J}_\alpha \zeta^k)(t;p) < \infty$ and $(\mathfrak{J}_\alpha \eta^x)(t;p) < \infty$. If $0 < m_1 \leq \zeta(s) \leq M_1$ and $0 \leq m_2 \leq \eta(s) \leq M_2$, for $s \in [a, t]$, then following inequalities hold:

$$\big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\big]^{\frac{1}{k}} + \big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{1}{k}} \leq M\big[(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p)\big]^{\frac{1}{k}}, \tag{5.11}$$

where

$$M = \frac{M_1(m_1+M_2)+M_2(m_2+M_1)}{(m_1+M_2)(m_2+M_1)}.$$

Proof. Since $m_2 \leq \eta(s) \leq M_2$, then we have

$$\frac{1}{M_2} \leq \frac{1}{\eta(s)} \leq \frac{1}{m_2}. \tag{5.12}$$

Also, we have

$$m_1 \leq \zeta(s) \leq M_1. \tag{5.13}$$

By using (5.12) and (5.13), we can write

$$\frac{m_1}{M_2} \leq \frac{\zeta(s)}{\eta(s)} \leq \frac{M_1}{m_2}. \tag{5.14}$$

Now from (5.14), we get

$$\eta(s) \leq \frac{M_2\zeta(s)}{m_1} \ \Rightarrow\ \eta(s)\big(1+\frac{M_2}{m_1}\big) \leq \frac{M_2}{m_1}(\zeta(s)+\eta(s)). \tag{5.15}$$

From (5.15), it follows that

$$(\eta(s))^k \leq \Big(\frac{M_2}{m_1+M_2}\Big)^k(\zeta(s)+\eta(s))^k. \tag{5.16}$$

Similarly by using the condition $\frac{\zeta(s)}{\eta(s)} \leq \frac{M_1}{m_2}$, we get

$$(\zeta(s))^k \leq \Big(\frac{M_1}{m_2+M_1}\Big)^k(\zeta(s)+\eta(s))^k. \tag{5.17}$$

Multiplying both the sides of (5.16) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) and integrating with respect to s from a to t, we obtain

$$(\mathfrak{J}_\alpha \eta^k)(t;p) \leq \Big(\frac{M_2}{m_1+M_2}\Big)^k(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p). \tag{5.18}$$

Taking $(\frac{1}{k})^{th}$ power on both the sides of (5.18), we get

$$\big[(\mathfrak{J}_\alpha \eta^k)(t;p)\big]^{\frac{1}{k}} \leq \frac{M_2}{m_1+M_2}\big[(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p)\big]^{\frac{1}{k}}. \tag{5.19}$$

Similarly from (5.17), we have

$$\left[(\mathfrak{J}_\alpha \zeta^k)(t;p)\right]^{\frac{1}{k}} \leq \frac{M_1}{m_2+M_1}\left[(\mathfrak{J}_\alpha(\zeta+\eta)^k)(t;p)\right]^{\frac{1}{k}}. \tag{5.20}$$

Adding inequalities (5.19) and (5.20), we get required inequality (5.11). □

Theorem 5.3. Let $\alpha > 0$, ζ and η be two positive functions in $[0,\infty)$ such that for all $t \geq 0$, $(\mathfrak{J}_\alpha \zeta)(t;p) < \infty$ and $(\mathfrak{J}_\alpha \eta)(t;p) < \infty$. If $0 < l \leq \frac{\zeta(s)}{\eta(s)} \leq Ł$ for $s \in [a,t]$, then following inequalities hold:

$$\frac{1}{Ł}(\mathfrak{J}_\alpha(\zeta\eta))(t;p) \leq \frac{1}{(l+1)(Ł+1)}(\mathfrak{J}_\alpha(\zeta+\eta)^2)(t;p) \leq \frac{1}{l}(\mathfrak{J}_\alpha(\zeta\eta))(t;p). \tag{5.21}$$

Proof. Since

$$l \leq \frac{\zeta(s)}{\eta(s)} \leq Ł. \tag{5.22}$$

Therefore

$$\frac{1}{Ł} \leq \frac{\eta(s)}{\zeta(s)} \leq \frac{1}{l}.$$

From above, it follows that

$$\Big(\frac{Ł+1}{Ł}\Big)\zeta(s) \leq \eta(s)+\zeta(s) \leq \Big(\frac{l+1}{l}\Big)\zeta(s). \tag{5.23}$$

Also, from (5.22), we have

$$(l+1)\eta(s) \leq \eta(s)+\zeta(s) \leq (Ł+1)\eta(s). \tag{5.24}$$

Taking multiplication between (5.23) and (5.24), we obtain following inequality

$$\frac{\eta(s)\zeta(s)}{Ł} \leq \frac{(\eta(s)+\zeta(s))^2}{(l+1)(Ł+1)} \leq \frac{\eta(s)\zeta(s)}{l}. \tag{5.25}$$

Multiplying both the sides of (5.25) by $(t-s)^{\alpha-1}\boldsymbol{E}_\alpha(\omega(t-s)^\vartheta;p)$, $s \in$ (a, t) and integrating with respect to s from a to t, we obtain

$$\frac{1}{Ł}(\mathfrak{J}_\alpha(\eta\zeta))(t;p) \leq \frac{1}{(l+1)(Ł+1)}(\mathfrak{J}_\alpha(\eta+\zeta)^2)(t;p) \leq \frac{1}{l}(\mathfrak{J}_\alpha(\eta\zeta))(t;p). \quad \square$$

Theorem 5.4. Let $\alpha > 0$, $k \geq 1$. Let ζ and η be two positive functions defined on $[0,\infty)$ such that for all $t > a \geq 0$, $(\mathfrak{J}_\alpha \zeta^k)(t;p) < \infty$, $(\mathfrak{J}_\alpha \eta^k)(t;p) < \infty$. If $0 < \delta < l \leq \frac{\zeta(s)}{\eta(s)} \leq Ł$, $s \in [a,t]$, then following inequalities hold:

$$\begin{aligned}\frac{Ł+1}{Ł-\delta}\Big[(\mathfrak{J}_\alpha(\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}} \leq & \Big[(\mathfrak{J}_\alpha \zeta^k)(t;p)\Big]^{\frac{1}{k}} + \Big[(\mathfrak{J}_\alpha \eta^k)(t;p)\Big]^{\frac{1}{k}} \\ \leq & \frac{l+1}{l-\delta}\Big[(\mathfrak{J}_\alpha(\zeta-\delta\eta)^k)(t;p)\Big]^{\frac{1}{k}}. \end{aligned} \tag{5.26}$$

Proof. If we put z=1 in theorem (3.3), we get inequality (5.26). □

6. Examples

Example 1. Jordan inequality [11] is given by

$$\frac{2}{\pi} \leq \frac{sinr}{r} < 1, \quad 0 < r \leq \pi/2.$$

Then by using theorem (2.1), we have

$$(\mathfrak{J}_\alpha\ sin^k r)(t;p) < \infty \text{ and } (\mathfrak{J}_\alpha\ r^k)(t;p) < \infty, \quad k \geq 1.$$

Applying theorem (3.1), we get

$$\begin{aligned} & \left[(\mathfrak{J}_\alpha\ sin^k r)(t;p)\right]^{\frac{1}{k}} + \left[(\mathfrak{J}_\alpha\ r^k)(t;p)\right]^{\frac{1}{k}} \\ \leq \quad & \left[\frac{\frac{3\pi}{2}+1}{2(\pi+1)}\right]\left[(\mathfrak{J}_\alpha\ (sinr+r)^k(t;p)\right]^{\frac{1}{k}}. \end{aligned}$$

Also, using theorem (3.2), we obtain

$$\begin{aligned} & \frac{4}{\pi}\left[(\mathfrak{J}_\alpha\ sin^k r)(t;p)\right]^{\frac{1}{k}}\left[(\mathfrak{J}_\alpha\ r^k)(t;p)\right]^{\frac{1}{k}} \\ \leq \quad & \left[(\mathfrak{J}_\alpha\ sin^k r)(t;p)\right]^{\frac{2}{k}} + \left[(\mathfrak{J}_\alpha\ r^k)(t;p)\right]^{\frac{2}{k}}. \end{aligned}$$

Example 2. Since

$$\frac{r}{1+r} \leq 1 - e^{-r} \leq \frac{4}{3}\frac{r}{1+r}, \quad 0 < r < \infty.$$

By using theorem (2.1) and theorem (3.1), for $k \geq 1$, we have

$$\begin{aligned} & \left[\Big(\mathfrak{J}_\alpha\ \big(1-e^{-r}\big)^k\Big)(t;p)\right]^{\frac{1}{k}} + \left[\Big(\mathfrak{J}_\alpha\ \big(\frac{r}{1+r}\big)^k\Big)(t;p)\right]^{\frac{1}{k}} \\ \leq \quad & \frac{15}{14}\left[\Big(\mathfrak{J}_\alpha\ \big((1-e^{-r})+(\frac{r}{1+r})\big)^k\Big)(t;p)\right]^{\frac{1}{k}}. \end{aligned}$$

Next, by theorem (2.1) and theorem (3.2), for $k \geq 1$, we have

$$\begin{aligned} & \frac{3}{2}\left[\Big(\mathfrak{J}_\alpha\ \big(1-e^{-r}\big)^k\Big)(t;p)\right]^{\frac{1}{k}}\left[\Big(\mathfrak{J}_\alpha\ \big(\frac{r}{1+r}\big)^k\Big)(t;p)\right]^{\frac{1}{k}} \\ \leq \quad & \left[\Big(\mathfrak{J}_\alpha\ \big(1-e^{-r}\big)^k\Big)(t;p)\right]^{\frac{2}{k}} + \left[\Big(\mathfrak{J}_\alpha\ \big(\frac{r}{1+r}\big)^k\Big)(t;p)\right]^{\frac{2}{k}}. \end{aligned}$$

Example 3. In [16], Qi and Mahmoud proved following inequality:

$$\frac{tan(\frac{\pi}{4}r)}{ar} \leq \Gamma(r+1) < \frac{tan(\frac{\pi}{4}r)}{br}, \quad 0 < r \leq 1,$$

where $a = 1$ and $b = \frac{\pi}{4}$ are the best possible.
From above we have

$$1 \leq \frac{r\Gamma(r+1)}{tan(\frac{\pi}{4}r)} < \frac{4}{\pi}.$$

Then from theorem (2.1) and (4.1), for $k = j = 2$, we have

$$\Big[\Big(\mathfrak{J}_\alpha\, r\Gamma(r+1)\Big)(t;p)\Big]^{\frac{1}{2}}\Big[\Big(\mathfrak{J}_\alpha\, tan(\frac{\pi}{4}r)\Big)(t;p)\Big]^{\frac{1}{2}}$$

$$\leq \quad \left(\frac{4}{\pi}\right)^{\frac{1}{4}}\Big[\Big(\mathfrak{J}_\alpha\, \big(r\Gamma(r+1)\big)^{\frac{1}{2}}\big(tan(\frac{\pi}{4}r)\big)^{\frac{1}{2}}\Big)(t;p)\Big].$$

References

[1] G. A. Anastassiou, Advances on frcational inequalities, *Springer, New York*, 2011.

[2] M. Andric, G. Farid, J. Pecaric, A further extension of Mittag-Leffler function, *Fract. Calc. Appl. Anal.*, 21(5), 2018, 1377-1395.

[3] B. Benaissa, More on reverses of Minkowski's inequalities and Hardy's integral inequalities, *Asian Eur. J. Math.*, 13(1), 2018, 1-7.

[4] B. Benaissa, H. Budak, More on reverse of Holder's integral inequality, *Korean J. Math.*, 28(1), 2020, 9-15.

[5] L. Bougoffa, On Minkowski and Hardy integral inequalities, *J. inequal. Pure and Appl. Math.*, 7(2), art. 60, 2006.

[6] V. L. Chinchane, D. B. Pachpatte, New fractional inequalities involving Saigo fractional integral operator, *Math. Sci. Lett.*, 3(3), 2014, 133-139.

[7] Z. Dahmani, On Minkowski and Hermite-Hadmard integral inequalities via fractional integration, Ann. Funct. Anal., 1(1), 2010, 51-58.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, *Cambridge University Press, England*, 1988, 146-150.

[9] H. Khan, C. Tune, A. Alkhazan, B. Ameen, A. Khan, A generaliztion of Minkowski's inequality by Hahn integral operator, *J. Taibh Univ. Sci.*, 12(5), 2018, 506-519.

[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, *Elsevier, Amsterdam*, 204, 2006.

[11] D. S. Mitrinovic, Analytic inequalities, *Springer Verlag, Berlin*, 1970.

[12] M. C. Ozdemir, H. Kuvurmaci, M. A. Ardic, New inequalities of Ostrowski type for mappings whose derivatives are (α, m)-convex via fractional integrals, *Thai J. Math.*, 16(3), 2018, 723-731.

[13] D. B. Pachpatte, On Some ψ Caputo fractional Čebyšev like inequalities for functions of two and three variables, *AIMS Mathematics*, 5(3), 2020.

[14] I. Podlubny, Fractional Differential equations, *Academic Press, San Diego*, 198, 1999.

[15] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.*, 19(1971), 715.

[16] F. Qi, M. Mahmoud, Bounding the gamma function in terms of the Trignometric and Exponential functions, *Int. J. Optim. Control Theor. Appl.*, 8(2), 2018, 137-144.

[17] G. Rahman, A. Khan, T. Abdeljawad, K. S. Nisar, The Minkowski inequalities via generalized proportional fractional integral operators, *Adv. Differ.Equ.*, 287(2019), 2019.

[18] S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf, Y. M. Chu, New investigation on the generalized k-fractional integral operators, *Front. Phys.*, 8(25), 2020.

[19] J. E. Restrepo, V. L. Chinchane, P. Agarwal, Weighted reverse fractional inequalities of Minkowski's and Holder's type, *TWMS J. Pure Appl. Math.*, 10(2), 2019, 188-198.

[20] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, F. Ahmad, (k, s)-Riemann-Liouville fractional integral and applications, *Hacet. J. Math. Stat.*, 45(1), 2016, 77-89.

[21] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hetmite-Hadamard inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, 57(10), 2013, 2403-2407.

[22] E. Set, M. Tomar, M. Z. Sarikaya, On generalized Grüss type inequalities for k- fractional integrals, *Appl. Math. Comput.*, 269(2015), 2015, 29-34.

[23] J. Sousa, E. Olivera, The Minkowski's inequality by means of a generalized fractional integral, *AIMS Mathematics*, 3(1), 2018, 131-147.

[24] H. M. Srivastava, Z. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *Appl. Math. Comput. 211*, no 1 (2009), 198-210.

[25] B. Sroysang, More on reverses of Minkowski's integral inequality, *Math. Aeterna*, 3(7), 2013, 597-600.

[26] W. T. Sulaiman, Reverses of Minkowski's, Holder's and Hardy's integral inequalities, *Int. J. Mod. Sci.*, 1(1), 2012, 14-24.

[27] S. Taf, K. Brahim, Some new results using Hadamard fractional integral, *Int. J. Nonlinear Anal. Appl. Math.*, 7(1), 2015, 103-109.

[28] F. Usta, H. Budak, F. Ertugral, M. Z. Sarikaya, The Minkowski inequalities utilizing newly defined generalized fractional integral operators, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 68(1), 2019, 686-701.

[29] A. Yassine, J. Fahd, A. Thabet, On generalization of fractional operators and a gronwall type inequality with applications, *Filomat*, 31(17), 2017, 5457-5473.

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