Note on Dragomir's theorems

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1. Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. In paticular, we denote $A > 0$ if A is positive and invertible. The positivity of operators induces the order $A \geq B$ among selfadjoint operators by $A - B \geq 0$. A continuous function f on an interval I is said to be operator monotone if f is order preserving, i.e., $A \geq B$ for A, B whose spectra are included in I implies $f(A) \geq f(B)$.

Now, motivated by a recent work due to Dragomir, we discuss the positivity of

$$(B - A)(f(B) - f(A))$$

for $A, B > 0$ and operator monotone function f on $(0, \infty)$, in this note.

2. Very recently, Dragomir [1, Theorem 2] (an old version) mentioned the following theorem:

Theorem A. If f is operator monotone on $(0, \infty)$, then

$$(B - A)\{f(B) - f(A)\} \geq 0 \qquad (D)$$

for all $A, B > 0$.

Now we consider Theorem A for the operator monotone function $f(t) = -t^{-1}$. Since

$$(B - A)(-B^{-1} + A^{-1}) = AB^{-1} + BA^{-1} - 2,$$

it follows that Theorem A in this case is equivalent to

$$\frac{AB^{-1} + BA^{-1}}{2} \geq 1. \qquad (C)$$

The following example shows that (C) does not hold in general:

Example 1. Even if $A \geq B > 0$, the inequality (C) does not hold.

Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \geq B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

$$AB^{-1} + BA^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 23 & 6 \\ 3 & 26 \end{pmatrix},$$

so that $AB^{-1} + BA^{-1}$ is not selfadjoint.

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This example suggests that Theorem A needs some additional assumptions.

Proposition 2. For invertible positive operators A and B, if $AB^{-1} + BA^{-1}$ is selfadjoint, then

$$AB^{-1} + BA^{-1} \geq 2.$$

Proof. If we put $H = AB^{-1}$, then $C = H + H^{-1}$ is selfadjoint by the assumption. We note that the spectrum $\sigma(H)$ of H is included in $(0, \infty)$ because $A, B > 0$ and $\sigma(H) = \sigma(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})$. Hence, since $\sigma(C) = \{t + \frac{1}{t}; t \in \sigma(H)\}$ by the spectral mapping theorem for rational functions, we have $H + H^{-1} \geq 2$.

Proposition 2 ensures that Theorem A holds for the operator monotone function $f(t) = -\dfrac{1}{t}$ on $(0, 1)$, and so it suggests the following conjecture as a modification of Theorem A.

Conjecture 3. If f is operator monotone on $(0, \infty)$ and $A, B > 0$ such that $(B - A)\{f(B) - f(A)\}$ is selfadjoint, then

$$(D) \qquad (B - A)\{f(B) - f(A)\} \geq 0.$$

Incidentally, we cite the following example:

Example 4. Take $g(t) = \sqrt{t}$. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$. Then

$$(A - B)(A^{1/2} - B^{1/2}) = \begin{pmatrix} 4 & 3 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -4 & 17 \end{pmatrix} \not\geq 0.$$

That is, $(A - B)(A^{1/2} - B^{1/2})$ is not selfadjoint; Theorem A does not hold for the geometric mean function $g(t) = \sqrt{t}$.

Dragomir also mentions a similar result in [2, Theorem 3]:

Theorem B. If f is operator monotone on $(-1, 1)$, then

$$(D) \qquad (B - A)\{f(B) - f(A)\} \geq 0$$

for all selfadjoint A, B whose spectra contained in $(-1, 1)$.

For this, we can give a counterexample to it.

Example 5. Let A and B be as in Example 1, i.e., $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \geq B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

We take the function $f(t) = (4-t)^{-1}$. Then it is trivial that f is opertor monotone on $(-\infty, 4)$, and we have

$$(B-A)(f(B)-f(A)) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3/2 & -1 \\ -1 & 2/3 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/3 \\ 1/2 & -1/3 \end{pmatrix},$$

so that $(B-A)(f(B)-f(A))$ is not selfadjoint.

Based on the above example, we pose a counterexample of Theorem B. Let $A_1 = \frac{A}{4}$, $B_1 = \frac{B}{4}$ and $g(t) = (1-t)^{-1}$. Then g is operator monotone on $(-1,1)$ and

$$g(A_1) - g(B_1) = 4(f(A) - f(B)).$$

Hence it follows that $(B_1 - A_1)(f(B_1) - f(A_1)) = (B-A)(f(A)-f(B))$ is not selfadjoint.

Now we claim that Proposition 2 implies Conjecture 3 by birtue of the integral representation for operator montone functions.

Theorem 6. Let f be an operator monotone function on $(0,\infty)$ and $A, B > 0$ such that $A(B+s)^{-1} + B(A+s)^{-1}$ is selfadjoint for all $s \geq 0$. Then

$$(D) \qquad (B-A)\{f(B) - f(A)\} \geq 0.$$

Proof. Since f be an operator monotone function on $(0,\infty)$, it has the integral representation

$$f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} d\mu(s),$$

where $a \in \mathbb{R}$, $b \geq 0$ and μ is a positive masure on $(0,\infty)$ with $\int_0^\infty \frac{s}{1+s} d\mu(s) < +\infty$. So we may assume that $f(t) = \frac{ts}{t+s}$ for some $s > 0$ and moreover $f(t) = \frac{t}{t+s} = 1 - \frac{s}{t+s}$ for some $s > 0$.

Namely it suffices to show that $(B-A)(f(B)-f(A)) \geq 0$ for $f(t) = -\frac{1}{t+s}$ under the assumption $A(B+s)^{-1} + B(A+s)^{-1}$ is selfadjoint. Put $X_s = X + s$.

$$\begin{aligned} (B-A)(f(B)-f(A)) &= (B-A)((A+s)^{-1} - (B+s)^{-1}) \\ &= ((B+s) - (A+s))((A+s)^{-1} - (B+s)^{-1}) \\ &= B_s A_s^{-1} + A_s B_s^{-1} - (A_s A_s^{-1} + B_s B_s^{-1}) \\ &= B_s A_s^{-1} + A_s B_s^{-1} - 2 \geq 0 \end{aligned}$$

by Proposition 2.

As a corollary of Theorem 6, we have an extension of Proposition 2.

Proposition 7. If $A(B+s)^{-1}+B(A+s)^{-1}$ is selfadjoint for given $A, B>0$ and $s \geq 0$, then

$$A(B+s)^{-1}+B(A+s)^{-1} \geq \frac{m_A}{m_A+s}+\frac{m_B}{m_B+s},$$

where $m_T=\|T^{-1}\|^{-1}$.

Proof. Fix $s>0$. As in the proof of Theorem 6, we have

$$(B-A)((A+s)^{-1}-(B+s)^{-1}) \geq 0,$$

so that

$$B(A+s)^{-1}+A(B+s)^{-1}) \geq A(A+s)^{-1}+B(B+s)^{-1} \geq \frac{m_A}{m_A+s}+\frac{m_B}{m_B+s}.$$

3. We present an example of a pair $A, B>0$ satisfying the assumption in above that $A(B+s)^{-1}+B(A+s)^{-1}$ is selfadjoint for all $s \geq 0$.

Define a positive map on the C^*-algebra $\mathbb{M}_2$ of the 2×2 matrices by

$$\hat{X}=\begin{pmatrix} x & -y \\ -z & w \end{pmatrix} \quad \text{and} \quad X=\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Lemma 8. For $A=\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \in \mathbb{M}_2$, $A\hat{A}=\hat{A}A$ if and only if $(a-b)c=0$.

As a matter of fact, it follows from

$$A\hat{A}=\begin{pmatrix} a^2-|c|^2 & -(a-b)c \\ (a-b)c^* & b^2-|c|^2 \end{pmatrix} \quad \text{for} \quad \hat{A}A=\begin{pmatrix} a^2-|c|^2 & (a-b)c \\ -(a-b)c^* & b^2-|c|^2 \end{pmatrix}.$$

Next, since $\hat{X}=JXJ$ for $J=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have the invariance of the determinant and spectra. We denote the spectrum of X by $\sigma(X)$.

Lemma 9. $\det(X)=\det(\hat{X})$ and $\sigma(X)=\sigma(\hat{X})$ for $X \in \mathbb{M}_2$.

Proposition 10. If $A=\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \in \mathbb{M}_2$ is selfadjoint, i.e., $a, b \in \mathbb{R}$, then so is $A(\hat{A}+s)^{-1}+\hat{A}(A+s)^{-1}$ for $s \in \mathbb{R}\setminus\sigma(-A)$.

Proof. For a fixed $s \in \mathbb{R} \setminus \sigma(A)$, we have $d = \det(A+s) = \det(\hat{A}+s)$ by Lemma 9. Hence it is easily checked that

$$\begin{aligned} A(\hat{A}+s)^{-1} + \hat{A}(A+s)^{-1} &= \frac{1}{d}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}\begin{pmatrix} b+s & c \\ c^* & a+s \end{pmatrix} + \frac{1}{d}\begin{pmatrix} a & -c \\ -c^* & b \end{pmatrix}\begin{pmatrix} b+s & -c \\ -c^* & a+s \end{pmatrix} \\ &= \frac{2}{d}\begin{pmatrix} a(b+s)+|c|^2 & 0 \\ 0 & b(a+s)+|c|^2 \end{pmatrix}. \end{aligned}$$

Corollary 11. If $A \in \mathbb{M}_2$ is positive, then so is $A(\hat{A}+s)^{-1} + \hat{A}(A+s)^{-1}$ for $s \geq 0$.

References

[1] S. S. Dragomir, *Some inequalities for operator monotone functions*, RGMIA, V23a57.

[2] S. S. Dragomir, *Several inequalities for operator monotone functions on finite intervals*, RGMIA, V23a58.

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