OPERATOR MONOTONICITY OF THE LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *logarithmic integral transform*

$$\mathcal{L}og\,(w,\mu)\,(T) := \int_0^\infty w(\lambda)\ln(\lambda + T)\,d\mu(\lambda)\,,$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $B \geq A > 0$, then $\mathcal{L}og\,(w,\mu)\,(B) \geq \mathcal{L}og\,(w,\mu)\,(A)$, namely $\mathcal{L}og\,(w,\mu)\,(w,\mu)$ is operator monotone on $(0,\infty)$. If $\delta \geq A$, $B \geq \alpha > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then

$$0 \leq m\mathcal{D}(w,\mu)(\delta) \leq \mathcal{L}og\,(w,\mu)\,(B) - \mathcal{L}og\,(w,\mu)\,(A) \leq M\mathcal{D}(w,\mu)(\alpha)\,,$$

where

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t}d\mu(\lambda) = \frac{d\mathcal{L}og\,(w,\mu)\,(t)}{dt}.$$

Some examples for integral transforms $\mathcal{L}og\,(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda + t}d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t}\ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_{0}^{\infty}\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce [2], for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, then we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

We define the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda>0$ by

$$\mathcal{L}og(w,\mu)(t):=\int_{0}^{\infty}w(\lambda)\ln(\lambda+t)\,d\mu(\lambda), \tag{1.9}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t):=\int_{0}^{\infty}w(\lambda)\ln(\lambda+t)\,d\lambda. \tag{1.10}$$

If we consider the positive kernel $w_{\exp(-a\cdot)}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$, then, after some calculations

$$\mathcal{L}og(\exp(-a\cdot))(t)=\int_{0}^{\infty}\exp(-a\lambda)\ln(\lambda+t)\,d\lambda=\frac{1}{a}\left[\ln t+E_1(at)\exp(at)\right],$$

for $t>0$,where the *exponential integral* is given by

$$E_1(t):=\int_{t}^{\infty}\frac{e^{-u}}{u}du.$$

For $a=1$ we have

$$\mathcal{L}og\left(\exp(-\cdot)\right)(t)=\int_0^\infty \exp(-\lambda)\ln(\lambda+t)\,d\lambda=\ln t+E_1(t)\exp(t),$$

For $t=0$, we derive

$$\mathcal{L}og\left(\exp(-\cdot)\right)(0)=\int_0^\infty \exp(-\lambda)\ln(\lambda)\,d\lambda=-\gamma,$$

where γ is Euler–Mascheroni constant.

For $a>0$, by changing the variable $a\lambda=\nu$, then

$$\begin{aligned}\int_0^\infty \exp(-a\lambda)\ln(\lambda)\,d\lambda&=\int_0^\infty \exp(-\nu)\ln\left(\frac{\nu}{a}\right)\frac{1}{a}d\nu\\&=\frac{1}{a}\int_0^\infty\left[\exp(-\nu)\ln\nu-\exp(-\nu)\ln a\right]d\nu\\&=\frac{1}{a}\left(-\gamma-\ln a\right)=-\frac{\ln a+\gamma}{a}\end{aligned}$$

and we have for any $a>0$ that

$$\mathcal{L}og\left(\exp(-a\cdot)\right)(0)=-\frac{\ln a+\gamma}{a}.$$

If we consider the positive kernel $w_{(\cdot+a)^{-2}}(\lambda):=\frac{1}{(\lambda+a)^2}$, $\lambda\geq 0$, $a>0$, then, after some calculations

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(t):=\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2}d\lambda=\begin{cases}\frac{t\ln t-a\ln a}{a(t-a)}, & \text{if } t\neq a,\\ \\ \frac{\ln a+1}{a}, & \text{if } t=a\end{cases}$$

for $t>0$.

If $a=1$, then

$$\mathcal{L}og\left(w_{(\cdot+1)^{-2}}\right)(t):=\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+1)^2}d\lambda=\begin{cases}\frac{t\ln t}{t-1}, & \text{if } t\neq 1,\\ \\ 1, & \text{if } t=1\end{cases}$$

for $t>0$.

For $t=0$, we derive

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(0):=\int_0^\infty \frac{\ln(\lambda)}{(\lambda+a)^2}d\lambda=\frac{\ln a}{a}$$

for $a>0$.

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{L}og\left(w,\mu\right)(T)=\int_0^\infty w(\lambda)\ln(\lambda+T)\,d\mu(\lambda)\,, \tag{1.11}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og\left(w\right)(T)=\int_0^\infty w(\lambda)\ln(\lambda+T)\,d\lambda \tag{1.12}$$

In this paper, we show among others that, if $B\geq A>0$, then $\mathcal{L}og\left(w,\mu\right)\left(B\right)\geq\mathcal{L}og\left(w,\mu\right)\left(A\right)$, namely $\mathcal{L}og\left(w,\mu\right)\left(w,\mu\right)$ is operator monotone on $\left(0,\infty\right)$. If $\delta\geq A$, $B\geq\alpha>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M, then

$$0\leq m\mathcal{D}\left(w,\mu\right)\left(\delta\right)\leq\mathcal{L}og\left(w,\mu\right)\left(B\right)-\mathcal{L}og\left(w,\mu\right)\left(A\right)\leq M\mathcal{D}\left(w,\mu\right)\left(\alpha\right),$$

where

$$\mathcal{D}\left(w,\mu\right)\left(t\right):=\int_{0}^{\infty}\frac{w\left(\lambda\right)}{\lambda+t}d\mu\left(\lambda\right)=\frac{d\mathcal{L}og\left(w,\mu\right)\left(t\right)}{dt}.$$

Some examples for integral transforms $\mathcal{L}og\left(\cdot,\cdot\right)$ related to the exponential and logarithmic functions are also provided.

2. Some Identities

Start to the following identity for the logarithmic function:

Lemma 1. *For all A, $B>0$ we have the identity:*

$$\begin{aligned}(2.1)\quad&\ln B-\ln A\\&=\int_{0}^{\infty}\left(\int_{0}^{1}\left(s+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(s+\left(1-t\right)A+tB\right)^{-1}dt\right)ds.\end{aligned}$$

Proof. We have from (1.6) for A, $B>0$ that

$$\ln B-\ln A=\int_{0}^{\infty}\frac{1}{s+1}\left[\left(B-1\right)\left(s+B\right)^{-1}-\left(A-1\right)\left(s+A\right)^{-1}\right]ds. \tag{2.2}$$

Since

$$\begin{aligned}&\left(B-1\right)\left(s+B\right)^{-1}-\left(A-1\right)\left(s+A\right)^{-1}\\&=B\left(s+B\right)^{-1}-A\left(s+A\right)^{-1}-\left(\left(s+B\right)^{-1}-\left(s+A\right)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&B\left(s+B\right)^{-1}-A\left(s+A\right)^{-1}\\&=\left(B+s-s\right)\left(s+B\right)^{-1}-\left(A+s-s\right)\left(s+A\right)^{-1}\\&=1-s\left(s+B\right)^{-1}-1+s\left(s+A\right)^{-1}=s\left(s+A\right)^{-1}-s\left(s+B\right)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&\left(B-1\right)\left(s+B\right)^{-1}-\left(A-1\right)\left(s+A\right)^{-1}\\&=s\left(s+A\right)^{-1}-s\left(s+B\right)^{-1}-\left(\left(s+B\right)^{-1}-\left(s+A\right)^{-1}\right)\\&=\left(s+1\right)\left[\left(s+A\right)^{-1}-\left(s+B\right)^{-1}\right]\end{aligned}$$

and by (2.2) we get

$$\ln B-\ln A=\int_{0}^{\infty}\left[\left(s+A\right)^{-1}-\left(s+B\right)^{-1}\right]ds. \tag{2.3}$$

Let T, $S>0$. The function $f\left(t\right)=-t^{-1}$ is operator monotone on $\left(0,\infty\right)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_{T}\left(S\right):=\lim_{t\to0}\left[\frac{f\left(T+tS\right)-f\left(T\right)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D]$: $\{(1-t)C + tD,\ t \in [0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD),\ t \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Since, by (2.6) we have

$$\begin{aligned}&(s+A)^{-1} - (s+B)^{-1} \\ &= \int_0^1 (s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}\,dt,\end{aligned} \tag{2.7}$$

for all $s \geq 0$, hence by (2.3) and (2.7) we get (2.1). □

Theorem 1. *For all $A,\ B > 0$ we have the identity:*

$$\begin{aligned}&\mathcal{L}og(w,\mu)(B) - \mathcal{L}og(w,\mu)(A) \\ &= \int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\ &\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}\,dt\right)ds\right)d\mu(\lambda).\end{aligned} \tag{2.8}$$

Proof. For all $A,\ B > 0$ we have

$$\begin{aligned}&\mathcal{L}og(w,\mu)(B) - \mathcal{L}og(w,\mu)(A) \\ &= \int_0^\infty w(\lambda)\ln(\lambda+B)\,d\mu(\lambda) - \int_0^\infty w(\lambda)\ln(\lambda+A)\,d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)[\ln(\lambda+B) - \ln(\lambda+A)]\,d\mu(\lambda).\end{aligned} \tag{2.9}$$

Since, by (2.1) we get

$$\begin{aligned}&\ln(\lambda+B) - \ln(\lambda+A) \\ &= \int_0^\infty\left(\int_0^1 (s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\right. \\ &\left.\times(\lambda+B-(\lambda+A))(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\,dt\right)ds\end{aligned}$$

for all $\lambda \geq 0$, then by multiplying with $w(\lambda)$ and integrating over $\mu(\lambda)$ we obtain

$$\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]\,d\mu(\lambda) = \left(\int_0^\infty \left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A) \times (s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda). \tag{2.10}$$

Finally, by (2.9) and (2.10) we get (2.8). □

Corollary 1. *If* $B \geq A > 0$, *then* $\mathcal{L}og(w,\mu)(B) \geq \mathcal{L}og(w,\mu)(A)$, *namely* $\mathcal{L}og(w,\mu)(\cdot)$ *is operator monotone on* $(0,\infty)$.

Proof. If $B-A \geq 0$, then by multiplying both sides with $(s+\lambda+(1-t)A+tB)^{-1}$ we get

$$(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1} \geq 0$$

for all $t \in [0,1]$ and $s,\lambda \geq 0$.

If we integrate of over $t \in [0,1]$ and $s \in [0,\infty)$ we obtain

$$\int_0^\infty \left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds \geq 0.$$

Further, if we multiply this inequality by $w(\lambda) \geq 0$, integrate over the positive measure $\mu(\lambda)$ and use the identity (2.8) we derive the desired inequality. □

For some recent results related to operator monotone functions we refer to [3], [4] [5], [8], [9] and the references therein.

3. Some Inequalities

When more assumptions for the operators A and B are imposed, then we have the following improvement of monotonicity property:

Theorem 2. *Assume that* $A \geq \alpha > 0$, $\delta \geq B > 0$ *and* $0 < m \leq B-A \leq M$ *for some constants* α, δ, m, M. *Then*

$$\begin{aligned} 0 &\leq \mathcal{L}og(w,\mu)(\delta)-\mathcal{L}og(w,\mu)(\delta-m) \\ &\leq \mathcal{L}og(w,\mu)(B)-\mathcal{L}og(w,\mu)(A) \\ &\leq \frac{M}{m}[\mathcal{L}og(w,\mu)(m+\alpha)-\mathcal{L}og(w,\mu)(\alpha)]. \end{aligned} \tag{3.1}$$

Proof. Observe that for $t \in [0,1]$

$$(1-t)A+tB = A+t(B-A).$$

Since

$$\alpha+tm \leq A+t(B-A),$$

hence

$$s+\lambda+\alpha+tm \leq s+\lambda+(1-t)A+tB,$$

which implies that

$$(s+\lambda+(1-t)A+tB)^{-1} \leq (s+\lambda+\alpha+tm)^{-1} \tag{3.2}$$

for all $t \in [0,1]$ and s, $\lambda \in [0,\infty)$.

Also,

$$0<(1-t)A+tB=B-(1-t)(B-A)\le\delta-(1-t)m=tm+\delta-m,$$

which implies that

$$0<s+\lambda+(1-t)A+tB\le s+\lambda+tm+\delta-m$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$. This implies that

$$(s+\lambda+tm+\delta-m)^{-1}\le(s+\lambda+(1-t)A+tB)^{-1}. \tag{3.3}$$

Since $m\le B-A\le M$, then by multiplying both sides with $(s+\lambda+(1-t)A+tB)^{-1}$ we get

$$\begin{aligned}&m(s+\lambda+(1-t)A+tB)^{-2}\\&\le(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}\\&\le M(s+\lambda+(1-t)A+tB)^{-2}\end{aligned} \tag{3.4}$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$.

By (3.2) we get

$$M(s+\lambda+(1-t)A+tB)^{-2}\le M(s+\lambda+\alpha+tm)^{-2}$$

and by (3.3) we have

$$m(s+\lambda+tm+\delta-m)^{-2}\le m(s+\lambda+(1-t)A+tB)^{-1},$$

therefore, by (3.4), we get

$$\begin{aligned}&m(s+\lambda+tm+\delta-m)^{-2}\\&\le(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}\\&\le M(s+\lambda+\alpha+tm)^{-2}\end{aligned} \tag{3.5}$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$.

If we multiply (3.5) by $w(\lambda)\ge0$ and integrate over t, s and λ, we get

$$\begin{aligned}0&\le m\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+tm+\delta-m)^{-2}dt\right)ds\right)d\mu(\lambda)\\&\le\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\&\quad\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda)\\&\le M\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+\alpha+tm)^{-2}dt\right)ds\right)d\mu(\lambda)\end{aligned}$$

and by (2.8) we obtain

$$\begin{aligned}0&\le m\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+tm+\delta-m)^{-2}dt\right)ds\right)d\mu(\lambda)\\&\le\mathcal{L}og(w,\mu)(B)-\mathcal{L}og(w,\mu)(A)\\&\le M\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+\alpha+tm)^{-2}dt\right)ds\right)d\mu(\lambda).\end{aligned} \tag{3.6}$$

Since

$$tm+\delta-m=(1-t)(\delta-m)+t\delta,$$

then

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+tm+\delta-m)^{-2}dt\right)ds\right)d\mu(\lambda) \tag{3.7}$$
$$=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)(\delta-m)+t\delta)^{-2}dt\right)ds\right)d\mu(\lambda)$$
$$=\frac{1}{m}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)(\delta-m)+t\delta)\right.\right.$$
$$\times(\delta-(\delta-m))(s+\lambda+(1-t)(\delta-m)+t\delta)\,dt)\,ds)\,d\mu(\lambda)$$
$$=\frac{1}{m}\left[\mathcal{L}og(w,\mu)(\delta)-\mathcal{L}og(w,\mu)(\delta-m)\right],$$

where the last equality follows by (2.8) for $B=\delta$ and $A=\delta-m$.

Since

$$\alpha+tm=(1-t)\alpha+t(m+\alpha),$$

then

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+\alpha+tm)^{-2}dt\right)ds\right)d\mu(\lambda) \tag{3.8}$$
$$=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)\alpha+t(m+\alpha))^{-1}\right.\right.$$
$$\left.\left.\times((m+\alpha)-\alpha)(s+\lambda+(1-t)\alpha+t(m+\alpha))^{-1}dt\right)ds\right)d\mu(\lambda)$$
$$=\frac{1}{m}\left[\mathcal{L}og(w,\mu)(m+\alpha)-\mathcal{L}og(w,\mu)(\alpha)\right]$$

where the last equality follows by (2.8) for $B=m+\alpha$ and $A=\alpha$.

By making use of (3.6)-(3.8) we derive the desired result (3.1). □

We also have:

Theorem 3. *Assume that $\delta\ge A$, $B\ge\alpha>0$ and $0<m\le B-A\le M$ for some constants α, δ, m, M. Then*

$$0\le m\mathcal{D}(w,\mu)(\delta)\le\mathcal{L}og(w,\mu)(B)-\mathcal{L}og(w,\mu)(A)\le M\mathcal{D}(w,\mu)(\alpha), \tag{3.9}$$

where

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda)=\frac{d\mathcal{L}og(w,\mu)(t)}{dt}.$$

Proof. Since $\delta\ge A$, $B\ge\alpha>0$, then

$$\alpha\le(1-t)A+tB\le\delta \text{ for all } t\in[0,1],$$

which implies that

$$0<s+\lambda+\alpha\le s+\lambda+(1-t)A+tB\le s+\lambda+\delta$$

for all $t\in[0,1]$ and s, $\lambda\in[0,\infty)$.

From this double inequality we derive that

$$0<(s+\lambda+\delta)^{-1}\le(s+\lambda+(1-t)A+tB)^{-1}\le(s+\lambda+\alpha)^{-1},$$

which implies that

$$0<(s+\lambda+\delta)^{-2}\le(s+\lambda+(1-t)A+tB)^{-2}\le(s+\lambda+\alpha)^{-2}. \tag{3.10}$$

Therefore

$$0 < m(s+\lambda+\delta)^{-2} \le m(s+\lambda+(1-t)A+tB)^{-2}$$

and

$$M(s+\lambda+(1-t)A+tB)^{-2} \le M(s+\lambda+\alpha)^{-2} \tag{3.11}$$

for all $t \in [0,1]$ and $s, \lambda \in [0,\infty)$.

By employing inequality (3.4), (3.10) and (3.11), we deduce

$$\begin{aligned} 0 &< m(s+\lambda+\delta)^{-2} \\ &\le (s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1} \\ &\le M(s+\lambda+\alpha)^{-2} \end{aligned} \tag{3.12}$$

for all $t \in [0,1]$ and $s, \lambda \in [0,\infty)$.

If we multiply (3.12) by $w(\lambda) \ge 0$ and integrate over t, s and λ, we get

$$\begin{aligned} 0 &\le m\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2}ds\right)d\mu(\lambda) \\ &\le \int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\ &\quad \left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda) \\ &\le M\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2}ds\right)d\mu(\lambda). \end{aligned}$$

and by (2.8),

$$\begin{aligned} 0 &\le m\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2}ds\right)d\mu(\lambda) \\ &\le \mathcal{L}og(w,\mu)(B) - \mathcal{L}og(w,\mu)(A) \\ &\le M\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2}ds\right)d\mu(\lambda). \end{aligned} \tag{3.13}$$

Observe that

$$\int_0^\infty (s+\lambda+\delta)^{-2}ds = -(s+\lambda+\delta)^{-1}\Big|_0^\infty = (\lambda+\delta)^{-1}$$

and

$$\int_0^\infty (s+\lambda+\alpha)^{-2}ds = -(s+\lambda+\alpha)^{-1}\Big|_0^\infty = (\lambda+\alpha)^{-1}.$$

Therefore

$$\begin{aligned} \int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2}ds\right)d\mu(\lambda) &= \int_0^\infty w(\lambda)(\lambda+\delta)^{-1}d\mu(\lambda) \\ &= \mathcal{D}(w,\mu)(\delta) \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2}ds\right)d\mu(\lambda) &= \int_0^\infty w(\lambda)(\lambda+\alpha)^{-1}d\mu(\lambda) \\ &= \mathcal{D}(w,\mu)(\alpha). \end{aligned}$$

By making use of (3.13) we derive the desired result (3.9).

The fact that the derivative of $\mathcal{L}og\,(w,\mu)\,(t)$ is $\mathcal{D}\,(w,\mu)\,(t)$ follows by the properties of the derivative of a parameter integral, i.e.

$$\frac{d\mathcal{L}og\,(w,\mu)\,(t)}{dt} := \int_0^\infty w\,(\lambda)\,\frac{d\ln(\lambda+t)}{dt}d\mu\,(\lambda) = \int_0^\infty w\,(\lambda)\,(\lambda+t)^{-1}\,d\mu\,(\lambda)$$

for $t > 0$. □

The case of separated operators is as follows:

Theorem 4. *If the positive operators satisfy the separation condition*

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta \tag{3.14}$$

for some positive constants α, β, γ, δ, *then*

$$\begin{aligned} 0 &\le \frac{\gamma-\beta}{\delta-\beta}\left[\mathcal{L}og\,(w,\mu)\,(\delta) - \mathcal{L}og\,(w,\mu)\,(\beta)\right] \\ &\le \mathcal{L}og\,(w,\mu)\,(B) - \mathcal{L}og\,(w,\mu)\,(A) \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[\mathcal{L}og\,(w,\mu)\,(\gamma) - \mathcal{L}og\,(w,\mu)\,(\alpha)\right]. \end{aligned} \tag{3.15}$$

Proof. From (3.14) we derive that

$$(1-t)\,A + tB + \lambda + s \le (1-t)\,\beta + t\delta + \lambda + s,$$

which implies that

$$((1-t)\,A + tB + \lambda + s)^{-1} \ge ((1-t)\,\beta + t\delta + \lambda + s)^{-1}$$

and

$$((1-t)\,A + tB + \lambda + s)^{-2} \ge ((1-t)\,\beta + t\delta + \lambda + s)^{-2} \tag{3.16}$$

for all $t \in [0,1]$ and $\lambda \ge 0$.

Also

$$(1-t)\,A + tB + \lambda + s \ge (1-t)\,\alpha + t\gamma + \lambda + s,$$

which implies that

$$((1-t)\,A + tB + \lambda + s)^{-1} \le ((1-t)\,\alpha + t\gamma + \lambda + s)^{-1}$$

and

$$((1-t)\,A + tB + \lambda + s)^{-2} \le ((1-t)\,\alpha + t\gamma + \lambda + s)^{-2} \tag{3.17}$$

for all $t \in [0,1]$ and $\lambda, s \ge 0$.

From (3.14) we have

$$0 < \gamma - \beta \le B - A \le \delta - \alpha,$$

which implies that

$$\begin{aligned} 0 &\le (\gamma-\beta)\,((1-t)\,A + tB + \lambda + s)^{-2} \\ &\le ((1-t)\,A + tB + \lambda + s)^{-1}\,(B-A)\,((1-t)\,A + tB + \lambda + s)^{-1} \\ &\le (\delta-\alpha)\,((1-t)\,A + tB + \lambda + s)^{-2} \end{aligned} \tag{3.18}$$

for all $t \in [0,1]$ and $\lambda, s \ge 0$.

By making use of (3.16)-(3.18) we get

$$\begin{aligned}(3.19)\qquad 0 &\leq (\gamma-\beta)\left((1-t)\,\beta+t\delta+\lambda+s\right)^{-2}\\ &\leq \left((1-t)\,A+tB+\lambda+s\right)^{-1}(B-A)\left((1-t)\,A+tB+\lambda+s\right)^{-1}\\ &\leq (\delta-\alpha)\left((1-t)\,\alpha+t\gamma+\lambda+s\right)^{-2}\end{aligned}$$

for all $t\in[0,1]$ and $\lambda, s\geq 0$.

If we multiply (3.19) by $w(\lambda)$ integrate and use the identity (2.8), then we obtain

$$\begin{aligned}(3.20)\qquad 0 &\leq (\gamma-\beta)\int_0^\infty w(\lambda)\\ &\times\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\beta+t\delta+\lambda+s\right)^{-2}dt\right)ds\right)d\mu(\lambda)\\ &\leq \mathcal{L}og\,(w,\mu)\,(B)-\mathcal{L}og\,(w,\mu)\,(A)\\ &\leq (\delta-\alpha)\int_0^\infty w(\lambda)\\ &\times\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\alpha+t\gamma+\lambda+s\right)^{-2}dt\right)ds\right)d\mu(\lambda)\,.\end{aligned}$$

Observe that

$$\begin{aligned}(3.21)\qquad &\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\beta+t\delta+\lambda+s\right)^{-2}dt\right)ds\right)d\mu(\lambda)\\ &=\frac{1}{\delta-\beta}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\beta+t\delta+\lambda+s\right)^{-1}\right.\right.\\ &\left.\left.\times(\delta-\beta)\left((1-t)\,\beta+t\delta+\lambda+s\right)^{-1}dt\right)ds\right)d\mu(\lambda)\\ &=\frac{1}{\delta-\beta}\left[\mathcal{L}og\,(w,\mu)\,(\delta)-\mathcal{L}og\,(w,\mu)\,(\beta)\right]\ \text{ by (2.8)}\end{aligned}$$

and

$$\begin{aligned}(3.22)\qquad &\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\alpha+t\gamma+\lambda+s\right)^{-2}dt\right)ds\right)d\mu(\lambda)\\ &=\frac{1}{\gamma-\alpha}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left((1-t)\,\alpha+t\gamma+\lambda+s\right)^{-1}(\gamma-\alpha)\right.\right.\\ &\left.\left.\times\left((1-t)\,\alpha+t\gamma+\lambda+s\right)^{-1}dt\right)ds\right)d\mu(\lambda)\\ &=\frac{1}{\gamma-\alpha}\left[\mathcal{L}og\,(w,\mu)\,(\gamma)-\mathcal{L}og\,(w,\mu)\,(\alpha)\right]\ \text{ by (2.8).}\end{aligned}$$

By making use of (3.20)-(3.22) we deduce the desired result (3.15). □

4. Some Examples

From Introduction we have the following two classes of continuous functions defined on $[0,\infty)$,

$$\mathcal{L}og\left(\exp(-a\cdot)\right)(t):=\int_0^\infty \exp(-a\lambda)\ln(\lambda+t)\,d\lambda = \begin{cases} \frac{1}{a}\left[\ln t+E_1(at)\exp(at)\right], & \text{if } t>0 \\ -\frac{\ln a+\gamma}{a}, & \text{if } t=0 \end{cases}$$

and

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(t):=\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2}d\lambda = \begin{cases} \frac{t\ln t-a\ln a}{a(t-a)}, & \text{if } t\neq a, \\ \frac{\ln a+1}{a}, & \text{if } t=a \\ \frac{\ln a}{a}, & \text{if } t=0 \end{cases}$$

for all $a>0$.

By making use of Theorem 1 we conclude that the functions $\mathcal{L}og\left(\exp(-a\cdot)\right)$ and $\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)$ are operator monotone on $[0,\infty)$ for all $a>0$.

Assume that $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M. Then by Theorem 2, we derive the following operator inequalities

$$\begin{aligned} 0&\leq\left[\mathcal{L}og\left(\exp(-a\cdot)\right)(\delta)-\mathcal{L}og\left(\exp(-a\cdot)\right)(\delta-m)\right] \\ &\leq \mathcal{L}og\left(\exp(-a\cdot)\right)(B)-\mathcal{L}og\left(\exp(-a\cdot)\right)(A) \\ &\leq \frac{M}{m}\left[\mathcal{L}og\left(\exp(-a\cdot)\right)(m+\alpha)-\mathcal{L}og\left(\exp(-a\cdot)\right)(\alpha)\right] \end{aligned} \tag{4.1}$$

and the similar ones for $\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)$.

For the kernel $w_{\exp(-a\cdot)}(\lambda):=\exp(-a\lambda)$, $\lambda\geq0$, we obtain

$$\mathcal{D}\left(w_{\exp(-a\cdot)}\right)(t):=\int_0^\infty\frac{\exp(-a\lambda)}{\lambda+t}d\lambda=E_1(at)\exp(at),$$

for a, $t>0$.

For the kernel $w_{(\cdot+a)^{-2}}(\lambda):=\frac{1}{(\lambda+a)^2}$, $\lambda\geq0$, $a>0$, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-2}}\right)(t):=\int_0^\infty\frac{1}{(\lambda+t)(\lambda+a)^2}d\lambda=\begin{cases}\frac{t-a-a(\ln t-\ln a)}{a(t-a)^2}, & \text{if } t\neq a \\ \frac{1}{2a^2}, & t=a\end{cases}$$

for a, $t>0$.

Assume that $\delta\geq A$, $B\geq\alpha>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M. Then by Theorem 3.9 we have the operator inequalities

$$\begin{aligned} 0&\leq mE_1(a\delta)\exp(a\delta) \\ &\leq\frac{1}{a}\left[\ln B+E_1(aB)\exp(aB)\right]-\frac{1}{a}\left[\ln A+E_1(aA)\exp(aA)\right] \\ &\leq ME_1(a\alpha)\exp(a\alpha), \end{aligned} \tag{4.2}$$

for $a>0$ and the similar ones for $\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)$.

Similar inequalities may be stated if we apply Theorem 4 for the transforms $\mathcal{L}og\left(\exp(-a\cdot)\right)$ and $\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)$. The details are omitted.

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