SOME POWER INEQUALITIES FOR THE DISTANCE IN METRIC SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this note we provide some upper and lower bounds for the sum

$$\sum_{1\leq i<j\leq n} p_i p_j d^s\left(x_i, x_j\right),$$

where (X,d) is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$ and $s > 0$.

1. Introduction

Let X be a nonempty set. A function $d : X \times X \to [0,\infty)$ is called a *distance* on X if the following properties are satisfied:

(d) $d(x,y) = 0$ if and only if $x = y$;
(dd) $d(x,y) = d(y,x)$ for any $x,y \in X$ (the *symmetry* of the distance);
(ddd) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x,y,z \in X$ (the *triangle inequality*).

The pair (X,d) is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space E over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\| : E \to [0,\infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties:

(n) $\|x\| = 0$ if and only if $x = 0$;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha \in \mathbb{K}$ and any vector $x \in E$;
(nnn) $\|x+y\| \leq \|x\| + \|y\|$ for each $x,y \in E$ (the triangle inequality).

Further, we recall that, the linear space H over the real or complex number field $\mathbb{K}$ endowed with an application $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ is called an *inner product space*, if the function $\langle\cdot,\cdot\rangle$, called the *inner product*, satisfies the following properties:

(i) $\langle x,x\rangle \geq 0$ for any $x \in H$ and $\langle x,x\rangle = 0$ if and only if $x = 0$;
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x,z\rangle + \beta \langle y,z\rangle$ for any scalars α, β and any vectors x,y,z;
(iii) $\langle y,x\rangle = \overline{\langle x,y\rangle}$ for any $x,y \in H$.

It is well know that the function $\|x\| := \sqrt{\langle x,x\rangle}$ defines a norm on H and thus an important example of normed spaces are the inner product spaces.

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalised triangle inequality*, or the *polygonal inequality* which states that: for any points $x_1, x_2, ..., x_{n-1}, x_n$ $(n \geq 3)$ in a metric space (X,d), we have the inequality

$$d(x_1,x_n) \leq d(x_1,x_2) + ... + d(x_{n-1},x_n). \tag{1.1}$$

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The following result in the general setting of metric spaces holds.

Theorem 1. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d\left(x_i, x_j\right) \leq \inf_{x\in X}\left[\sum_{i=1}^{n} p_i d\left(x_i, x\right)\right]. \tag{1.2}$$

The inequality is sharp in the sense that the multiplicative constant $c=1$ in front of " inf " cannot be replaced by a smaller quantity.

We have:

Corollary 1. *Let (X,d) be a metric space and $x_i \in X$, $i \in \{1,...,n\}$. If there exists a closed ball of radius $r>0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x,r) := \{y \in X : d(x,y) \leq r\}$, then for any $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$ we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d\left(x_i, x_j\right) \leq r. \tag{1.3}$$

The inequality (1.2) and its consequences were extended to the case of b-metric spaces in [3] and for natural powers of the distance in [1].

In this note we provide some upper and lower bounds for the sum

$$\sum_{1\leq i<j\leq n} p_i p_j d^s\left(x_i, x_j\right)$$

where (X,d) is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $s>0$.

2. Main Results

We have the following generalization of the inequality (1.2).

Theorem 2. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d^s\left(x_i, x_j\right) \leq \begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{i=1}^{n} p_i d^s\left(x_i, x\right)\right], & \text{if } s \geq 1, \\ \\ \inf_{x\in X}\left[\sum_{i=1}^{n} p_i d^s\left(x_i, x\right)\right], & \text{if } 0<s<1. \end{cases} \tag{2.1}$$

Proof. We know that, by the convexity property of the power function $f(t) = t^s$, $s \geq 1$ on $[0,\infty)$, we have for a, $b \geq 0$ that

$$(a+b)^s \leq 2^{s-1}\left(a^s + b^s\right).$$

We consider the function $f_s : [0,\infty) \to \mathbb{R}$, $f_s(t) = (t+1)^s - t^s$ we have $f_s'(t) = s\left[(t+1)^{s-1} - t^{s-1}\right]$. Observe that for $0<s<1$ and $t>0$ we have that $f_s'(t) < 0$ showing that f_s is strictly decreasing on the interval $[0,\infty)$. Now for $t_0 = \frac{a}{b}$ $(b>0, a\geq 0)$ we have $f_s(t_0) < f_s(0)$ giving that $\left(\frac{a}{b}+1\right)^s - \left(\frac{a}{b}\right)^s < 1$, i.e., the inequality

$$(a+b)^s \leq a^s + b^s.$$

Using the triangle inequality, we have for any $x \in X$ and i, $j \in \{1,...,n\}$, that

$$d\left(x_i, x_j\right) \leq d\left(x_i, x\right) + d\left(x, x_j\right). \tag{2.2}$$

If we take the power $s>0$ in (2.2) we get

$$d^{s}(x_i,x_j)\leq[d(x_i,x)+d(x,x_j)]^{s}\leq\begin{cases}2^{s-1}\left(d^{s}(x_i,x)+d(x_j,x)^{s}\right), & s\geq1\\ d^{s}(x_i,x)+d(x_j,x)^{s}, & 0<s<1\end{cases}\tag{2.3}$$

for any $x\in X$ and $i,\ j\in\{1,...,n\}$.

If we multiply (2.3) by $p_ip_j\geq0$ and sum over $i,\ j$ from 1 to n, we get

$$\sum_{1\leq i,j\leq n}p_ip_jd^{s}(x_i,x_j)\leq\begin{cases}2^{s-1}\sum_{1\leq i,j\leq n}p_ip_j\left(d^{s}(x_i,x)+d(x_j,x)^{s}\right), & s\geq1\\ \sum_{1\leq i,j\leq n}\left(d^{s}(x_i,x)+d(x_j,x)^{s}\right), & 0<s<1.\end{cases}\tag{2.4}$$

Since

$$\sum_{1\leq i,j\leq n}p_ip_jd^{s}(x_i,x_j)=2\sum_{1\leq i<j\leq n}p_ip_jd^{s}(x_i,x_j)$$

and

$$\begin{aligned}\sum_{1\leq i,j\leq n}p_ip_j\left(d^{s}(x_i,x)+d(x_j,x)^{s}\right)&=\sum_{1\leq i,j\leq n}p_ip_jd^{s}(x_i,x)+\sum_{1\leq i,j\leq n}p_ip_jd(x_j,x)^{s}\\&=\sum_{i=1}^{n}p_id^{s}(x_i,x)\sum_{j=1}^{n}p_j+\sum_{i=1}^{n}p_i\sum_{j=1}^{n}p_jd(x_j,x)^{s}\\&=2\sum_{i=1}^{n}p_id^{s}(x_i,x),\end{aligned}$$

then by (2.4) we get the desired result (2.1). □

Corollary 2. *Let (X,d) be a metric space and $x_i\in X$, $i\in\{1,...,n\}$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n}d^{s}(x_i,x_j)\leq\begin{cases}2^{s-1}n\inf_{x\in X}\left[\sum_{i=1}^{n}d^{s}(x_i,x)\right], & s\geq1,\\ n\inf_{x\in X}\left[\sum_{i=1}^{n}d^{s}(x_i,x)\right], & 0<s<1.\end{cases}\tag{2.5}$$

Corollary 3. *Let (X,d) be a metric space and $x_i\in X$, $i\in\{1,...,n\}$. If there exists a closed ball of radius $r>0$ centered in a point x containing all the points x_i, i.e., $x_i\in\overline{B}(x,r):=\{y\in X:d(x,y)\leq r\}$, then for any $p_i\geq0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$ we have the inequalities*

$$\sum_{1\leq i<j\leq n}p_ip_jd^{s}(x_i,x_j)\leq\begin{cases}2^{s-1}r^{s}, & \text{if } s\geq1,\\ r^{s}, & \text{if } 0<s<1.\end{cases}\tag{2.6}$$

We also have the following lower bound:

Theorem 3. *Let (X,d) be a metric space and $x_i\in X$, $p_i\geq0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$. Then we have the inequality*

$$\left(\frac{2}{1-\sum_{i=1}^{n}p_i^{2}}\right)^{s-1}\left(\sum_{1\leq i<j\leq n}p_ip_jd(x_i,x_j)\right)^{s}\leq\sum_{1\leq i<j\leq n}p_ip_jd^{s}(x_i,x_j).\tag{2.7}$$

Proof. We use Jensen's discrete inequality for the power function $f(t) = t^s$, $s > 1$ to write

$$\frac{\sum_{1 \le i < j \le n} p_i p_j d^s(x_i, x_j)}{\sum_{1 \le i < j \le n} p_i p_j} \ge \left(\frac{\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j)}{\sum_{1 \le i < j \le n} p_i p_j} \right)^s . \tag{2.8}$$

Observe that

$$2 \sum_{1 \le i < j \le n} p_i p_j + \sum_{1 \le i = j \le n} p_i p_j = \sum_{1 \le i, j \le n} p_i p_j = \left(\sum_{i=1}^n p_i \right)^2 = 1,$$

which gives that

$$2 \sum_{1 \le i < j \le n} p_i p_j + \sum_{i=1}^n p_i^2 = 1,$$

namely

$$\sum_{1 \le i < j \le n} p_i p_j = \frac{1}{2} \left(1 - \sum_{i=1}^n p_i^2 \right).$$

By (2.8) we get

$$\begin{aligned} \sum_{1 \le i < j \le n} p_i p_j d^s(x_i, x_j) &\ge \left(\sum_{1 \le i < j \le n} p_i p_j \right)^{1-s} \left(\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \right)^s \\ &= \left[\frac{1}{2} \left(1 - \sum_{i=1}^n p_i^2 \right) \right]^{1-s} \left(\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \right)^s \\ &= 2^{s-1} \left(1 - \sum_{i=1}^n p_i^2 \right)^{1-s} \left(\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \right)^s , \end{aligned} \tag{2.9}$$

and the inequality (2.7) is proved. □

Corollary 4. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. Then we have the inequality*

$$\left(\frac{2}{n(n-1)} \right)^{s-1} \left(\sum_{1 \le i < j \le n} d(x_i, x_j) \right)^s \le \sum_{1 \le i < j \le n} d^s(x_i, x_j). \tag{2.10}$$

3. Applications

If $(E, \|\cdot\|)$ is a normed linear space and $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \ge 0$ $(i \in \{1, ..., n\})$ with $\sum_{i=1}^n p_i = 1$, then by (2.1) we have the inequality

$$\sum_{1 \le i < j \le n} p_i p_j \|x_i - x_j\|^s \le \begin{cases} 2^{s-1} \inf_{x \in X} \left[\sum_{i=1}^n p_i \|x_i - x\|^s \right], & \text{if } s \ge 1, \\ \\ \inf_{x \in X} \left[\sum_{i=1}^n p_i \|x_i - x\|^s \right], & \text{if } 0 < s < 1. \end{cases} \tag{3.1}$$

In particular, for the uniform distribution $p_i = \frac{1}{n}$, we have

$$\sum_{1 \le i < j \le n} \|x_i - x_j\|^s \le \begin{cases} 2^{s-1} n \inf_{x \in X} \left[\sum_{i=1}^n \|x_i - x\|^s \right], & s \ge 1, \\ \\ n \inf_{x \in X} \left[\sum_{i=1}^n \|x_i - x\|^s \right], & 0 < s < 1. \end{cases} \tag{3.2}$$

We can state the following results as well.

Proposition 1. *Let* $(E, \|\cdot\|)$ *be a normed linear space and* $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \geq 0\,(i \in \{1, ..., n\})$ *with* $\sum_{i=1}^{n} p_i = 1$. *Denote* $\overline{x_p} := \sum_{i=1}^{n} p_i x_i$. *Then we have the inequalities*

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s \leq \begin{cases} 2^{s-1} \inf_{x\in X} \left[\sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s\right], & \text{if } s \geq 1, \\ \\ \inf_{x\in X} \left[\sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s\right], & \text{if } 0 < s < 1 \end{cases} \tag{3.3}$$

and

$$\frac{1}{2}\sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s \leq \sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s, \ \text{if } s \geq 1. \tag{3.4}$$

Proof. Since

$$\begin{cases} 2^{s-1} \inf_{x\in X} \left[\sum_{i=1}^{n} p_i \|x_i - x\|^s\right], & \text{if } s \geq 1, \\ \\ \inf_{x\in X} \left[\sum_{i=1}^{n} p_i \|x_i - x\|^s\right], & \text{if } 0 < s < 1 \end{cases}$$
$$\leq \begin{cases} 2^{s-1} \sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s, & \text{if } s \geq 1, \\ \\ \sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s, & \text{if } 0 < s < 1 \end{cases},$$

hence by (3.1) we deduce the second inequality in (3.3).

By Jensen's inequality we have for $s \geq 1$ that

$$\sum_{j=1}^{n} p_j \|x_i - x_j\|^s \geq \left\|\sum_{j=1}^{n} p_j (x_i - x_j)\right\|^s = \left\|x_i - \sum_{j=1}^{n} p_j x_j\right\|^s$$
$$= \|x_i - \overline{x_p}\|^s.$$

Therefore

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \|x_i - x_j\|^s \geq \sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\|^s \tag{3.5}$$

and since

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \|x_i - x_j\|^s = 2 \sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s$$

hence by (3.5) we get (3.4). □

Corollary 5. *Let* $(E, \|\cdot\|)$ *be a normed linear space and* $x_i \in E$, $i \in \{1, ..., n\}$. *If*

$$\overline{x} := \frac{x_1 + ... + x_n}{n}$$

denotes the gravity center of the vectors x_i, $i \in \{1, ..., n\}$, *then we have the inequality*

$$\sum_{1\leq i<j\leq n} \|x_i - x_j\|^s \leq \begin{cases} 2^{s-1} n \inf_{x\in X} \left[\sum_{i=1}^{n} \|x_i - \overline{x}\|^s\right], & \text{if } s \geq 1, \\ \\ n \inf_{x\in X} \left[\sum_{i=1}^{n} \|x_i - \overline{x}\|^s\right], & \text{if } 0 < s < 1 \end{cases} \tag{3.6}$$

and

$$\frac{1}{2} n \sum_{i=1}^{n} \|x_i - \overline{x}\|^s \leq \sum_{1\leq i<j\leq n} \|x_i - x_j\|^s, \ \text{if } s \geq 1. \tag{3.7}$$

Proposition 2. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space,* $x_i \in H, (i \in \{1, ..., n\})$ *and assume that there exists the vectors* a, $A \in H$ *so that either*

$$\operatorname{Re}\langle A - x_i, x_i - a\rangle \geq 0, \text{ for } i \in \{1, ..., n\},$$

or, equivalently,

$$\left\| x_i - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|, \text{ for } i \in \{1, ..., n\}.$$

Then for any $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *one has the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2}\|A - a\|^s, & \text{if } s \geq 1, \\ \\ \frac{1}{2^s}\|A - a\|^s, & \text{if } 0 < s < 1. \end{cases} \tag{3.8}$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$ then by (3.8) we get

$$\sum_{1 \leq i < j \leq n} \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2} n^2 \|A - a\|^s, & \text{if } s \geq 1, \\ \\ \frac{1}{2^s} n^2 \|A - a\|^s, & \text{if } 0 < s < 1. \end{cases} \tag{3.9}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.