OPERATOR MONOTONICITY OF THE $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following $\mathcal{D}$-*logarithmic integral transform*

$$\mathcal{DL}og\,(w)\,(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + t}{\lambda}\right) d\lambda,$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $B \geq A > 0$, then $\mathcal{DL}og\,(w,\mu)\,(B) \geq \mathcal{DL}og\,(w,\mu)\,(A)$, namely $\mathcal{DL}og\,(w,\mu)\,(w,\mu)$ is operator monotone on $(0,\infty)$. If $\delta \geq A$, $B \geq \alpha > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then

$$0 \leq m\mathcal{D}(w,\mu)(\delta) \leq \mathcal{DL}og\,(w,\mu)\,(B) - \mathcal{DL}og\,(w,\mu)\,(A) \leq M\mathcal{D}(w,\mu)(\alpha),$$

where

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda) = \frac{d\mathcal{DL}og\,(w,\mu)\,(t)}{dt}.$$

Some examples for integral transforms $\mathcal{DL}og\,(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.$$

If f is operator monotone in $[0,\infty)$, then $a = f(0)$ in (1.1).

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We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$s^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+s}d\lambda. \tag{1.2}$$

Observe that for $s>0$, $s\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+s)(\lambda+1)}=\frac{\ln s}{s-1}+\frac{1}{1-s}\ln\left(\frac{u+s}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln s}{s-1}=\int_0^\infty\frac{d\lambda}{(\lambda+s)(\lambda+1)}, \tag{1.3}$$

which gives the representation for the logarithm

$$\ln s=(s-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+s)} \tag{1.4}$$

If we integrate (1.2) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\frac{t^r}{r}&=\frac{\sin(r\pi)}{\pi}\int_0^\infty\left(\int_0^t\left(\frac{1}{\lambda+s}\right)ds\right)\lambda^{r-1}d\lambda\\&=\frac{\sin(r\pi)}{\pi}\int_0^\infty\lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda\end{aligned}$$

giving the identity of interest

$$t^r=\frac{r\sin(r\pi)}{\pi}\int_0^\infty\lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda,\ t>0\text{ and }r\in(0,1].$$

Recall the *dilogarithmic function* $\operatorname{dilog}:[0,\infty)\to\mathbb{R}$ defined by

$$\operatorname{dilog}(t):=\int_1^t\frac{\ln s}{1-s}ds,\ t\geq 0.$$

Some particular values of interest are

$$\operatorname{dilog}(1)=0,\ \operatorname{dilog}(0)=\int_1^0\frac{\ln s}{1-s}ds=\int_0^1\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2,$$

and

$$\operatorname{dilog}\left(\frac{1}{2}\right)=\frac{1}{12}\pi^2-\frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t\frac{\ln s}{s-1}ds=\int_0^\infty\left(\int_0^t\frac{1}{\lambda+s}ds\right)\frac{1}{(\lambda+1)}d\lambda=\int_0^\infty\frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda$$

and since

$$\begin{aligned}\int_0^t\frac{\ln s}{s-1}ds&=\int_0^1\frac{\ln s}{s-1}ds+\int_1^t\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2-\int_1^t\frac{\ln s}{1-s}ds\\&=\frac{1}{6}\pi^2-\operatorname{dilog}(t)\end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \operatorname{dilog}(t) = \int_0^\infty \frac{1}{(\lambda+1)} \ln\left(\frac{t+\lambda}{\lambda}\right) d\lambda,\ t>0.$$

Motivated by the above representations, we define the $\mathcal{D}$-*logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda>0$ by

$$\mathcal{D}\mathcal{L}og(w,\mu)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda), \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t>0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}\mathcal{L}og(w)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right) d\lambda. \tag{1.6}$$

Obviously,

$$\begin{aligned}\mathcal{D}\mathcal{L}og(w,\mu)(t) &= \int_0^\infty w(\lambda)\ln\left(1+\frac{t}{\lambda}\right) d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\left[\ln(\lambda+t)-\ln(\lambda)\right] d\mu(\lambda)\end{aligned}$$

and one can use either of these representations when is needed.

If we use the $\mathcal{D}$-*logarithmic transform* for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$ we have

$$\mathcal{D}\mathcal{L}og(w_{\ell^{r-1}})(t) = t^r,\ t\geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{D}\mathcal{L}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \operatorname{dilog}(t),\ t\geq 0. \tag{1.7}$$

In the recent paper [2] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\mu(\lambda),\ s>0, \tag{1.8}$$

for a continuous and positive function $w(\lambda)$, $\lambda>0$,where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $s>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\lambda,\ s>0. \tag{1.9}$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [2].

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\int_0^t \mathcal{D}(w,\mu)(s)\,ds &:= \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s}ds\right) w(\lambda)\,d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda)\end{aligned} \tag{1.10}$$

for $t>0$, which provides the equality of interest

$$\mathcal{D}\mathcal{L}og(w,\mu)(t) = \int_0^t \mathcal{D}(w,\mu)(s)\,ds,\ t>0, \tag{1.11}$$

provided that the integral on the right side exists for all $t>0$.

In this paper, we show among others that, if $B \geq A > 0$, then $\mathcal{DL}og(w,\mu)(B) \geq \mathcal{DL}og(w,\mu)(A)$, namely $\mathcal{DL}og(w,\mu)(w,\mu)$ is operator monotone on $(0,\infty)$. If $\delta \geq A$, $B \geq \alpha > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then

$$0 \leq m\mathcal{D}(w,\mu)(\delta) \leq \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A) \leq M\mathcal{D}(w,\mu)(\alpha),$$

where

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda) = \frac{d\mathcal{DL}og(w,\mu)(t)}{dt}.$$

Some examples for integral transforms $\mathcal{DL}og(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Some Identities

Start to the following identity for the logarithmic function:

Lemma 1. *For all A, $B > 0$ we have the identity:*

$$\ln B - \ln A = \int_0^\infty \left(\int_0^1 (s + (1-t)A + tB)^{-1} (B - A)(s + (1-t)A + tB)^{-1} dt \right) ds. \tag{2.1}$$

Proof. We have from (1.4) for A, $B > 0$ that

$$\ln B - \ln A = \int_0^\infty \frac{1}{s+1} \left[(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \right] ds. \tag{2.2}$$

Since

$$\begin{aligned} &(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \\ &= B(s+B)^{-1} - A(s+A)^{-1} - \left((s+B)^{-1} - (s+A)^{-1} \right) \end{aligned}$$

and

$$\begin{aligned} &B(s+B)^{-1} - A(s+A)^{-1} \\ &= (B+s-s)(s+B)^{-1} - (A+s-s)(s+A)^{-1} \\ &= 1 - s(s+B)^{-1} - 1 + s(s+A)^{-1} = s(s+A)^{-1} - s(s+B)^{-1}, \end{aligned}$$

hence

$$\begin{aligned} &(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \\ &= s(s+A)^{-1} - s(s+B)^{-1} - \left((s+B)^{-1} - (s+A)^{-1} \right) \\ &= (s+1)\left[(s+A)^{-1} - (s+B)^{-1} \right] \end{aligned}$$

and by (2.2) we get

$$\ln B - \ln A = \int_0^\infty \left[(s+A)^{-1} - (s+B)^{-1} \right] ds. \tag{2.3}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = T^{-1} S T^{-1} \tag{2.4}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Since, by (2.6) we have

$$\begin{aligned}&(s+A)^{-1}-(s+B)^{-1}\\ &=\int_0^1 (s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}\,dt,\end{aligned} \tag{2.7}$$

for all $s\geq 0$, hence by (2.3) and (2.7) we get (2.1). □

Theorem 2. *For all A, $B>0$ we have the identity:*

$$\begin{aligned}&\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\ &\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}\,dt\right)ds\right)d\mu(\lambda).\end{aligned} \tag{2.8}$$

Proof. For all A, $B>0$ we have

$$\begin{aligned}&\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln\lambda]\,d\mu(\lambda)-\int_0^\infty w(\lambda)[\ln(\lambda+A)-\ln\lambda]\,d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]\,d\mu(\lambda).\end{aligned} \tag{2.9}$$

Since, by (2.1) we get

$$\begin{aligned}&\ln(\lambda+B)-\ln(\lambda+A)\\ &=\int_0^\infty\left(\int_0^1 (s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\right.\\ &\left.\times(\lambda+B-(\lambda+A))(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\,dt\right)ds\end{aligned}$$

for all $\lambda \geq 0$, then by multiplying with $w(\lambda)$ and integrating over $\mu(\lambda)$ we obtain

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left[\ln(\lambda+B)-\ln(\lambda+A)\right]d\mu(\lambda) \\
&=\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\
&\quad\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda).
\end{aligned} \tag{2.10}$$

Finally, by (2.9) and (2.10) we get (2.8). □

Corollary 1. *If $B \geq A > 0$, then $\mathcal{DL}og(w,\mu)(B) \geq \mathcal{DL}og(w,\mu)(A)$, namely $\mathcal{DL}og(w,\mu)(\cdot)$ is operator monotone on $(0,\infty)$.*

Proof. If $B-A \geq 0$, then by multiplying both sides with $(s+\lambda+(1-t)A+tB)^{-1}$ we get

$$(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1} \geq 0$$

for all $t \in [0,1]$ and $s,\ \lambda \geq 0$.

If we integrate of over $t \in [0,1]$ and $s \in [0,\infty)$ we obtain

$$\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds \geq 0.$$

Further, if we multiply this inequality by $w(\lambda) \geq 0$, integrate over the positive measure $\mu(\lambda)$ and use the identity (2.8) we derive the desired inequality. □

Remark 1. *Since, by (1.7),*

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t)=\frac{1}{6}\pi^2-\operatorname{dilog}(t),\ t \geq 0$$

and $\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)$ is operator monotone, then the function $-\operatorname{dilog}$ is is operator monotone on $(0,\infty)$.

For some recent results related to operator monotone functions we refer to [3], [4] [5], [8], [9] and the references therein.

3. Some Inequalities

When more assumptions for the operators A and B are imposed, then we have the following improvement of monotonicity property:

Theorem 3. *Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B-A \leq M$ for some constants α, δ, m, M. Then*

$$\begin{aligned}
0 &\leq \mathcal{DL}og(w,\mu)(\delta)-\mathcal{DL}og(w,\mu)(\delta-m) \\
&\leq \mathcal{DL}og(w,\mu)(B)-\mathcal{DL}og(w,\mu)(A) \\
&\leq \frac{M}{m}\left[\mathcal{DL}og(w,\mu)(m+\alpha)-\mathcal{DL}og(w,\mu)(\alpha)\right].
\end{aligned} \tag{3.1}$$

Proof. Observe that for $t \in [0,1]$

$$(1-t)A+tB=A+t(B-A).$$

Since

$$\alpha+tm \leq A+t(B-A),$$

hence

$$s+\lambda+\alpha+tm\leq s+\lambda+(1-t)A+tB,$$

which implies that

$$(s+\lambda+(1-t)A+tB)^{-1}\leq(s+\lambda+\alpha+tm)^{-1} \tag{3.2}$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$.

Also,

$$0<(1-t)A+tB=B-(1-t)(B-A)\leq\delta-(1-t)m=tm+\delta-m,$$

which implies that

$$0<s+\lambda+(1-t)A+tB\leq s+\lambda+tm+\delta-m$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$. This implies that

$$(s+\lambda+tm+\delta-m)^{-1}\leq(s+\lambda+(1-t)A+tB)^{-1}. \tag{3.3}$$

Since $m\leq B-A\leq M$, then by multiplying both sides with $(s+\lambda+(1-t)A+tB)^{-1}$ we get

$$\begin{aligned} &m(s+\lambda+(1-t)A+tB)^{-2}\\ &\leq(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}\\ &\leq M(s+\lambda+(1-t)A+tB)^{-2} \end{aligned} \tag{3.4}$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$.

By (3.2) we get

$$M(s+\lambda+(1-t)A+tB)^{-2}\leq M(s+\lambda+\alpha+tm)^{-2}$$

and by (3.3) we have

$$m(s+\lambda+tm+\delta-m)^{-2}\leq m(s+\lambda+(1-t)A+tB)^{-1},$$

therefore, by (3.4), we get

$$\begin{aligned} &m(s+\lambda+tm+\delta-m)^{-2}\\ &\leq(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}\\ &\leq M(s+\lambda+\alpha+tm)^{-2} \end{aligned} \tag{3.5}$$

for all $t\in[0,1]$ and $s,\ \lambda\in[0,\infty)$.

If we multiply (3.5) by $w(\lambda)\geq0$ and integrate over t, s and λ, we get

$$\begin{aligned} 0&\leq m\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+tm+\delta-m)^{-2}dt\right)ds\right)d\mu(\lambda)\\ &\leq\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\ &\quad\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda)\\ &\leq M\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+\alpha+tm)^{-2}dt\right)ds\right)d\mu(\lambda) \end{aligned}$$

and by (2.8) we obtain

$$0 \le m \int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + tm + \delta - m)^{-2} dt \right) ds \right) d\mu(\lambda) \tag{3.6}$$
$$\le \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)$$
$$\le M \int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + \alpha + tm)^{-2} dt \right) ds \right) d\mu(\lambda).$$

Since

$$tm + \delta - m = (1-t)(\delta - m) + t\delta,$$

then

$$\int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + tm + \delta - m)^{-2} dt \right) ds \right) d\mu(\lambda) \tag{3.7}$$
$$= \int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + (1-t)(\delta - m) + t\delta)^{-2} dt \right) ds \right) d\mu(\lambda)$$
$$= \frac{1}{m} \int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + (1-t)(\delta - m) + t\delta) \right.\right.$$
$$\times (\delta - (\delta - m))(s + \lambda + (1-t)(\delta - m) + t\delta)\, dt)\, ds)\, d\mu(\lambda)$$
$$= \frac{1}{m} \left[\mathcal{DL}og(w,\mu)(\delta) - \mathcal{DL}og(w,\mu)(\delta - m) \right],$$

where the last equality follows by (2.8) for $B = \delta$ and $A = \delta - m$.

Since

$$\alpha + tm = (1-t)\alpha + t(m + \alpha),$$

then

$$\int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + \alpha + tm)^{-2} dt \right) ds \right) d\mu(\lambda) \tag{3.8}$$
$$= \int_0^\infty w(\lambda) \left(\int_0^\infty \left(\int_0^1 (s + \lambda + (1-t)\alpha + t(m+\alpha))^{-1} \right.\right.$$
$$\left.\left. \times ((m+\alpha) - \alpha)(s + \lambda + (1-t)\alpha + t(m+\alpha))^{-1} dt \right) ds \right) d\mu(\lambda)$$
$$= \frac{1}{m} \left[\mathcal{DL}og(w,\mu)(m+\alpha) - \mathcal{DL}og(w,\mu)(\alpha) \right]$$

where the last equality follows by (2.8) for $B = m + \alpha$ and $A = \alpha$.

By making use of (3.6)-(3.8) we derive the desired result (3.1). □

Remark 2. *With the assumptions of Theorem 3 for operators A and B, we have the* dilog *inequalities*

$$0 \le \operatorname{dilog}(\delta - m) - \operatorname{dilog}(\delta) \le \operatorname{dilog}(A) - \operatorname{dilog}(B) \tag{3.9}$$
$$\le \frac{M}{m} \left[\operatorname{dilog}(\alpha) - \operatorname{dilog}(m + \alpha) \right].$$

We also have:

Theorem 4. *Assume that $\delta \ge A,\ B \ge \alpha > 0$ and $0 < m \le B - A \le M$ for some constants $\alpha,\ \delta,\ m,\ M$. Then*

$$0 \le m\mathcal{D}(w,\mu)(\delta) \le \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A) \le M\mathcal{D}(w,\mu)(\alpha), \tag{3.10}$$

where

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda) = \frac{d\mathcal{D}\mathcal{L}og(w,\mu)(t)}{dt}.$$

Proof. Since $\delta \geq A,\ B \geq \alpha > 0$, then

$$\alpha \leq (1-t)A + tB \leq \delta \text{ for all } t \in [0,1],$$

which implies that

$$0 < s+\lambda+\alpha \leq s+\lambda+(1-t)A+tB \leq s+\lambda+\delta$$

for all $t \in [0,1]$ and $s,\ \lambda \in [0,\infty)$.

From this double inequality we derive that

$$0 < (s+\lambda+\delta)^{-1} \leq (s+\lambda+(1-t)A+tB)^{-1} \leq (s+\lambda+\alpha)^{-1},$$

which implies that

$$0 < (s+\lambda+\delta)^{-2} \leq (s+\lambda+(1-t)A+tB)^{-2} \leq (s+\lambda+\alpha)^{-2}. \tag{3.11}$$

Therefore

$$0 < m(s+\lambda+\delta)^{-2} \leq m(s+\lambda+(1-t)A+tB)^{-2}$$

and

$$M(s+\lambda+(1-t)A+tB)^{-2} \leq M(s+\lambda+\alpha)^{-2} \tag{3.12}$$

for all $t \in [0,1]$ and $s,\ \lambda \in [0,\infty)$.

By employing inequality (3.4), (3.11) and (3.12), we deduce

$$\begin{aligned} 0 &< m(s+\lambda+\delta)^{-2} \\ &\leq (s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1} \\ &\leq M(s+\lambda+\alpha)^{-2} \end{aligned} \tag{3.13}$$

for all $t \in [0,1]$ and $s,\ \lambda \in [0,\infty)$.

If we multiply (3.13) by $w(\lambda) \geq 0$ and integrate over t, s and λ, we get

$$\begin{aligned} 0 &\leq m\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2} ds\right) d\mu(\lambda) \\ &\leq \int_0^\infty w(\lambda)\left(\int_0^\infty \left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\ &\quad \times \left.\left.(s+\lambda+(1-t)A+tB)^{-1} dt\right) ds\right) d\mu(\lambda) \\ &\leq M\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2} ds\right) d\mu(\lambda). \end{aligned}$$

and by (2.8),

$$\begin{aligned} 0 &\leq m\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2} ds\right) d\mu(\lambda) \\ &\leq \mathcal{D}\mathcal{L}og(w,\mu)(B) - \mathcal{D}\mathcal{L}og(w,\mu)(A) \\ &\leq M\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2} ds\right) d\mu(\lambda). \end{aligned} \tag{3.14}$$

Observe that

$$\int_0^\infty (s+\lambda+\delta)^{-2} ds = -\left.(s+\lambda+\delta)^{-1}\right|_0^\infty = (\lambda+\delta)^{-1}$$

and

$$\int_0^\infty (s+\lambda+\alpha)^{-2} ds = -(s+\lambda+\alpha)^{-1}\Big|_0^\infty = (\lambda+\alpha)^{-1}.$$

Therefore

$$\begin{aligned}\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\delta)^{-2} ds\right) d\mu(\lambda) &= \int_0^\infty w(\lambda)(\lambda+\delta)^{-1} d\mu(\lambda)\\ &= \mathcal{D}(w,\mu)(\delta)\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty w(\lambda)\left(\int_0^\infty (s+\lambda+\alpha)^{-2} ds\right) d\mu(\lambda) &= \int_0^\infty w(\lambda)(\lambda+\alpha)^{-1} d\mu(\lambda)\\ &= \mathcal{D}(w,\mu)(\alpha).\end{aligned}$$

By making use of (3.14) we derive the desired result (3.10).

The fact that the derivative of $\mathcal{DL}og(w,\mu)(t)$ is $\mathcal{D}(w,\mu)(t)$ follows by the properties of the derivative of a parameter integral, i.e.

$$\frac{d\mathcal{DL}og(w,\mu)(t)}{dt} := \int_0^\infty w(\lambda)\frac{d\ln(\lambda+t)}{dt} d\mu(\lambda) = \int_0^\infty w(\lambda)(\lambda+t)^{-1} d\mu(\lambda)$$

for $t>0$. □

Remark 3. *With the assumptions of Theorem 4 for operators A and B, we have the* dilog *inequalities*

$$0 \le mu(\delta) \le \operatorname{dilog}(A) - \operatorname{dilog}(B) \le Mu(\alpha), \tag{3.15}$$

where

$$u(t) = \begin{cases} \frac{\ln t}{t-1}, & t \ne 1,\\ 1, & t = 1.\end{cases}$$

The case of separated operators is as follows:

Theorem 5. *If the positive operators satisfy the separation condition*

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta \tag{3.16}$$

for some positive constants α, β, γ, δ, then

$$\begin{aligned}0 &\le \frac{\gamma-\beta}{\delta-\beta}\left[\mathcal{DL}og(w,\mu)(\delta) - \mathcal{DL}og(w,\mu)(\beta)\right]\\ &\le \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[\mathcal{DL}og(w,\mu)(\gamma) - \mathcal{DL}og(w,\mu)(\alpha)\right].\end{aligned} \tag{3.17}$$

Proof. From (3.16) we derive that

$$(1-t)A + tB + \lambda + s \le (1-t)\beta + t\delta + \lambda + s,$$

which implies that

$$((1-t)A + tB + \lambda + s)^{-1} \ge ((1-t)\beta + t\delta + \lambda + s)^{-1}$$

and

$$((1-t)A + tB + \lambda + s)^{-2} \ge ((1-t)\beta + t\delta + \lambda + s)^{-2} \tag{3.18}$$

for all $t \in [0,1]$ and $\lambda \ge 0$.

Also

$$(1-t)A+tB+\lambda+s\geq(1-t)\alpha+t\gamma+\lambda+s,$$

which implies that

$$((1-t)A+tB+\lambda+s)^{-1}\leq((1-t)\alpha+t\gamma+\lambda+s)^{-1}$$

and

$$((1-t)A+tB+\lambda+s)^{-2}\leq((1-t)\alpha+t\gamma+\lambda+s)^{-2} \tag{3.19}$$

for all $t\in[0,1]$ and $\lambda,s\geq0$.

From (3.16) we have

$$0<\gamma-\beta\leq B-A\leq\delta-\alpha,$$

which implies that

$$\begin{aligned}0&\leq(\gamma-\beta)((1-t)A+tB+\lambda+s)^{-2}\\&\leq((1-t)A+tB+\lambda+s)^{-1}(B-A)((1-t)A+tB+\lambda+s)^{-1}\\&\leq(\delta-\alpha)((1-t)A+tB+\lambda+s)^{-2}\end{aligned} \tag{3.20}$$

for all $t\in[0,1]$ and $\lambda,s\geq0$.

By making use of (3.18)-(3.20) we get

$$\begin{aligned}0&\leq(\gamma-\beta)((1-t)\beta+t\delta+\lambda+s)^{-2}\\&\leq((1-t)A+tB+\lambda+s)^{-1}(B-A)((1-t)A+tB+\lambda+s)^{-1}\\&\leq(\delta-\alpha)((1-t)\alpha+t\gamma+\lambda+s)^{-2}\end{aligned} \tag{3.21}$$

for all $t\in[0,1]$ and $\lambda,s\geq0$.

If we multiply (3.21) by $w(\lambda)$ integrate and use the identity (2.8), then we obtain

$$\begin{aligned}0&\leq(\gamma-\beta)\int_0^\infty w(\lambda)\\&\times\left(\int_0^\infty\left(\int_0^1((1-t)\beta+t\delta+\lambda+s)^{-2}dt\right)ds\right)d\mu(\lambda)\\&\leq\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\&\leq(\delta-\alpha)\int_0^\infty w(\lambda)\\&\times\left(\int_0^\infty\left(\int_0^1((1-t)\alpha+t\gamma+\lambda+s)^{-2}dt\right)ds\right)d\mu(\lambda).\end{aligned} \tag{3.22}$$

Observe that

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)\beta+t\delta+\lambda+s)^{-2}dt\right)ds\right)d\mu(\lambda)\\&=\frac{1}{\delta-\beta}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)\beta+t\delta+\lambda+s)^{-1}\right.\right.\\&\left.\left.\times(\delta-\beta)((1-t)\beta+t\delta+\lambda+s)^{-1}dt\right)ds\right)d\mu(\lambda)\\&=\frac{1}{\delta-\beta}[\mathcal{D}\mathcal{L}og(w,\mu)(\delta)-\mathcal{D}\mathcal{L}og(w,\mu)(\beta)]\text{ by (2.8)}\end{aligned} \tag{3.23}$$

and

$$\int_0^\infty w(\lambda)\left(\int_0^\infty \left(\int_0^1 ((1-t)\alpha + t\gamma + \lambda + s)^{-2} dt\right) ds\right) d\mu(\lambda) \tag{3.24}$$
$$= \frac{1}{\gamma - \alpha}\int_0^\infty w(\lambda)\left(\int_0^\infty \left(\int_0^1 ((1-t)\alpha + t\gamma + \lambda + s)^{-1} (\gamma - \alpha)\right.\right.$$
$$\left.\left.\times ((1-t)\alpha + t\gamma + \lambda + s)^{-1} dt\right) ds\right) d\mu(\lambda)$$
$$= \frac{1}{\gamma - \alpha}\left[\mathcal{DL}og(w,\mu)(\gamma) - \mathcal{DL}og(w,\mu)(\alpha)\right] \text{ by (2.8).}$$

By making use of (3.22)-(3.24) we deduce the desired result (3.17). □

Remark 4. *With the assumptions of Theorem 5 for operators A and B, we have the* dilog *inequalities*

$$0 \le \frac{\gamma - \beta}{\delta - \beta}\left[\operatorname{dilog}(\beta) - \operatorname{dilog}(\delta)\right] \le \operatorname{dilog}(A) - \operatorname{dilog}(B) \tag{3.25}$$
$$\le \frac{\delta - \alpha}{\gamma - \alpha}\left[\operatorname{dilog}(\alpha) - \operatorname{dilog}(\gamma)\right].$$

4. Some Examples Via Operator Monotone Functions

We have:

Proposition 1. *Assume that function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1), where $b \ge 0$ and μ is a positive measure on $(0,\infty)$. Then*

$$\mathcal{DL}og(\ell,\mu)(t) = F_f(t) - bt \tag{4.1}$$

provided the function

$$F_f(t) := \int_0^t \frac{f(s) - f(0)}{s} ds \tag{4.2}$$

is defined for all $t \in (0,\infty)$.

Proof. From (1.1) we have

$$\frac{f(s) - f(0)}{s} - b = \int_0^\infty \frac{\lambda}{s+\lambda} d\mu(\lambda) = \mathcal{D}(\ell,\mu)(s) \tag{4.3}$$

where $\ell(\lambda) = \lambda$, $\lambda \ge 0$.

By taking the integral over s on $(0,t)$, we have

$$\int_0^t \frac{f(s) - f(0)}{s} ds - bt = \int_0^t \mathcal{D}(\ell,\mu)(s)\, ds = \mathcal{DL}og(\ell,\mu)(t)$$

for $t > 0$, and the proposition is proved. □

Corollary 2. *With the assumptions of Proposition 1 and if $B \ge A > 0$, we have*

$$F_f(B) - F_f(A) = \mathcal{DL}og(\ell,\mu)(B) - \mathcal{DL}og(\ell,\mu)(A) + b(B - A) \tag{4.4}$$
$$\ge b(B - A) \ge 0.$$

In particular, F_f is operator monotone on $(0,\infty)$.

The proof follows by Corollary 1 and the identity (4.1).

Remark 5. *If we take $f(t) = \ln(t+a)$, for a, $t > 0$, then we have*

$$F_{\ln(t+a)}(t) := \int_0^t \frac{\ln(s+a) - \ln(a)}{s} ds = \int_0^t \frac{1}{s} \ln\left(\frac{s}{a}+1\right) ds.$$

If we change the variable $u = \frac{s}{a}$, then we get

$$\int_0^t \frac{1}{s} \ln\left(\frac{s}{a}+1\right) ds = \int_0^{t/a} \frac{1}{ua} \ln(u+1)\, adu = \int_0^{t/a} \frac{1}{u} \ln(u+1)\, du$$
$$= -\operatorname{dilog}\left(\frac{t}{a}+1\right),$$

which gives

$$F_{\ln(t+a)}(t) = -\operatorname{dilog}\left(\frac{t}{a}+1\right), \ t > 0.$$

If $f(t) = t^r$, $r \in (0,1]$, then $F_f(t) := \frac{t^r}{r}$, $t > 0$.

Proposition 2. *Assume that function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1), where $b \geq 0$ and μ is a positive measure on $(0,\infty)$. If $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then*

$$\begin{aligned} 0 &\leq F_f(\delta) - F_f(\delta - m) \\ &\leq F_f(\delta) - F_f(\delta - m) + b[(B-A) - m] \\ &\leq F_f(B) - F_f(A) \\ &\leq \frac{M}{m}[F_f(m+\alpha) - F_f(\alpha)] + b[(B-A) - M] \\ &\leq \frac{M}{m}[F_f(m+\alpha) - F_f(\alpha)]. \end{aligned} \tag{4.5}$$

Proof. From (3.1) and (4.1) we have

$$\begin{aligned} 0 &\leq F_f(\delta) - F_f(\delta - m) - bm \\ &\leq F_f(B) - F_f(A) - b(B-A) \\ &\leq \frac{M}{m}[F_f(m+\alpha) - bm - F_f(\alpha)], \end{aligned}$$

which implies that

$$\begin{aligned} 0 &\leq F_f(\delta) - F_f(\delta - m) + b[(B-A) - m] \\ &\leq F_f(B) - F_f(A) \\ &\leq \frac{M}{m}[F_f(m+\alpha) - F_f(\alpha)] + b[(B-A) - M]. \end{aligned}$$

Since $b[(B-A) - m] \geq 0$, $b[(B-A) - M] \leq 0$ and F_f is operator monotone, hence the inequality (4.5) is proved. □

Proposition 3. *With the assumptions of Proposition 3 for f and if $\delta \geq A$, $B \geq \alpha > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then*

$$0 \leq m\left(\frac{f(\delta)-f(0)}{\delta}\right) \leq m\left(\frac{f(\delta)-f(0)}{\delta}\right) + b\left[(B-A)-m\right] \leq F_f(B) - F_f(A) \leq M\left(\frac{f(\delta)-f(0)}{\delta}\right) + b\left[(B-A)-M\right] \leq M\left(\frac{f(\delta)-f(0)}{\delta}\right). \tag{4.6}$$

Proof. From (3.10) and (4.1) we have

$$0 \leq m\left(\frac{f(\delta)-f(0)}{\delta} - b\right) \leq F_f(B) - F_f(A) - b(B-A) \leq M\left(\frac{f(\delta)-f(0)}{\delta} - b\right),$$

which implies that

$$0 \leq m\left(\frac{f(\delta)-f(0)}{\delta}\right) + b\left[(B-A)-m\right] \leq F_f(B) - F_f(A) \leq M\left(\frac{f(\delta)-f(0)}{\delta}\right) + b\left[(B-A)-M\right].$$

Since $b\left[(B-A)-m\right] \geq 0$, $b\left[(B-A)-M\right] \leq 0$ and f is operator monotone, hence the inequality (4.6) is proved. □

Proposition 4. *With the assumptions of Proposition 3 for f and if $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$, then*

$$\begin{aligned} 0 &\leq (\gamma-\beta)\left[\frac{F_f(\delta)-F_f(\beta)}{\delta-\beta}\right] \\ &\leq (\gamma-\beta)\left[\frac{F_f(\delta)-F_f(\beta)}{\delta-\beta}\right] + b\left[(B-A)-(\gamma-\beta)\right] \\ &\leq F_f(B) - F_f(A) \\ &\leq (\delta-\alpha)\left[\frac{F_f(\gamma)-F_f(\alpha)}{\gamma-\alpha}\right] + b\left[(B-A)-(\delta-\alpha)\right] \\ &\leq (\delta-\alpha)\left[\frac{F_f(\gamma)-F_f(\alpha)}{\gamma-\alpha}\right] \end{aligned} \tag{4.7}$$

Proof. From (3.17) and (4.1) we have

$$\begin{aligned} 0 &\leq \frac{\gamma-\beta}{\delta-\beta}\left[F_f(\delta) - F_f(\beta) - b(\delta-\beta)\right] \\ &\leq F_f(B) - F_f(A) - b(B-A) \\ &\leq \frac{\delta-\alpha}{\gamma-\alpha}\left[F_f(\gamma) - F_f(\alpha) - b(\gamma-\alpha)\right], \end{aligned}$$

namely

$$\begin{aligned} 0 &\le (\gamma-\beta)\left[\frac{F_f(\delta)-F_f(\beta)}{\delta-\beta}\right]+b\left[(B-A)-(\gamma-\beta)\right] \\ &\le F_f(B)-F_f(A) \\ &\le (\delta-\alpha)\left[\frac{F_f(\gamma)-F_f(\alpha)}{\gamma-\alpha}\right]+b\left[(B-A)-(\delta-\alpha)\right]. \end{aligned}$$

Since $b\left[(B-A)-(\gamma-\beta)\right] \ge 0$, $b\left[(B-A)-(\delta-\alpha)\right] \le 0$ and F_f is operator monotone, hence the inequality (4.7) is proved. □

5. More Examples

If we consider the positive kernel $w_{\exp(-a\cdot)}(\lambda) := \exp(-a\lambda)$, $\lambda \ge 0$, then, after some calculations

$$\int_0^\infty \exp(-a\lambda)\ln(\lambda+t)\,d\lambda = \frac{1}{a}\left[\ln t + E_1(at)\exp(at)\right],$$

for $t > 0$,where the *exponential integral* is given by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

For $a = 1$ we have

$$\int_0^\infty \exp(-\lambda)\ln(\lambda+t)\,d\lambda = \ln t + E_1(t)\exp(t),$$

For $t = 0$, we derive

$$\int_0^\infty \exp(-\lambda)\ln(\lambda)\,d\lambda = -\gamma,$$

where γ is Euler–Mascheroni constant.

For $a > 0$, by changing the variable $a\lambda = \nu$, then

$$\begin{aligned} \int_0^\infty \exp(-a\lambda)\ln(\lambda)\,d\lambda &= \int_0^\infty \exp(-\nu)\ln\left(\frac{\nu}{a}\right)\frac{1}{a}d\nu \\ &= \frac{1}{a}\int_0^\infty \left[\exp(-\nu)\ln\nu - \exp(-\nu)\ln a\right]d\nu \\ &= \frac{1}{a}(-\gamma-\ln a) = -\frac{\ln a+\gamma}{a}. \end{aligned}$$

We then have

$$\begin{aligned} \mathcal{DL}og\left(w_{\exp(-a\cdot)}\right)(t) &= \int_0^\infty \exp(-a\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda) \\ &= \frac{1}{a}\left[\ln(at)+E_1(at)\exp(at)+\gamma\right] \end{aligned}$$

and, for $a = 1$,

$$\begin{aligned} \mathcal{DL}og\left(w_{\exp(-\cdot)}\right)(t) &= \int_0^\infty \exp(-\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda) \\ &= \ln(t)+E_1(t)\exp(t)+\gamma. \end{aligned}$$

Using Corollary 1 we conclude that the function $\ln(t)+E_1(t)\exp(t)$ is *operator monotone on* $(0,\infty)$.

If we consider the positive kernel $w_{(\cdot+a)^{-2}}(\lambda) := \frac{1}{(\lambda+a)^2}$, $\lambda \geq 0$, $a > 0$, then, after some calculations

$$\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2} d\lambda = \begin{cases} \frac{t\ln t - a\ln a}{a(t-a)}, \text{ if } t \neq a, \\ \\ \frac{\ln a + 1}{a}, \text{ if } t = a \end{cases}$$

for $t > 0$.

If $a = 1$, then

$$\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+1)^2} d\lambda = \begin{cases} \frac{t\ln t}{t-1}, \text{ if } t \neq 1, \\ \\ 1, \text{ if } t = 1 \end{cases}$$

for $t > 0$.

For $t = 0$, we derive

$$\int_0^\infty \frac{\ln(\lambda)}{(\lambda+a)^2} d\lambda = \frac{\ln a}{a}$$

for $a > 0$.

Therefore

$$\mathcal{DL}og\left(w_{(\cdot+a)^{-2}}\right)(t) = \begin{cases} \frac{t(\ln t - \ln a)}{a(t-a)}, \text{ if } t \neq a, \\ \\ \frac{1}{a}, \text{ if } t = a. \end{cases}$$

For $a = 1$,

$$\mathcal{DL}og\left(w_{(\cdot+1)^{-2}}\right)(t) = \begin{cases} \frac{t\ln t}{t-1}, \text{ if } t \neq 1, \\ \\ 1, \text{ if } t = 1. \end{cases}$$

Using Corollary 1 we conclude that the function $\mathcal{DL}og\left(w_{(\cdot+1)^{-2}}\right)$ is *operator monotone on* $(0, \infty)$.

If we write the operator inequalities (3.1), (3.10) and (3.17) for the transforms $\mathcal{DL}og\left(w_{\exp(-a\cdot)}\right)$ and $\mathcal{DL}og\left(w_{(\cdot+a)^{-2}}\right)$ we can get similar results as above. The details are omitted.

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