NONCOMMUTATIVE PERSPECTIVES OF OPERATOR MONOTONE FUNCTIONS IN HILBERT SPACES

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Abstract. Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} dw(\lambda),$$

where $a \in \mathbb{R}$, $b \geq 0$ and w is a positive measure on $(0, \infty)$. In this paper we obtained among others that

$$\begin{aligned}&\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P)\\&= b(B - A) + \int_0^\infty \lambda^2 \left[\int_0^1 P((1-t)A + tB + \lambda P)^{-1}(B - A)\right.\\&\left.\times((1-t)A + tB + \lambda P)^{-1} P dt\right] dw(\lambda)\end{aligned}$$

for all A, B, $P > 0$. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

In 1934, K. Löwner [10] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}dw(\lambda) \tag{1.3}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure w on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}dw(\lambda)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [9]. The function ln is also operator monotone on $(0,\infty)$.

For other examples of operator monotone functions, see [7] and [8].

Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A)=Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset\mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x)=xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [12]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B)=\mathcal{P}_f(B,A). \tag{1.4}$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [12].

The following inequality is of interest, see [12]:

Theorem 2. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_f(A,B)\geq\mathcal{P}_f(C,D). \tag{1.5}$$

It is well known that (see [3] and [2] or [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto\mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_\nu : [0,\infty) \to [0,\infty),\ f_\nu(t) = t^\nu,\ \nu \in [0,1]$, then

$$P_{f_\nu}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2} =: A\sharp_\nu B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight ν.

We define the *weighted operator arithmetic mean* by

$$A\nabla_\nu B := (1-\nu)A + \nu B,\ \nu \in [0,1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_\nu B \le A\nabla_\nu B$$

for any $\nu \in [0,1]$.

If we take the function $f = \ln$, then

$$P_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [5], [6] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [11].

2. Main Results

We start to the following identity of interest:

Lemma 1. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all U, $V > 0$ we have*

$$\begin{aligned} f(V) - f(U) &= b(V-U) \\ &\quad + \int_0^\infty \lambda^2 \left[\int_0^1 ((1-t)U + tV + \lambda)^{-1} \right. \\ &\quad \left. \times (V-U)((1-t)U + tV + \lambda)^{-1} dt\right] dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. Since the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3), then for U, $V > 0$ we have the representation

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda\left[V(V+\lambda)^{-1} - U(U+\lambda)^{-1}\right] dw(\lambda). \tag{2.2}$$

Observe that for $\lambda > 0$

$$\begin{aligned} &V(V+\lambda)^{-1} - U(U+\lambda)^{-1} \\ &= (V+\lambda-\lambda)(V+\lambda)^{-1} - (U+\lambda-\lambda)(U+\lambda)^{-1} \\ &= (V+\lambda)(V+\lambda)^{-1} - \lambda(V+\lambda)^{-1} - (U+\lambda)(U+\lambda)^{-1} + \lambda(U+\lambda)^{-1} \\ &= \lambda\left[(U+\lambda)^{-1} - (V+\lambda)^{-1}\right]. \end{aligned}$$

Therefore, (2.2) becomes, see also [8]

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda^2\left[(U+\lambda)^{-1} - (V+\lambda)^{-1}\right] dw(\lambda). \tag{2.3}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we replace in (2.6) $C=U+\lambda$ and $D=V+\lambda$ for $\lambda>0$, then

$$\begin{aligned}&(U+\lambda)^{-1}-(V+\lambda)^{-1}\\&=\int_0^1((1-t)U+tV+\lambda)^{-1}(V-U)((1-t)U+tV+\lambda)^{-1}\,dt.\end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

Theorem 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all A, B, $P>0$ we have*

$$\begin{aligned}&\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\\&=b(B-A)+\int_0^\infty\lambda^2\left[\int_0^1P((1-t)A+tB+\lambda P)^{-1}(B-A)\right.\\&\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda).\end{aligned} \tag{2.8}$$

Proof. If we take $V=P^{-1/2}BP^{-1/2}$ and $U=P^{-1/2}AP^{-1/2}$ in (2.1), then we get

$$\begin{aligned}&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right)\\&=b\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)\\&+\int_0^\infty\lambda^2\left[\int_0^1\left((1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\right.\\&\times\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)\\&\left.\times\left((1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}dt\right]dw(\lambda).\end{aligned} \tag{2.9}$$

Observe that

$$P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}=P^{-1/2}(B-A)P^{-1/2},$$

and

$$(1-t)\,P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda=P^{-1/2}\left((1-t)\,A+tB+\lambda P\right)P^{-1/2},$$

which gives

$$\begin{aligned}&\left((1-t)\,P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\\&=P^{1/2}\left((1-t)\,A+tB+\lambda P\right)^{-1}P^{1/2}\end{aligned}$$

and by (2.9),

$$\begin{aligned}&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1 P^{1/2}\left((1-t)\,A+tB+\lambda P\right)^{-1}P^{1/2}P^{-1/2}\left(B-A\right)P^{-1/2}\right.\\&\left.\times P^{1/2}\left((1-t)\,A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1 P^{1/2}\left((1-t)\,A+tB+\lambda P\right)^{-1}\left(B-A\right)\right.\\&\left.\times\left((1-t)\,A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right).\end{aligned}\tag{2.10}$$

If we multiply both sides of (2.10) by $P^{1/2}$ we obtain the desired identity (2.8). □

Lemma 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all U, $V>0$ we have*

$$\begin{aligned}&\tilde f\left(V\right)-\tilde f\left(U\right)\\&=a\left(V-U\right)+\int_0^\infty\lambda\left(\int_0^1\left(1+\lambda\left[(1-t)\,U+tV\right]\right)^{-1}\right.\\&\left.\times\left(V-U\right)\left(1+\lambda\left[(1-t)\,U+tV\right]\right)^{-1}dt\right)dw\left(\lambda\right).\end{aligned}\tag{2.11}$$

Proof. From (1.3) we have

$$f\left(t\right)=a+bt+t\int_0^\infty\frac{\lambda}{t+\lambda}dw\left(\lambda\right),\ t>0.$$

If we put $\frac{1}{t}$ instead of t we get

$$\begin{aligned}f\left(\frac{1}{t}\right)&=a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty\frac{\lambda}{\frac{1}{t}+\lambda}dw\left(\lambda\right)\\&=a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty\frac{t\lambda}{1+t\lambda}dw\left(\lambda\right)\end{aligned}$$

and by multiplication with $t>0$, we get

$$\tilde f\left(t\right)=b+ta+\int_0^\infty\frac{t\lambda}{1+t\lambda}dw\left(\lambda\right)=b+ta+\int_0^\infty\left(1-\frac{1}{1+t\lambda}\right)dw\left(\lambda\right).$$

Therefore

$$\tilde f\left(V\right)-\tilde f\left(U\right)=a\left(V-U\right)+\int_0^\infty\left[\left(1+U\lambda\right)^{-1}-\left(1+V\lambda\right)^{-1}\right]dw\left(\lambda\right).\tag{2.12}$$

From (2.6) we get

$$\begin{aligned}(2.13)\quad &(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\\&=\int_0^1((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}((1+V\lambda)-(1+U\lambda))\\&\times((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}dt\\&=\int_0^1\lambda(1+\lambda[(1-t)U+tV])^{-1}(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt.\end{aligned}$$

Therefore, by (2.12) we get

$$\begin{aligned}(2.14)\quad &\tilde{f}(V)-\tilde{f}(U)\\&=a(V-U)+\int_0^\infty\left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda)\\&=a(V-U)+\int_0^\infty\lambda\left(\int_0^1(1+\lambda[(1-t)U+tV])^{-1}\right.\\&\left.\times(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt\right)dw(\lambda)\end{aligned}$$

and the identity (2.11) is proved. □

Theorem 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all C, D, $Q>0$ we have*

$$\begin{aligned}(2.15)\quad &\mathcal{P}_{\tilde{f}}(D,Q)-\mathcal{P}_{\tilde{f}}(C,Q)\\&=a(D-C)+\int_0^\infty\lambda\left(\int_0^1Q[(Q+\lambda[(1-t)C+tD])]^{-1}(D-C)\right.\\&\left.\times[(Q+\lambda[(1-t)C+tD])]^{-1}Qdt\right)dw(\lambda).\end{aligned}$$

Proof. If we take $V=Q^{-1/2}DQ^{-1/2}$ and $U=Q^{-1/2}CQ^{-1/2}$ in (2.11), then we get

$$\begin{aligned}(2.16)\quad &\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)-\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)\\&=a\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\&+\int_0^\infty\lambda\left(\int_0^1\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}\right.\\&\times\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\&\left.\times\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}dt\right)dw(\lambda)\end{aligned}$$

$$= aQ^{-1/2}(D-C)Q^{-1/2}$$
$$+\int_0^\infty \lambda \left(\int_0^1 \left[Q^{-1/2}(Q+\lambda[(1-t)C+tD])\right]^{-1}\right.$$
$$\left.\times Q^{-1/2}(D-C)Q^{-1/2}\left[Q^{-1/2}(Q+\lambda[(1-t)C+tD])Q^{-1/2}\right]^{-1}dt\right)dw(\lambda)$$
$$= aQ^{-1/2}(D-C)Q^{-1/2}$$
$$+\int_0^\infty \lambda \left(\int_0^1 Q^{1/2}[(Q+\lambda[(1-t)C+tD])]^{-1}(D-C)\right.$$
$$\left.\times[(Q+\lambda[(1-t)C+tD])]^{-1}Q^{1/2}dt\right)dw(\lambda).$$

If we multiply both sides by $Q^{1/2}$ we get the desired result (2.15). □

Corollary 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all C, D, $Q>0$ we have*

$$\begin{aligned}(2.17)\qquad &\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\\ &= a(D-C)+\int_0^\infty \lambda\left(\int_0^1 Q[(Q+\lambda[(1-t)C+tD])]^{-1}(D-C)\right.\\ &\left.\times[(Q+\lambda[(1-t)C+tD])]^{-1}Qdt\right)dw(\lambda).\end{aligned}$$

We also have:

Corollary 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all A, B, C, $D>0$ we have*

$$\begin{aligned}(2.18)\qquad &\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)\\ &= b(A-C)+a(B-D)\\ &+\int_0^\infty \lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\ &\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\\ &+\int_0^\infty \lambda\left(\int_0^1 C[(C+\lambda[(1-t)D+tB])]^{-1}(B-D)\right.\\ &\left.\times[(C+\lambda[(1-t)D+tB])]^{-1}Cdt\right)dw(\lambda).\end{aligned}$$

Proof. Observe that

$$(2.19)\quad \mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)=\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)+\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D).$$

Since, by (2.8),

$$\begin{aligned}(2.20)\qquad &\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)\\ &= b(A-C)+\int_0^\infty \lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\ &\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\end{aligned}$$

and by (2.17),

$$\begin{aligned}(2.21)\qquad &\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D)\\ &=a(B-D)+\int_0^\infty \lambda\left(\int_0^1 C\left[(C+\lambda[(1-t)D+tB])\right]^{-1}(B-D)\right.\\ &\left.\times\left[(C+\lambda[(1-t)D+tB])\right]^{-1}Cdt\right)dw(\lambda),\end{aligned}$$

hence by (2.19)-(2.21) we obtain (2.18). □

As a natural consequence of the above representations, we derive the following inequalities:

Theorem 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If $B\ge A>0$ and $P>0$, then*

$$\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\ge b(B-A)\ge 0 \tag{2.22}$$

and

$$\mathcal{P}_f(P,B)-\mathcal{P}_f(P,A)\ge a(B-A). \tag{2.23}$$

If $A\ge C>0$ and $B\ge D>0$, then

$$\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)\ge b(A-C)+a(B-D). \tag{2.24}$$

Proof. If $B-A\ge 0$, then by multiplying both sides by $((1-t)A+tB+\lambda P)^{-1}$ for $t\in[0,1]$ and $\lambda\ge 0$ we get

$$((1-t)A+tB+\lambda P)^{-1}(B-A)((1-t)A+tB+\lambda P)^{-1}\ge 0.$$

Also by multiplying both sides by $P>0$, we get

$$P((1-t)A+tB+\lambda P)^{-1}(B-A)((1-t)A+tB+\lambda P)^{-1}P\ge 0,$$

for $t\in[0,1]$ and $\lambda\ge 0$.

If we multiply this inequality by λ^2 integrate over $t\in[0,1]$ and over the measure $w(\lambda)$ on $[0,\infty)$ we get

$$\begin{aligned}&\int_0^\infty \lambda^2\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right.\\ &\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda)\ge 0\end{aligned}$$

and by representation (2.8) we deduce (2.22).

The inequality (2.23) follows in a similar way by (2.17). The inequality (2.24) follows by the representation (3.2). □

Remark 1. *If $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and nonnegative, then in representation (1.3) the parameter a must be nonnegative and in this situation we have*

$$\mathcal{P}_f(P,B)-\mathcal{P}_f(P,A)\ge a(B-A)\ge 0, \tag{2.25}$$

if $B\ge A>0$ and $P>0$.

If f is defined on $[0,\infty)$, then we can take $a=f(0)$ in (2.23) and (2.25). If the parameters a and b are positive in representation (1.3), then the inequality (2.24) improves (1.5).

3. Some Examples of Interest

We also have identities for the *weighted operator geometric mean:*

Proposition 1. *For all* A, B, $P>0$ *and* $r\in(0,1]$ *we have*

$$\begin{aligned}(3.1)\qquad & P\sharp_r B - P\sharp_r A \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r+1}\left[\int_0^1 P\left((1-t)A+tB+\lambda P\right)^{-1}(B-A)\right.\\ &\left.\times\left((1-t)A+tB+\lambda P\right)^{-1}Pdt\right]d\lambda.\end{aligned}$$

The proof follows by (2.8) and (1.1) for the measure $dw(\lambda)=\frac{\sin(r\pi)}{\pi}\lambda^{r-1}d\lambda$. The dual case follows by (2.17) and (1.1).

Proposition 2. *For all* C, D, $Q>0$ *and* $r\in(0,1]$ *we have*

$$\begin{aligned}(3.2)\qquad & D\sharp_r Q - C\sharp_r Q \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r}\left(\int_0^1 Q\left[(Q+\lambda\left[(1-t)C+tD\right])\right]^{-1}(D-C)\right.\\ &\left.\times\left[(Q+\lambda\left[(1-t)C+tD\right])\right]^{-1}Qdt\right)d\lambda.\end{aligned}$$

The following identity for the logarithmic function also holds:

Lemma 3. *For all* U, $V>0$ *we have the identity:*

$$\begin{aligned}(3.3)\qquad & \ln V-\ln U \\ &= \int_0^\infty\left(\int_0^1(\lambda+(1-t)U+tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}dt\right)d\lambda.\end{aligned}$$

Proof. We have from the representation of logarithm (1.2) that

$$(3.4)\qquad \ln V-\ln U=\int_0^\infty\frac{1}{\lambda+1}\left[(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\right]d\lambda$$

for U, $V>0$.

Since

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\ &=V(\lambda+V)^{-1}-U(\lambda+U)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&V(\lambda+V)^{-1}-U(\lambda+U)^{-1}\\ &=(V+\lambda-\lambda)(\lambda+V)^{-1}-(U+\lambda-\lambda)(\lambda+U)^{-1}\\ &=1-\lambda(\lambda+V)^{-1}-1+\lambda(\lambda+U)^{-1}=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\ &=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\\ &=(\lambda+1)\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]\end{aligned}$$

and by (3.4) we get

$$\ln V-\ln U=\int_0^\infty \left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]d\lambda. \tag{3.5}$$

Since, by (2.6) we have

$$\begin{aligned}(\lambda+U)^{-1}&-(\lambda+V)^{-1}\\ &=\int_0^1 (\lambda+(1-t)U+tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}dt,\end{aligned} \tag{3.6}$$

for all $\lambda\geq 0$, hence by (3.5) and (3.6) we get (3.3). □

Theorem 6. *For all A, B, $P>0$ we have*

$$\begin{aligned}S(P|B)-S(P|A)=\int_0^\infty &\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right.\\ &\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]d\lambda.\end{aligned} \tag{3.7}$$

Proof. Follows by Lemma 3 by taking $V=P^{-1/2}BP^{-1/2}$ and $U=P^{-1/2}AP^{-1/2}$ and multiplying both sides by $P^{1/2}$. □

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