REFINED INEQUALITIES FOR THE DISTANCE IN METRIC SPACES

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Abstract. In this note we prove among others that

$$\sum_{1\leq i<j\leq n} p_i p_j d^s(x_i,x_j) \leq \begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)\,d^s(x_k,x)\right], & s\geq 1\\ \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)\,d^s(x_k,x)\right], & 0<s<1,\end{cases}$$

where (X,d) is a metric space, $x_i\in X$, $p_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $s>0$. This generalizes and improves some early upper bounds for the sum $\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j)$.

1. Introduction

Let X be a nonempty set. A function $d: X\times X\to[0,\infty)$ is called a *distance* on X if the following properties are satisfied:

(d) $d(x,y)=0$ if and only if $x=y$;
(dd) $d(x,y)=d(y,x)$ for any $x,y\in X$ (the *symmetry* of the distance);
(ddd) $d(x,y)\leq d(x,z)+d(z,y)$ for any $x,y,z\in X$ (the *triangle inequality*).

The pair (X,d) is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space E over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\|: E\to[0,\infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties:

(n) $\|x\|=0$ if and only if $x=0$;
(nn) $\|\alpha x\|=|\alpha|\,\|x\|$ for any scalar $\alpha\in\mathbb{K}$ and any vector $x\in E$;
(nnn) $\|x+y\|\leq\|x\|+\|y\|$ for each $x,y\in E$ (the triangle inequality).

Further, we recall that, the linear space H over the real or complex number field $\mathbb{K}$ endowed with an application $\langle\cdot,\cdot\rangle: H\times H\to\mathbb{K}$ is called an *inner product space*, if the function $\langle\cdot,\cdot\rangle$, called the *inner product*, satisfies the following properties:

(i) $\langle x,x\rangle\geq 0$ for any $x\in H$ and $\langle x,x\rangle=0$ if and only if $x=0$;
(ii) $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$ for any scalars α,β and any vectors x,y,z;
(iii) $\langle y,x\rangle=\overline{\langle x,y\rangle}$ for any $x,y\in H$.

It is well know that the function $\|x\|:=\sqrt{\langle x,x\rangle}$ defines a norm on H and thus an important example of normed spaces are the inner product spaces.

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalised triangle inequality*, or

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the *polygonal inequality* which states that: for any points $x_1, x_2, ..., x_{n-1}, x_n$ $(n \geq 3)$ in a metric space (X, d), we have the inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + ... + d(x_{n-1}, x_n). \tag{1.1}$$

The following result in the general setting of metric spaces holds.

Theorem 1. *Let (X, d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \inf_{x \in X} \left[\sum_{i=1}^n p_i d(x_i, x) \right]. \tag{1.2}$$

The inequality is sharp in the sense that the multiplicative constant $c = 1$ in front of " inf " cannot be replaced by a smaller quantity.

We have:

Corollary 1. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. If there exists a closed ball of radius $r > 0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$, then for any $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq r. \tag{1.3}$$

The inequality (1.2) and its consequences were extended to the case of b-metric spaces in [3] and for natural powers of the distance in [1].

In this note we provide some new and improved upper and lower bounds for the sum

$$\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j)$$

where (X, d) is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ and $s > 0$.

2. Main Results

We have the following generalization of the inequality (1.2).

Theorem 2. *Let (X, d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j) \leq \begin{cases} 2^{s-1} \sum_{k=1}^n p_k (1 - p_k) d^s(x_k, x), & s \geq 1 \\ \\ \sum_{k=1}^n p_k (1 - p_k) d^s(x_k, x), & 0 < s < 1, \end{cases} \tag{2.1}$$

$$\leq \frac{1}{4} \begin{cases} 2^{s-1} \sum_{k=1}^n d^s(x_k, x), & s \geq 1 \\ \\ \sum_{k=1}^n d^s(x_k, x), & 0 < s < 1. \end{cases}$$

Proof. We know that, by the convexity property of the power function $f(t) = t^s$, $s \geq 1$ on $[0, \infty)$, we have for a, $b \geq 0$ that

$$(a + b)^s \leq 2^{s-1} (a^s + b^s).$$

We consider the function $f_s : [0,\infty) \to \mathbb{R}$, $f_s(t) = (t+1)^s - t^s$ we have $f_s'(t) = s\left[(t+1)^{s-1} - t^{s-1}\right]$. Observe that for $0 < s < 1$ and $t > 0$ we have that $f_s'(t) < 0$ showing that f_s is strictly decreasing on the interval $[0,\infty)$. Now for $t_0 = \frac{a}{b}$ $(b > 0, a \geq 0)$ we have $f_s(t_0) < f_s(0)$ giving that $\left(\frac{a}{b}+1\right)^s - \left(\frac{a}{b}\right)^s < 1$, i.e., the inequality

$$(a+b)^s \leq a^s + b^s.$$

Using the triangle inequality, we have for any $x \in X$ and $i, j \in \{1,...,n\}$, that

$$d(x_i,x_j) \leq d(x_i,x) + d(x,x_j). \tag{2.2}$$

If we take the power $s > 0$ in (2.2) we get

$$\begin{aligned} d^s(x_i,x_j) &\leq [d(x_i,x) + d(x,x_j)]^s \\ &\leq \begin{cases} 2^{s-1}\left(d^s(x_i,x) + d(x_j,x)^s\right), & s \geq 1 \\ \\ d^s(x_i,x) + d(x_j,x)^s, & 0 < s < 1 \end{cases} \end{aligned} \tag{2.3}$$

for any $x \in X$ and $i, j \in \{1,...,n\}$.

If we multiply (2.3) by $p_ip_j \geq 0$ and sum over $1 \leq i < j \leq n$ from 1 to n, we get

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_ip_j d^s(x_i,x_j) \\ &\leq \begin{cases} 2^{s-1}\sum_{1\leq i<j\leq n} p_ip_j\left(d^s(x_i,x) + d(x_j,x)^s\right), & s \geq 1 \\ \\ \sum_{1\leq i<j\leq n} p_ip_j\left(d^s(x_i,x) + d(x_j,x)^s\right), & 0 < s < 1. \end{cases} \end{aligned} \tag{2.4}$$

Observe that, in general, if $a_{ij} = a_{ji}$ for $1 \leq i,j \leq n$ then

$$\sum_{1\leq i,j\leq n} a_{ij} = \sum_{1\leq i<j\leq n} a_{ij} + \sum_{1\leq j<i\leq n} a_{ij} + \sum_{k=1}^{n} a_{kk} = 2\sum_{1\leq i<j\leq n} a_{ij} + \sum_{k=1}^{n} a_{kk},$$

which implies that

$$\sum_{1\leq i<j\leq n} a_{ij} = \frac{1}{2}\left(\sum_{1\leq i,j\leq n} a_{ij} - \sum_{k=1}^{n} a_{kk}\right).$$

Therefore

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_ip_j\left(d^s(x_i,x) + d(x_j,x)^s\right) \\ &= \frac{1}{2}\left(\sum_{1\leq i,j\leq n} p_ip_j\left(d^s(x_i,x) + d(x_j,x)^s\right) - 2\sum_{k=1}^{n} p_k^2 d^s(x_k,x)\right) \\ &= \sum_{k=1}^{n} p_k d^s(x_k,x) - \sum_{k=1}^{n} p_k^2 d^s(x_k,x) = \sum_{k=1}^{n} p_k(1-p_k) d^s(x_k,x) \end{aligned}$$

and by (2.4) the first inequality in (2.1).

The second part follows by the fact that

$$p_k(1-p_k) \leq \frac{1}{4}(p_k + 1 - p_k)^2 = \frac{1}{4}$$

for all $k \in \{1,...,n\}$. □

Remark 1. *By taking the infimum over $x \in X$ in (2.1), we get*

(2.5)
$$\sum_{1\le i<j\le n} p_i p_j d^s(x_i,x_j) \le \begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)d^s(x_k,x)\right], & s\ge 1 \\ \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)d^s(x_k,x)\right], & 0<s<1, \end{cases}$$
$$\le \frac{1}{4}\begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n d^s(x_k,x)\right], & s\ge 1 \\ \inf_{x\in X}\left[\sum_{k=1}^n d^s(x_k,x)\right], & 0<s<1. \end{cases}$$

For $s=1$ we derive

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)d(x_k,x)\right],$$

which is a better inequality than (1.2) since

$$\sum_{k=1}^n p_k(1-p_k)d(x_k,x) \le \sum_{i=1}^n p_i d(x_i,x).$$

Corollary 2. *Let (X,d) be a metric space and $x_i\in X$, $i\in\{1,...,n\}$. Then we have the inequality*

(2.6)
$$\sum_{1\le i<j\le n} d^s(x_i,x_j) \le (n-1)\begin{cases} 2^{s-1}\sum_{k=1}^n d^s(x_k,x), & s\ge 1 \\ \sum_{k=1}^n d^s(x_k,x), & 0<s<1. \end{cases}$$

Follows by the first inequality in (2.1) for $p_k=\frac{1}{n}$, $k\in\{1,...,n\}$.

Corollary 3. *Let (X,d) be a metric space and $x_i\in X$, $i\in\{1,...,n\}$. If there exists a closed ball of radius $r>0$ centered in a point x containing all the points x_i, i.e., $x_i\in\overline{B}(x,r) := \{y\in X : d(x,y)\le r\}$, then for any $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ we have the inequalities*

(2.7)
$$\sum_{1\le i<j\le n} p_i p_j d^s(x_i,x_j) \le \sum_{k=1}^n p_k(1-p_k)\begin{cases} 2^{s-1}r^s, & s\ge 1 \\ r^s, & 0<s<1. \end{cases}$$

We also have the following lower bound:

Theorem 3. *Let (X,d) be a metric space and $x_i\in X$, $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$. Then we have the inequality*

(2.8)
$$2^{s-1}\left(\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j)\right)^s \le \sum_{1\le i<j\le n} p_i p_j d^s(x_i,x_j), \quad s>1.$$

Proof. We use Jensen's discrete inequality for the power function $f(t)=t^s$, $s>1$ to write

(2.9)
$$\frac{\sum_{1\le i,j\le n} p_i p_j d^s(x_i,x_j)}{\sum_{1\le i,j\le n} p_i p_j} \ge \left(\frac{\sum_{1\le i,j\le n} p_i p_j d(x_i,x_j)}{\sum_{1\le i,j\le n} p_i p_j}\right)^s.$$

Observe that

$$\sum_{1\le i,j\le n} p_i p_j = \left(\sum_{i=1}^n p_i\right)^2 = 1,$$

$$\sum_{1\leq i,j\leq n} p_i p_j d^s (x_i, x_j) = 2 \sum_{1\leq i<j\leq n} p_i p_j d^s (x_i, x_j)$$

and

$$\sum_{1\leq i,j\leq n} p_i p_j d (x_i, x_j) = 2 \sum_{1\leq i<j\leq n} p_i p_j d^s (x_i, x_j) .$$

By (2.9) we get

$$2 \sum_{1\leq i<j\leq n} p_i p_j d^s (x_i, x_j) \geq \left(2 \sum_{1\leq i<j\leq n} p_i p_j d (x_i, x_j)\right)^s = 2^s \left(\sum_{1\leq i<j\leq n} p_i p_j d (x_i, x_j)\right)^s , \tag{2.10}$$

and the inequality (2.8) is proved. □

Corollary 4. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. Then we have the inequality*

$$\left(\frac{2}{n^2}\right)^{s-1} \left(\sum_{1\leq i<j\leq n} d (x_i, x_j)\right)^s \leq \sum_{1\leq i<j\leq n} d^s (x_i, x_j) , \ s > 1. \tag{2.11}$$

3. Applications

If $(E, \|\cdot\|)$ is a normed linear space and $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \geq 0$ $(i \in \{1, ..., n\})$ with $\sum_{i=1}^n p_i = 1$, then by (2.1) we have the inequality

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s \\ &\leq \begin{cases} 2^{s-1} \inf_{x\in X} \left[\sum_{k=1}^n p_k (1 - p_k) \|x_k - x\|^s\right], \ s \geq 1 \\ \inf_{x\in X} \left[\sum_{k=1}^n p_k (1 - p_k) \|x_k - x\|^s\right], \ 0 < s < 1, \end{cases} \\ &\leq \frac{1}{4} \begin{cases} 2^{s-1} \inf_{x\in X} \left[\sum_{k=1}^n \|x_k - x\|^s\right], \ s \geq 1 \\ \inf_{x\in X} \left[\sum_{k=1}^n \|x_k - x\|^s\right], \ 0 < s < 1. \end{cases} \end{aligned} \tag{3.1}$$

In particular, for the uniform distribution $p_i = \frac{1}{n}$, we have

$$\sum_{1\leq i<j\leq n} \|x_i - x_j\|^s \leq \begin{cases} 2^{s-1} (n - 1) \inf_{x\in X} \left[\sum_{i=1}^n \|x_i - x\|^s\right], \ s \geq 1, \\ (n - 1) \inf_{x\in X} \left[\sum_{i=1}^n \|x_i - x\|^s\right], \ 0 < s < 1. \end{cases} \tag{3.2}$$

Denote $\overline{x_p} := \sum_{i=1}^n p_i x_i$, then we have the inequalities

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s \leq \begin{cases} 2^{s-1}\sum_{k=1}^n p_k (1-p_k) \|x_k - \overline{x_p}\|^s, & s\geq 1 \\ \\ \sum_{k=1}^n p_k (1-p_k) \|x_k - \overline{x_p}\|^s, & 0<s<1, \end{cases} \leq \frac{1}{4}\begin{cases} 2^{s-1}\sum_{k=1}^n \|x_k - \overline{x_p}\|^s, & s\geq 1 \\ \\ \sum_{k=1}^n \|x_k - \overline{x_p}\|^s, & 0<s<1. \end{cases} \tag{3.3}$$

By triangle inequality we have that

$$\sum_{j=1}^n p_j \|x_i - x_j\| \geq \left\| \sum_{j=1}^n p_j (x_i - x_j) \right\| = \left\| x_i - \sum_{j=1}^n p_j x_j \right\| = \|x_i - \overline{x_p}\|.$$

Therefore

$$\sum_{i=1}^n p_i \sum_{j=1}^n p_j \|x_i - x_j\| \geq \sum_{i=1}^n p_i \|x_i - \overline{x_p}\| \tag{3.4}$$

and since

$$\begin{aligned} 2^{s-1}\left(\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|\right)^s &= 2^{s-1}\left(\frac{1}{2}\sum_{i=1}^n p_i \sum_{j=1}^n p_j \|x_i - x_j\|\right)^s \\ &= \frac{1}{2}\left(\sum_{i=1}^n p_i \sum_{j=1}^n p_j \|x_i - x_j\|\right)^s \\ &\geq \frac{1}{2}\left(\sum_{i=1}^n p_i \|x_i - \overline{x_p}\|\right)^s \end{aligned}$$

and by (2.8) we derive

$$\frac{1}{2}\left(\sum_{i=1}^n p_i \|x_i - \overline{x_p}\|\right)^s \leq \sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s.$$

If

$$\overline{x} := \frac{x_1 + ... + x_n}{n}$$

denotes the gravity center of the vectors x_i, $i \in \{1, ..., n\}$, then we have the inequality

$$\sum_{1\leq i<j\leq n} \|x_i - x_j\|^s \leq \begin{cases} 2^{s-1}(n-1)\sum_{i=1}^n \|x_i - \overline{x}\|^s, & \text{if } s\geq 1, \\ \\ (n-1)\sum_{i=1}^n \|x_i - \overline{x}\|^s, & \text{if } 0<s<1. \end{cases} \tag{3.5}$$

Proposition 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in H, (i \in \{1, ..., n\})$ and assume that there exists the vectors $a, A \in H$ so that either*

$$\operatorname{Re}\langle A - x_i, x_i - a\rangle \geq 0, \text{ for } i \in \{1, ..., n\},$$

or, equivalently,

$$\left\| x_i - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|, \text{ for } i \in \{1, ..., n\}.$$

Then for any $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^{n} p_i = 1$ *one has the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2} \|A - a\|^s \sum_{k=1}^{n} p_k (1 - p_k), & s \geq 1 \\ \\ \frac{1}{2^s} \|A - a\|^s \sum_{k=1}^{n} p_k (1 - p_k), & 0 < s < 1. \end{cases} \tag{3.6}$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$ then by (3.6) we get

$$\sum_{1 \leq i < j \leq n} \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2} (n-1) \|A - a\|^s, & s \geq 1 \\ \\ \frac{1}{2^s} (n-1) \|A - a\|^s, & 0 < s < 1. \end{cases} \tag{3.7}$$

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