SOME COMPANIONS OF OSTROWSKI TYPE INEQUALITY FOR TWICE DIFFERENTIABLE $r-$CONVEX FUNCTIONS

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Abstract. We present some inequalities for $r-$convex functions which are companions of Ostrowski type inequality.

1. Introduction

The following inequality gives us upper bounds was established by Ostrowski (see [1]) for differentiable functions with bounded derivatives.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior of I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

holds for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible .

Ostrowski inequality gives us an inequality which contains the left hand side of Hermite-Hadamard inequality for $x = \frac{a+b}{2}$.

Before the definition of $r-$convex functions, let us recall the power mean $M_r(x, y; \lambda)$.([2]) The power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1-\lambda) y^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ x^\lambda y^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

Definition 1. [2] *A positive function f is $r-$convex on $[a, b]$ if*

$$f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

In [2], Gill et al. obtained the following result for $r-$convex functions.

Theorem 2. *Suppose f is a positive $r-$convex function on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(u)du \leq L_r(f(a), f(b)). \tag{1.1}$$

Key words and phrases. Ostrowski inequality, $r-$convex function, Hölder inequality.
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RGMIA Res. Rep. Coll. 23 (2020), Art. 12, 12 pp. Received 16/01/20

Here L_r is the generalized logarithmic mean of order r of positive numbers x,y is defined by

$$L_r(x,y)=\begin{cases}\frac{r}{r+1}.\frac{x^{r+1}-y^{r+1}}{x^r-y^r}, & r\neq 0,-1,\ x\neq y\\ \frac{x-y}{\ln x-\ln y}, & r=0,\ x\neq y\\ xy.\frac{\ln x-\ln y}{x-y}, & r=-1,\ x\neq y\\ x, & x=y.\end{cases}$$

There are some results for $r-$convex functions in the references [2]-[9].

In this paper, our main aim is to establish several inequalities for a companion of Ostrowski inequality for functions whose second derivatives absolute value are $r-$convex. In order to establish our results we need the following identity which is emboided in the following lemma:

Lemma 1. [10] *Let $f:[a,b]\to\mathbb{R}$ be a function such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the equality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right.\\&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right|\\=\ &\frac{1}{2(b-a)}\left[\int_a^x(t-a)^2\left|f''(t)\right|dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t)\right|dt\right.\\&\left.+\int_{a+b-x}^b(t-b)^2\left|f''(t)\right|dt\right]\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

2. Main Results

Theorem 3. *Let $f:[a,b]\to\mathbb{R}^+$ be a function such that f' is absolutely continuous on $[a,b]$, $f''\in L[a,b]$. If $\left|f''\right|^q$ is $r-$convex on $[a,b]$, we obtain the following inequality for all $x\in\left[a,\frac{a+b}{2}\right]$ and $q>1$, $\frac{1}{p}+\frac{1}{q}=1$:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right.\\&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right|\\\leq\ &\frac{1}{2(b-a)}\left\{\frac{(x-a)^3}{(2p+1)^{\frac{1}{p}}}\left[\left(L_r\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right)\right)^{\frac{1}{q}}\right.\right.\\&\left.+\left(L_r\left(\left|f''(a+b-x)\right|^q,\left|f''(b)\right|^q\right)\right)^{\frac{1}{q}}\right]\\&\left.+\frac{(a+b-2x)^3}{4(2p+1)^{\frac{1}{p}}}\left(L_r\left(\left|f''(x)\right|^q,\left|f''(a+b-x)\right|^q\right)\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.1}$$

Proof. We can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\;& \frac{1}{2(b-a)}\left[\int_a^x |t-a|^2\left|f''(t)\right|dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^2\left|f''(t)\right|dt\right. \\
&\left.+\int_{a+b-x}^b |t-b|^2\left|f''(t)\right|dt\right]
\end{aligned}\tag{2.2}$$

via Lemma 1 and property of modulus. If we use Hölder inequality in (2.2) we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\;& \frac{1}{2(b-a)}\left[\left(\int_a^x |t-a|^{2p}dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f''(t)\right|^q dt\right)^{\frac{1}{q}}+\right. \\
&+\left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^{2p}dt\right)\left(\int_x^{a+b-x}\left|f''(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{a+b-x}^b |t-b|^{2p}dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b \left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}\tag{2.3}$$

Since $|f''|^q$ is $r-$convex on $[a,b]$, we have

$$\frac{1}{(x-a)}\int_a^x \left|f''(t)\right|^q dt\leq L_r\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right),\tag{2.4}$$

$$\frac{1}{a+b-x}\int_x^{a+b-x}\left|f''(t)\right|^q dt\leq L_r\left(\left|f''(a+b-x)\right|^q,\left|f''(b)\right|^q\right)\tag{2.5}$$

and

$$\frac{1}{x-a}\int_{a+b-x}^b \left|f''(t)\right|^q dt\leq L_r\left(\left|f''(x)\right|^q,\left|f''(a+b-x)\right|^q\right)\tag{2.6}$$

via the inequality in (1.1). If we use (2.4)-(2.6) in (2.2) and calculate the integrals in (2.2), we obtain the inequality in (2.1). □

Corollary 1. *In Theorem 3, if $f'(x)=f'(a+b-x)$ we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left\{\frac{(x-a)^3}{(2p+1)^{\frac{1}{p}}}\left[\left(L_r\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right)\right)^{\frac{1}{q}}\right.\right. \\
& \left.+\left(L_r\left(\left|f''(a+b-x)\right|^q,\left|f''(b)\right|^q\right)\right)^{\frac{1}{q}}\right] \\
& \left.+\frac{(a+b-2x)^3}{4\left(2p+1\right)^{\frac{1}{p}}}\left(L_r\left(\left|f''(x)\right|^q,\left|f''(a+b-x)\right|^q\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 2. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Corollary 1, we have the following Ostrowski type inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)\right| \\
\leq\ & \frac{1}{2(b-a)}\left\{\frac{(x-a)^3}{(2p+1)^{\frac{1}{p}}}\left[\left(L_r\left(\left|f''(a)\right|^q,\left|f''(x)\right|^q\right)\right)^{\frac{1}{q}}\right.\right. \\
& \left.+\left(L_r\left(\left|f''(a+b-x)\right|^q,\left|f''(b)\right|^q\right)\right)^{\frac{1}{q}}\right] \\
& \left.+\frac{(a+b-2x)^3}{4\left(2p+1\right)^{\frac{1}{p}}}\left(L_r\left(\left|f''(x)\right|^q,\left|f''(a+b-x)\right|^q\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 4. *Let $f:[a,b]\to\mathbb{R}^+$ be a function such that f' is absolutely continuous on $[a,b]$, $f''\in L[a,b]$. If $|f''|$ is $r-$convex on $[a,b]$, we obtain the following inequality for all $x\in\left[a,\frac{a+b}{2}\right]$ and $r>1$:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left[(x-a)^3\left(\frac{2r^3}{(1+r)(1+2r)(1+3r)}\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right. \\
& +\left((x-a)^3\left(\frac{r}{1+3r}\right)+(a+b-2x)^3\left(\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\right)\right) \\
& \left.\times\left(\left|f''(x)\right|+\left|f''(a+b-x)\right|\right)\right]
\end{aligned} \tag{2.7}$$

Proof. Using Lemma 1 and property of modulus, we can write

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\ &\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\ \leq\ &\frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t)\right|dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t)\right|dt\right. \\ &\left.+\int_{a+b-x}^b (t-b)^2\left|f''(t)\right|dt\right]. \end{aligned} \tag{2.8}$$

Since $|f''|$ is $r-$convex on $[a,b]$, we have

$$|f''(t)|\leq\left(\frac{t-a}{x-a}\left|f''(x)\right|^r+\frac{x-t}{x-a}\left|f''(a)\right|^r\right)^{\frac{1}{r}},\ t\in[a,x], \tag{2.9}$$

$$|f''(t)|\leq\left(\frac{t-x}{a+b-2x}\left|f''(a+b-x)\right|^r+\frac{a+b-x-t}{a+b-2x}\left|f''(x)\right|^r\right)^{\frac{1}{r}},\ t\in(x,a+b-x] \tag{2.10}$$

and

$$|f''(t)|\leq\left(\frac{t-a-b+x}{x-a}\left|f''(b)\right|^r+\frac{b-t}{x-a}\left|f''(a+b-x)\right|^r\right)^{\frac{1}{r}},\ t\in(a+b-x,b]. \tag{2.11}$$

If we use (2.9)-(2.11) in (2.8), we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\ &\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\ \leq\ &\frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left(\frac{t-a}{x-a}\left|f''(x)\right|^r+\frac{x-t}{x-a}\left|f''(a)\right|^r\right)^{\frac{1}{r}}dt\right. \\ &+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left(\frac{t-x}{a+b-2x}\left|f''(a+b-x)\right|^r\right. \\ &\left.+\frac{a+b-x-t}{a+b-2x}\left|f''(x)\right|^r\right)^{\frac{1}{r}}dt \\ &\left.+\int_{a+b-x}^b (t-b)^2\left(\frac{t-a-b+x}{x-a}\left|f''(b)\right|^r+\frac{b-t}{x-a}\left|f''(a+b-x)\right|^r\right)^{\frac{1}{r}}dt\right]. \end{aligned} \tag{2.12}$$

Using the fact that

$$\sum_{i=1}^n (x_i+y_i)^k\leq\sum_{i=1}^n x_i^k+\sum_{i=1}^n y_i^k \tag{2.13}$$

for $0<k<1, x_1, x_2, ..., x_n \geq 0$ and $y_1, y_2, ..., y_n \geq 0$, we have

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left(\left(\frac{t-a}{x-a}\right)^{\frac{1}{r}}|f''(x)|+\left(\frac{x-t}{x-a}\right)^{\frac{1}{r}}|f''(a)|\right)dt\right. \\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left(\left(\frac{t-x}{a+b-2x}\right)^{\frac{1}{r}}|f''(a+b-x)|\right. \\
&\left.+\left(\frac{a+b-x-t}{a+b-2x}\right)^{\frac{1}{r}}|f''(x)|\right)dt \\
&+\int_{a+b-x}^b (t-b)^2\left(\left(\frac{b-t}{x-a}\right)^{\frac{1}{r}}|f''(a+b-x)|\right. \\
&\left.\left.+\left(\frac{t-a-b+x}{x-a}\right)^{\frac{1}{r}}|f''(b)|\right)dt\right].
\end{aligned}
\tag{2.14}
$$

If we calculate the integrals in (2.14) we have

$$
\begin{aligned}
\int_a^x (t-a)^{2+\frac{1}{r}}dt &= \int_{a+b-x}^b (t-b)^2(b-t)^{\frac{1}{r}}dt \\
&= \frac{r(x-a)^{3+\frac{1}{r}}}{1+3r},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\int_a^x (t-a)^2(x-t)^{\frac{1}{r}}dt &= \int_{a+b-x}^b (t-b)^2(t-a-b+x)^{\frac{1}{r}}dt \\
&= \frac{2r^3(x-a)^{3+\frac{1}{r}}}{(1+r)(1+2r)(1+3r)}
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2(t-x)^{\frac{1}{r}}dt \\
=\ &\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2(a+b-x-t)^{\frac{1}{r}}dt \\
=\ &\frac{(a+b-2x)^{3+\frac{1}{r}}(r+r^2+2r^3)}{4(1+r)(1+2r)(1+3r)}.
\end{aligned}
\tag{2.17}
$$

If we use (2.15)-(2.17) in (2.14), we obtain the inequality in (2.7). □

Corollary 3. *In Theorem 4, if $f'(x)=f'(a+b-x)$ we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left[(x-a)^3\left(\frac{2r^3}{(1+r)(1+2r)(1+3r)}\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right. \\
&+\left((x-a)^3\left(\frac{r}{1+3r}\right)+(a+b-2x)^3\left(\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\right)\right) \\
&\left.\times\left(\left|f''(x)\right|+\left|f''(a+b-x)\right|\right)\right].
\end{aligned}$$

Corollary 4. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Corollary 3, we have the following Ostrowski type inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)\right| \\
\leq\ & \frac{1}{2(b-a)}\left[(x-a)^3\left(\frac{2r^3}{(1+r)(1+2r)(1+3r)}\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\right. \\
&+\left((x-a)^3\left(\frac{r}{1+3r}\right)+(a+b-2x)^3\left(\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\right)\right) \\
&\left.\times\left(\left|f''(x)\right|+\left|f''(a+b-x)\right|\right)\right].
\end{aligned}$$

Theorem 5. *Under the assumptions of Theorem 3, for $r>1$ we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{(x-a)^3\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&+\frac{(a+b-2x)^3}{4}\left[\left|f''(x)\right|^q+\left|f''(a+b-x)\right|^q\right]^{\frac{1}{q}} \\
&\left.+(x-a)^3\left[\left|f''(a+b-x)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}\tag{2.18}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we obtain the inequality in (2.3). If we use $r-$convexity of $\left|f''\right|^q$, we can write

$$\int_a^x \left|f''(t)\right|^q dt\leq\frac{r(x-a)}{1+r}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right],\tag{2.19}$$

$$\int_x^{a+b-x} \left|f''(t)\right|^q dt\leq\frac{r(a+b-2x)}{1+r}\left[\left|f''(x)\right|^q+\left|f''(a+b-x)\right|^q\right]\tag{2.20}$$

and

$$\int_{a+b-x}^b \left|f''(t)\right|^q dt\leq\frac{r(x-a)}{1+r}\left[\left|f''(a+b-x)\right|^q+\left|f''(b)\right|^q\right]\tag{2.21}$$

via inequality in (2.13). If we use (2.19)-(2.21) in (2.2) and calculate the integrals in (2.2), we obtain the inequality in (2.18). □

Corollary 5. *In Theorem 5, if $f'(x) = f'(a+b-x)$ we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{(x-a)^3\left[|f''(a)|^q+|f''(x)|^q\right]^{\frac{1}{q}}\right. \\
&+\frac{(a+b-2x)^3}{4}\left[|f''(x)|^q+|f''(a+b-x)|^q\right]^{\frac{1}{q}} \\
&\left.+(x-a)^3\left[|f''(a+b-x)|^q+|f''(b)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 6. *If $f(x) = f(a+b-x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ in Corollary 5, we have the following Ostrowski type inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right| \\
\leq\ & \frac{1}{2(b-a)}\left(\frac{r}{r+1}\right)^{\frac{1}{q}}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \\
&\times\left\{(x-a)^3\left[|f''(a)|^q+|f''(b)|^q\right]^{\frac{1}{q}}\right. \\
&+\frac{(a+b-2x)^3}{4}\left[|f''(x)|^q+|f''(a+b-x)|^q\right]^{\frac{1}{q}} \\
&\left.+(x-a)^3\left[|f''(a+b-x)|^q+|f''(b)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 6. *Under the assumptions of Theorem 3, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\ & \frac{1}{2(b-a)} \\
&\times\left\{\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{2r^3}{(1+r)(1+2r)(1+3r)}|f''(a)|^q+\frac{r}{1+3r}|f''(x)|^q\right]^{\frac{1}{q}}\right. \\
&+\frac{(a+b-2x)^3}{12^{\frac{1}{p}}}\left[\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\left(|f''(x)|^q+|f''(a+b-x)|^q\right)\right]^{\frac{1}{q}} \\
&\left.+\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{r}{1+3r}|f''(a+b-x)|^q+\frac{2r^3}{(1+r)(1+2r)(1+3r)}|f''(b)|^q\right]\right\}.
\end{aligned} \tag{2.22}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we obtain

$$\begin{aligned}
(2.23)\quad & \left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right. \\
& \left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right| \\
\leq\; & \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)^2\,dt\right)^{\frac{1}{p}}\left(\int_a^x (t-a)^2\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
& +\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right)^{\frac{1}{p}}\left(\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t)\right|^q dt\right)^{\frac{1}{q}} \\
& \left.+\left(\int_{a+b-x}^b (t-b)^2\,dt\right)^{\frac{1}{p}}\left(\int_{a+b-x}^b (t-b)^2\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we use $r-$convexity of $|f''|^q$, we can write

(2.24)
$$\int_a^x (t-a)^2\left|f''(t)\right|^q dt\leq (x-a)^3\left[\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(a)\right|^q+\frac{r}{1+3r}\left|f''(x)\right|^q\right],$$

(2.25)
$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t)\right|^q dt\leq (a+b-2x)^3\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\left[\left|f''(x)\right|^q+\left|f''(a+b-x)\right|^q\right]$$

and

(2.26)
$$\int_{a+b-x}^b (t-b)^2\left|f''(t)\right|^q dt\leq (x-a)^3\left[\frac{r}{1+3r}\left|f''(a+b-x)\right|^q\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(b)\right|^q\right]$$

via inequality in (2.13). If we use (2.24)-(2.26) in (2.23) and calculate the integrals in (2.23), we obtain the inequality in (2.22). □

Corollary 7. *In Theorem 6, if $f'(x)=f'(a+b-x)$ we have*

$$\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right| \\
\leq\; & \frac{1}{2(b-a)} \\
& \times\left\{\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(a)\right|^q+\frac{r}{1+3r}\left|f''(x)\right|^q\right]^{\frac{1}{q}}\right. \\
& +\frac{(a+b-2x)^3}{12^{\frac{1}{p}}}\left[\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\left(\left|f''(x)\right|^q+\left|f''(a+b-x)\right|^q\right)\right]^{\frac{1}{q}} \\
& \left.+\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{r}{1+3r}\left|f''(a+b-x)\right|^q+\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(b)\right|^q\right]\right\}.
\end{aligned}$$

Corollary 8. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Corollary 7, we have the following Ostrowski type inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)\right|\\
\leq\ &\frac{1}{2(b-a)}\\
&\times\left\{\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(a)\right|^q+\frac{r}{1+3r}\left|f''(x)\right|^q\right]^{\frac{1}{q}}\right.\\
&+\frac{(a+b-2x)^3}{12^{\frac{1}{p}}}\left[\frac{r+r^2+2r^3}{4(1+r)(1+2r)(1+3r)}\left(\left|f''(x)\right|^q+\left|f''(a+b-x)\right|^q\right)\right]^{\frac{1}{q}}\\
&\left.+\frac{(x-a)^3}{3^{\frac{1}{p}}}\left[\frac{r}{1+3r}\left|f''(a+b-x)\right|^q+\frac{2r^3}{(1+r)(1+2r)(1+3r)}\left|f''(b)\right|^q\right]\right\}.
\end{aligned}$$

Theorem 7. *Under the assumptions of Theorem 3, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right.\\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right|\\
\leq\ &\frac{1}{2(b-a)^{1-\frac{1}{q}}}\\
&\times\left(\frac{2\,(x-a)^{2p+1}}{2p+1}+\frac{(a+b-2x)^{2p+1}}{2^{2p}(2p+1)}\right)^{\frac{1}{p}}\\
&\times\left(L_r\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.27}$$

Proof. If we use Lemma 1, property of modulus and Hölder inequality we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f(x)+f(a+b-x)\right]\right.\\
&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)+f'(a+b-x)\right]\right|\\
\leq\ &\frac{1}{2(b-a)}\left(\int_a^x |t-a|^{2p}\,dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^{2p}dt\right.\\
&\left.+\int_{a+b-x}^b |t-b|^{2p}\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b \left|f''(t)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.28}$$

Since $|f''|^q$ is $r-$convex on $[a,b]$ we can write

$$\frac{1}{b-a}\int_a^b |f''(t)|^q\,dt \leq L_r\left(|f''(a)|^q, |f''(b)|^q\right) \tag{2.29}$$

via the inequality in (1.1). If we use (2.29) in (2.28) and calculate the integrals we obtain the inequality in (2.27). □

Corollary 9. *In Theorem 7, if $f'(x)=f'(a+b-x)$ we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right]\right| \\ \leq\ & \frac{1}{2(b-a)^{1-\frac{1}{q}}} \\ &\times\left(\frac{2\left(x-a\right)^{2p+1}}{2p+1}+\frac{\left(a+b-2x\right)^{2p+1}}{2^{2p}(2p+1)}\right)^{\frac{1}{p}} \\ &\times\left(L_r\left(|f''(a)|^q, |f''(b)|^q\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 10. *If $f(x)=f(a+b-x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$ in Corollary 9, we have the following Ostrowski type inequality*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x)\right| \\ \leq\ & \frac{1}{2(b-a)^{1-\frac{1}{q}}} \\ &\times\left(\frac{2\left(x-a\right)^{2p+1}}{2p+1}+\frac{\left(a+b-2x\right)^{2p+1}}{2^{2p}(2p+1)}\right)^{\frac{1}{p}} \\ &\times\left(L_r\left(|f''(a)|^q, |f''(b)|^q\right)\right)^{\frac{1}{q}}. \end{aligned}$$

References

[1] A. Ostrowski, Über die absolutabweichung einer differentiebaren function von ihren integralmittelwert. Commentarii Mathematici Helvetici, 10(1938) 226-227.

[2] P. M. Gill, C. E. M. Pearce and J. Pečarić, Hadamard's inequality for $r-$convex functions.Journal of Mathematical Analysis and Applications, 215 (1997) 461-470.

[3] M. W. N. Alomari, Several Inequalities of Hermite-Hadamard, Ostrowski and Simpson type for $s-$convex, quasi-convex and $r-$convex mappings and applications. Phd. Thesis, Faculty of Science and Technology, Universiti Kebangsaan, Malaysia 2011.

[4] F. Chen and X. Liu, Refinements on the Hermite-Hadamard Inequalities for$r-$Convex Functions. J. Appl. Math., vol. 2013, no. 1-2, p. 5, 2013, doi: 10.1155/2013/978493.

[5] S. Dragomir and C. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Its Applications. Victoria University: RGMIA Monograph, 2002.

[6] L. Han and L. Liu, Integral Inequalities of Hermite-Hadamard Type for $r-$Convex Functions. Appl. Math., no. 3, pp. 1967–1971, 2012, doi: 10.4236/am.2012.312270.

[7] N. Ngoc, N. Vinh, and P. Hien, Integral inequalities of Hadamard type for $r-$convex functions. Int. Math. Forum, vol. 4, no. 35, pp. 1723–1728, 2009.

[8] G. Yang and D. Hwang, Refinements of Hadamard inequality for $r-$convex functions. Indian J.Pure Appl.Math., vol. 32, no. 10, pp. 1571–1579, 2001.

[9] G. Zabandan, A. Bodagh, and A. Kilicman, "The Hermite-Hadamard inequality for $r-$convex functions." Journal of Inequalities and Applications, vol. 2012, no. 1, p. 215, 2012.

[10] Z. Liu, Some companions of an Ostrowski type inequality and applications. JIPAM vol.10, Issue 2, 2009.

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