UPPER AND LOWER BOUNDS FOR NONCOMMUTATIVE PERSPECTIVES OF OPERATOR MONOTONE FUNCTIONS: THE CASE OF FIRST VARIABLE

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Abstract. Assume that the function $f:(0,\infty)\rightarrow\mathbb{R}$ is operator monotone in $(0,\infty)$. We can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$. In this paper we show among others that, if $\beta\geq A\geq\alpha>0$, $B>0$, $\delta\geq P\geq\gamma>0$ and $0<m\leq B-A\leq M$ for some constants α, β, γ, δ, m, M, then

$$\begin{aligned}0&\leq\frac{m}{M}\left[\mathcal{P}_f(M+\beta,\delta)-\mathcal{P}_f(\beta,\delta)\right]\frac{P^2}{\delta^2}\\&\leq\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\leq\frac{M}{m}\left[\mathcal{P}_f(m+\alpha,\gamma)-\mathcal{P}_f(\alpha,\gamma)\right]\frac{P^2}{\gamma^2}.\end{aligned}$$

Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq0$) if $\langle Tx,x\rangle\geq0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\rightarrow\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

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which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

In 1934, K. Löwner [11] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} dw(\lambda) \tag{1.3}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure w on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [10]. The function ln is also operator monotone on $(0,\infty)$.

For other examples of operator monotone functions, see [8] and [9].

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset\mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [13]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A). \tag{1.4}$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [13].

The following inequality is of interest, see [13]:

Theorem 2. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_f(A,B) \geq \mathcal{P}_f(C,D). \tag{1.5}$$

It is well known that (see [4] and [3] or [5]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r : [0,\infty) \to [0,\infty),\ f_r(t) = t^r,\ r \in [0,1]$, then

$$P_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

We define the *weighted operator arithmetic mean* by

$$A\nabla_r B := (1-r)A + rB,\ r \in [0,1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_r B \le A\nabla_r B$$

for any $r \in [0,1]$.

If we take the function $f = \ln$, then

$$P_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [12].

In this paper we show among others that, if $\beta \ge A \ge \alpha > 0,\ B > 0,\ \delta \ge P \ge \gamma > 0$ and $0 < m \le B - A \le M$ for some constants $\alpha,\ \beta,\ \gamma,\ \delta,\ m,\ M$, then

$$\begin{aligned} 0 &\le \frac{m}{M}\left[\mathcal{P}_f(M+\beta,\delta) - \mathcal{P}_f(\beta,\delta)\right]\frac{P^2}{\delta^2} \\ &\le \mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) \le \frac{M}{m}\left[\mathcal{P}_f(m+\alpha,\gamma) - \mathcal{P}_f(\alpha,\gamma)\right]\frac{P^2}{\gamma^2}. \end{aligned}$$

Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

2. Some Preliminary Facts

We start to the following identity of interest [2]:

Lemma 1. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all $U,\ V > 0$ we have*

$$\begin{aligned} f(V) - f(U) &= b(V-U) \\ &\quad + \int_0^\infty \lambda^2\left[\int_0^1 ((1-t)U + tV + \lambda)^{-1}\right. \\ &\quad \left.\times (V-U)((1-t)U + tV + \lambda)^{-1}\,dt\right]dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. Since the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3), then for $U,\ V > 0$ we have the representation

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda\left[V(V+\lambda)^{-1} - U(U+\lambda)^{-1}\right]dw(\lambda). \tag{2.2}$$

Observe that for $\lambda > 0$

$$\begin{aligned} & V\left(V+\lambda\right)^{-1}-U\left(U+\lambda\right)^{-1} \\ & =\left(V+\lambda-\lambda\right)\left(V+\lambda\right)^{-1}-\left(U+\lambda-\lambda\right)\left(U+\lambda\right)^{-1} \\ & =\left(V+\lambda\right)\left(V+\lambda\right)^{-1}-\lambda\left(V+\lambda\right)^{-1}-\left(U+\lambda\right)\left(U+\lambda\right)^{-1}+\lambda\left(U+\lambda\right)^{-1} \\ & =\lambda\left[\left(U+\lambda\right)^{-1}-\left(V+\lambda\right)^{-1}\right]. \end{aligned}$$

Therefore, (2.2) becomes, see also [9]

$$f\left(V\right)-f\left(U\right)=b\left(V-U\right)+\int_{0}^{\infty}\lambda^{2}\left[\left(U+\lambda\right)^{-1}-\left(V+\lambda\right)^{-1}\right]dw\left(\lambda\right). \tag{2.3}$$

Let T, $S>0$. The function $f\left(t\right)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_{T}\left(S\right):=\lim_{t\rightarrow0}\left[\frac{f\left(T+tS\right)-f\left(T\right)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}\left(t\right)=f\left(\left(1-t\right)C+tD\right),\ t\in\left[0,1\right].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\left\{\left(1-t\right)C+tD,\ t\in\left[0,1\right]\right\}$, then we have, by the properties of the Bochner integral, that

$$f\left(D\right)-f\left(C\right)=\int_{0}^{1}\frac{d}{dt}\left(f_{C,D}\left(t\right)\right)dt=\int_{0}^{1}\nabla f_{\left(1-t\right)C+tD}\left(D-C\right)dt. \tag{2.5}$$

If we write this equality for the function $f\left(t\right)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_{0}^{1}\left(\left(1-t\right)C+tD\right)^{-1}\left(D-C\right)\left(\left(1-t\right)C+tD\right)^{-1}dt. \tag{2.6}$$

Now, if we replace in (2.6) $C=U+\lambda$ and $D=V+\lambda$ for $\lambda>0$, then

$$\begin{aligned} & \left(U+\lambda\right)^{-1}-\left(V+\lambda\right)^{-1} \\ & =\int_{0}^{1}\left(\left(1-t\right)U+tV+\lambda\right)^{-1}\left(V-U\right)\left(\left(1-t\right)U+tV+\lambda\right)^{-1}dt. \end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

The following representation for the difference in the first variable of the perspective may be stated [2]:

Theorem 3. *Assume that the function $f:(0,\infty)\rightarrow\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all A, B, $P>0$ we have*

$$\begin{aligned} & \mathcal{P}_{f}\left(B,P\right)-\mathcal{P}_{f}\left(A,P\right) \\ & =b\left(B-A\right)+\int_{0}^{\infty}\lambda^{2}\left[\int_{0}^{1}P\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\right. \\ & \left.\times\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}Pdt\right]dw\left(\lambda\right). \end{aligned} \tag{2.8}$$

Proof. If we take $V = P^{-1/2}BP^{-1/2}$ and $U = P^{-1/2}AP^{-1/2}$ in (2.1), then we get

$$\begin{aligned}
&f\left(P^{-1/2}BP^{-1/2}\right) - f\left(P^{-1/2}AP^{-1/2}\right) \\
&= b\left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\
&+ \int_0^\infty \lambda^2 \left[\int_0^1 \left((1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1}\right. \\
&\times \left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\
&\left.\times \left((1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1} dt\right] dw(\lambda).
\end{aligned} \tag{2.9}$$

Observe that

$$P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2} = P^{-1/2}(B-A)P^{-1/2},$$

and

$$(1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda = P^{-1/2}((1-t)A + tB + \lambda P)P^{-1/2},$$

which gives

$$\begin{aligned}
&\left((1-t)P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1} \\
&= P^{1/2}((1-t)A + tB + \lambda P)^{-1}P^{1/2}
\end{aligned}$$

and by (2.9),

$$\begin{aligned}
&f\left(P^{-1/2}BP^{-1/2}\right) - f\left(P^{-1/2}AP^{-1/2}\right) \\
&= bP^{-1/2}(B-A)P^{-1/2} \\
&+ \int_0^\infty \lambda^2 \left[\int_0^1 P^{1/2}((1-t)A + tB + \lambda P)^{-1} P^{1/2}P^{-1/2}(B-A)P^{-1/2}\right. \\
&\left.\times P^{1/2}((1-t)A + tB + \lambda P)^{-1}P^{1/2}dt\right] dw(\lambda) \\
&= bP^{-1/2}(B-A)P^{-1/2} \\
&+ \int_0^\infty \lambda^2 \left[\int_0^1 P^{1/2}((1-t)A + tB + \lambda P)^{-1}(B-A)\right. \\
&\left.\times ((1-t)A + tB + \lambda P)^{-1}P^{1/2}dt\right] dw(\lambda).
\end{aligned} \tag{2.10}$$

If we multiply both sides of (2.10) by $P^{1/2}$ we obtain the desired identity (2.8). □

We can state the following identity for the *weighted operator geometric mean:*

Proposition 1. *For all A, B, $P > 0$ and $r \in (0,1]$ we have*

$$\begin{aligned}
&P\sharp_r B - P\sharp_r A \\
&= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r+1}\left[\int_0^1 P((1-t)A + tB + \lambda P)^{-1}(B-A)\right. \\
&\left.\times ((1-t)A + tB + \lambda P)^{-1}Pdt\right] d\lambda.
\end{aligned} \tag{2.11}$$

The proof follows by (2.8) and (1.1) for the measure $dw(\lambda) = \frac{\sin(r\pi)}{\pi}\lambda^{r-1}d\lambda$.
The following identity for the logarithmic function also holds [2]:

Lemma 2. *For all U, $V > 0$ we have the identity:*

$$\ln V - \ln U = \int_0^\infty \left(\int_0^1 (\lambda + (1-t)U + tV)^{-1} (V - U) (\lambda + (1-t)U + tV)^{-1} dt \right) d\lambda. \tag{2.12}$$

Proof. We have from the representation of logarithm (1.2) that

$$\ln V - \ln U = \int_0^\infty \frac{1}{\lambda+1} \left[(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \right] d\lambda \tag{2.13}$$

for U, $V > 0$.

Since

$$\begin{aligned} &(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \\ &= V(\lambda+V)^{-1} - U(\lambda+U)^{-1} - \left((\lambda+V)^{-1} - (\lambda+U)^{-1} \right) \end{aligned}$$

and

$$\begin{aligned} &V(\lambda+V)^{-1} - U(\lambda+U)^{-1} \\ &= (V+\lambda-\lambda)(\lambda+V)^{-1} - (U+\lambda-\lambda)(\lambda+U)^{-1} \\ &= 1 - \lambda(\lambda+V)^{-1} - 1 + \lambda(\lambda+U)^{-1} = \lambda(\lambda+U)^{-1} - \lambda(\lambda+V)^{-1}, \end{aligned}$$

hence

$$\begin{aligned} &(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \\ &= \lambda(\lambda+U)^{-1} - \lambda(\lambda+V)^{-1} - \left((\lambda+V)^{-1} - (\lambda+U)^{-1} \right) \\ &= (\lambda+1)\left[(\lambda+U)^{-1} - (\lambda+V)^{-1} \right] \end{aligned}$$

and by (2.13) we get

$$\ln V - \ln U = \int_0^\infty \left[(\lambda+U)^{-1} - (\lambda+V)^{-1} \right] d\lambda. \tag{2.14}$$

Since, by (2.6) we have

$$\begin{aligned} &(\lambda+U)^{-1} - (\lambda+V)^{-1} \\ &= \int_0^1 (\lambda + (1-t)U + tV)^{-1} (V-U) (\lambda + (1-t)U + tV)^{-1} dt, \end{aligned} \tag{2.15}$$

for all $\lambda \geq 0$, hence by (2.14) and (2.15) we get (2.12). □

The case of *relative operator entropy* is as follows:

Proposition 2. *For all A, B, $P > 0$ we have*

$$\begin{aligned} S(P|B) - S(P|A) = \int_0^\infty \Bigg[\int_0^1 & P((1-t)A + tB + \lambda P)^{-1} (B-A) \\ & \times ((1-t)A + tB + \lambda P)^{-1} P dt \Bigg] d\lambda. \end{aligned} \tag{2.16}$$

Proof. Follows by Lemma 2 by taking $V = P^{-1/2}BP^{-1/2}$ and $U = P^{-1/2}AP^{-1/2}$ and multiplying both sides by $P^{1/2}$. □

3. Upper and Lower Bounds

We have the following upper and lower bounds for the difference of perspectives in the first variable:

Theorem 4. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.3). If* $\beta\geq A\geq\alpha>0$, $B>0$, $\delta\geq P\geq\gamma>0$ *and* $0<m\leq B-A\leq M$ *for some constants* α, β, γ, δ, m, M, *then*

$$0\leq m\left[\frac{\mathcal{P}_f\left(M+\beta,\delta\right)-\mathcal{P}_f\left(\beta,\delta\right)-bM}{\delta^2M}\right]P^2 \tag{3.1}$$
$$\leq\mathcal{P}_f\left(B,P\right)-\mathcal{P}_f\left(A,P\right)-b\left(B-A\right)$$
$$\leq M\left[\frac{\mathcal{P}_f\left(m+\alpha,\gamma\right)-\mathcal{P}_f\left(\alpha,\gamma\right)-bm}{\gamma^2m}\right]P^2.$$

Proof. Since $m\leq B-A\leq M$, then by multiplying both sides by $\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}$ we get

$$m\left(\left(1-t\right)A+tB+\lambda P\right)^{-2} \tag{3.2}$$
$$\leq\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}$$
$$\leq M\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}$$

for all $\lambda\geq0$ and $t\in[0,1]$.

If we multiply the inequality (3.2) both sides by P we obtain

$$mP\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}P \tag{3.3}$$
$$\leq P\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P$$
$$\leq MP\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}P.$$

Now, observe that

$$\left(1-t\right)A+tB+\lambda P=A+t\left(B-A\right)+\lambda P$$

for all $\lambda\geq0$ and $t\in[0,1]$. Then we get the double inequality

$$\alpha+tm+\lambda\gamma\leq A+t\left(B-A\right)+\lambda P\leq\beta+tM+\lambda\delta,$$

namely

$$\left(\beta+tM+\lambda\delta\right)^{-1}\leq\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\leq\left(\alpha+tm+\lambda\gamma\right)^{-1},$$

which implies that

$$\left(\beta+tM+\lambda\delta\right)^{-2}\leq\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}\leq\left(\alpha+tm+\lambda\gamma\right)^{-2}$$

for all $\lambda\geq0$ and $t\in[0,1]$.

This implies that

$$mP\left(\beta+tM+\lambda\delta\right)^{-2}P\leq mP\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}P$$

and

$$MP\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}P\leq MP\left(\alpha+tm+\lambda\gamma\right)^{-2}P$$

for all $\lambda\geq0$ and $t\in[0,1]$.

By utilising (3.3) we derive

$$mP(\beta+tM+\lambda\delta)^{-2}P \le P((1-t)A+tB+\lambda P)^{-1}(B-A)((1-t)A+tB+\lambda P)^{-1}P \le MP(\alpha+tm+\lambda\gamma)^{-2}P \tag{3.4}$$

for all $\lambda\ge 0$ and $t\in[0,1]$.

If we integrate (3.4) over t on $[0,1]$, multiply by λ^2 and integrate over the measure $w(\lambda)$ on $[0,\infty)$, then we get

$$\begin{aligned} 0 &\le mP\left[\int_0^\infty \lambda^2\left(\int_0^1(\beta+tM+\lambda\delta)^{-2}dt\right)dw(\lambda)\right]P \\ &\le \int_0^\infty \lambda^2\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right. \\ &\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda) \\ &\le M\left[P\int_0^\infty \lambda^2\left(\int_0^1(\alpha+tm+\lambda\gamma)^{-2}dt\right)dw(\lambda)\right]P, \end{aligned}$$

namely, by the identity (2.8),

$$\begin{aligned} 0 &\le mP\left[\int_0^\infty \lambda^2\left(\int_0^1(\beta+tM+\lambda\delta)^{-2}dt\right)dw(\lambda)\right]P \\ &\le \mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)-b(B-A) \\ &\le MP\left[\int_0^\infty \lambda^2\left(\int_0^1(\alpha+tm+\lambda\gamma)^{-2}dt\right)dw(\lambda)\right]P. \end{aligned} \tag{3.5}$$

Now, observe that

$$\begin{aligned} &\int_0^\infty \lambda^2\left(\int_0^1(\beta+tM+\lambda\delta)^{-2}dt\right)dw(\lambda) \\ &=\int_0^\infty \lambda^2\left(\int_0^1((1-t)\beta+t(M+\beta)+\lambda\delta)^{-2}dt\right)dw(\lambda). \end{aligned}$$

By the identity (2.8) for $A=\beta$, $B=M+\beta$ and $P=\delta$ we have

$$\begin{aligned} &\mathcal{P}_f(M+\beta,\delta)-\mathcal{P}_f(\beta,\delta) \\ &=bM+\int_0^\infty\lambda^2\left[\int_0^1\delta((1-t)\beta+t(M+\beta)+\lambda\delta)^{-1}M\right. \\ &\left.\times((1-t)\beta+t(M+\beta)+\lambda\delta)^{-1}\delta dt\right]dw(\lambda) \\ &=bM+\delta^2M\int_0^\infty\lambda^2\left[\int_0^1((1-t)\beta+t(M+\beta)+\lambda\delta)^{-2}dt\right]dw(\lambda), \end{aligned}$$

which gives

$$\begin{aligned} &\int_0^\infty\lambda^2\left[\int_0^1((1-t)\beta+t(M+\beta)+\lambda\delta)^{-2}dt\right]dw(\lambda) \\ &=\frac{\mathcal{P}_f(M+\beta,\delta)-\mathcal{P}_f(\beta,\delta)-bM}{\delta^2M}. \end{aligned} \tag{3.6}$$

Observe also that

$$\int_0^\infty \lambda^2 \left(\int_0^1 (\alpha + tm + \lambda\gamma)^{-2} dt \right) dw(\lambda)$$
$$\int_0^\infty \lambda^2 \left(\int_0^1 ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-2} dt \right) dw(\lambda).$$

By the identity (2.8) for $A = \alpha$, $B = m + \alpha$ and $P = \gamma$ we have

$$\begin{aligned}
&\mathcal{P}_f(m+\alpha,\gamma) - \mathcal{P}_f(\alpha,\gamma) \\
&= bm + \int_0^\infty \lambda^2 \left[\int_0^1 \gamma ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} (m) \right. \\
&\left. \times ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} \gamma dt \right] dw(\lambda) \\
&= bm + \gamma^2 m \int_0^\infty \lambda^2 \left[\int_0^1 ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} \right. \\
&\left. \times ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} dt \right] dw(\lambda),
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_0^\infty \lambda^2 \left[\int_0^1 ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} \right. \\
&\left. \times ((1-t)\alpha + t(m+\alpha) + \lambda\gamma)^{-1} dt \right] dw(\lambda) \\
&= \frac{\mathcal{P}_f(m+\alpha,\gamma) - \mathcal{P}_f(\alpha,\gamma) - bm}{\gamma^2 m}.
\end{aligned} \tag{3.7}$$

By utilising (3.5)-(3.7) we derive (3.1). □

If the integral representation for the operator monotone function is not available, then we can state the following result as well:

Corollary 1. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta \geq A \geq \alpha > 0$, $B > 0$, $\delta \geq P \geq \gamma > 0$ and $0 < m \leq B - A \leq M$ for some constants α, β, γ, δ, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m}{M} \left[\mathcal{P}_f(M+\beta,\delta) - \mathcal{P}_f(\beta,\delta) \right] \frac{P^2}{\delta^2} \\
&\leq \mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) \leq \frac{M}{m} \left[\mathcal{P}_f(m+\alpha,\gamma) - \mathcal{P}_f(\alpha,\gamma) \right] \frac{P^2}{\gamma^2}.
\end{aligned} \tag{3.8}$$

Proof. From (3.1) we get

$$\begin{aligned}
&m \left[\frac{\mathcal{P}_f(M+\beta,\delta) - \mathcal{P}_f(\beta,\delta) - bM}{\delta^2 M} \right] P^2 + b(B-A) \\
&\leq \mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) \\
&\leq M \left[\frac{\mathcal{P}_f(m+\alpha,\gamma) - \mathcal{P}_f(\alpha,\gamma) - bm}{\gamma^2 m} \right] P^2 + b(B-A),
\end{aligned}$$

namely

$$\begin{aligned} &\frac{m}{M}\left[\mathcal{P}_f\left(M+\beta,\delta\right)-\mathcal{P}_f\left(\beta,\delta\right)\right]\frac{P^2}{\delta^2}+b\left(B-A\right)-\frac{bm}{\delta^2}P^2 \\ &\leq \mathcal{P}_f\left(B,P\right)-\mathcal{P}_f\left(A,P\right)-b\left(B-A\right) \\ &\leq \frac{M}{m}\left[\mathcal{P}_f\left(m+\alpha,\gamma\right)-\mathcal{P}_f\left(\alpha,\gamma\right)\right]\frac{P^2}{\gamma^2}+b\left(B-A\right)-\frac{bM}{\gamma^2}P^2. \end{aligned} \tag{3.9}$$

Observe that

$$b\left(B-A\right)-\frac{bm}{\delta^2}P^2=b\left(B-A-m\frac{P^2}{\delta^2}\right).$$

Since $b\geq 0$, $1\geq \frac{P^2}{\delta^2}$ and $B-A\geq m$, hence

$$b\left(B-A\right)-\frac{bm}{\delta^2}P^2\geq 0$$

and the first inequality in (3.8) is proved.

Also

$$b\left(B-A\right)-\frac{bM}{\gamma^2}P^2=b\left(B-A-M\frac{P^2}{\gamma^2}\right).$$

Since $b\geq 0$, $\frac{P^2}{\gamma^2}\geq 1$ and $B-A\leq M$, hence

$$b\left(B-A-M\frac{P^2}{\gamma^2}\right)\leq 0$$

and the third inequality in (3.8) follows. □

We also have:

Theorem 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If $\beta\geq A$, $B\geq\alpha>0$, $\delta\geq P\geq\gamma>0$ and $0<m\leq B-A\leq M$ for some constants α, β, γ, δ, m, M, then*

$$\begin{aligned} 0&\leq \frac{m}{\delta^2}\left[f'\left(\frac{\beta}{\delta}\right)-b\right]P^2 \\ &\leq \mathcal{P}_f\left(B,P\right)-\mathcal{P}_f\left(A,P\right)-b\left(B-A\right)\leq \frac{M}{\gamma^2}\left[f'\left(\frac{\alpha}{\gamma}\right)-b\right]P^2. \end{aligned} \tag{3.10}$$

Proof. From the conditions $\beta\geq A$, $B\geq\alpha>0$ and $\delta\geq P\geq\gamma>0$ we have

$$\alpha+\lambda\gamma\leq\left(1-t\right)A+tB+\lambda P\leq\beta+\lambda\delta$$

for all $\lambda\geq 0$ and $t\in[0,1]$ which implies that

$$\left(\beta+\lambda\delta\right)^{-2}\leq\left(\left(1-t\right)A+tB+\lambda P\right)^{-2}\leq\left(\alpha+\lambda\gamma\right)^{-2}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

From (3.3) we derive

$$\begin{aligned} &m\left(\beta+\lambda\delta\right)^{-2}P^2 \\ &\leq P\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P \\ &\leq M\left(\alpha+\lambda\gamma\right)^{-2}P^2 \end{aligned} \tag{3.11}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we integrate (3.11) over t on $[0,1]$, multiply by λ^2 and integrate over the measure $w(\lambda)$ on $[0,\infty)$, then we get

$$\begin{aligned}
0 &\le m\left[\int_0^\infty \lambda^2 (\beta+\lambda\delta)^{-2} dw(\lambda)\right] P^2 \\
&\le \int_0^\infty \lambda^2 \left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right. \\
&\left.\times((1-t)A+tB+\lambda P)^{-1} Pdt\right] dw(\lambda) \\
&\le M\left[\int_0^\infty \lambda^2 (\alpha+\lambda\gamma)^{-2} dw(\lambda)\right] P^2,
\end{aligned}$$

namely, by the identity (2.8),

$$\begin{aligned}
0 &\le m\left[\int_0^\infty \lambda^2 (\beta+\lambda\delta)^{-2} dw(\lambda)\right] P^2 \\
&\le \mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)-b(B-A) \\
&\le M\left[\int_0^\infty \lambda^2 (\alpha+\lambda\gamma)^{-2} dw(\lambda)\right] P^2.
\end{aligned} \tag{3.12}$$

For $a,\ c>0$ we consider

$$I(a,c) := \int_0^\infty \frac{\lambda^2}{(a\lambda+c)^2} dw(\lambda) = \frac{1}{a^2}\int_0^\infty \frac{\lambda^2}{\left(\lambda+\frac{c}{a}\right)^2} dw(\lambda).$$

By taking the derivative over t in (1.3) we have

$$f'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda)-t\lambda}{(t+\lambda)^2} dw(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} dw(\lambda)$$

for $t>0$.

Therefore

$$f'\left(\frac{c}{a}\right) = b + \int_0^\infty \frac{\lambda^2}{\left(\frac{c}{a}+\lambda\right)^2} dw(\lambda),$$

namely

$$\frac{1}{a^2}\int_0^\infty \frac{\lambda^2}{\left(\frac{c}{a}+\lambda\right)^2} dw(\lambda) = \frac{1}{a^2}\left[f'\left(\frac{c}{a}\right)-b\right] = I(a,c).$$

We obtain

$$\int_0^\infty \lambda^2(\beta+\lambda\delta)^{-2} dw(\lambda) = I(\delta,\beta) = \frac{1}{\delta^2}\left[f'\left(\frac{\beta}{\delta}\right)-b\right]$$

and

$$\int_0^\infty \lambda^2(\alpha+\lambda\gamma)^{-2} dw(\lambda) = I(\gamma,\alpha) = \frac{1}{\gamma^2}\left[f'\left(\frac{\alpha}{\gamma}\right)-b\right].$$

By making use of (3.12) we derive (3.10). □

If the integral representation for the operator monotone function is not available, then we can state the following result as well:

Corollary 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A$, $B\geq\alpha>0$, $\delta\geq P\geq\gamma>0$ and $0<m\leq B-A\leq M$ for some constants α, β, γ, δ, m, M, then*

$$0\leq mf'\left(\frac{\beta}{\delta}\right)\frac{P^2}{\delta^2}\leq\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\leq Mf'\left(\frac{\alpha}{\gamma}\right)\frac{P^2}{\gamma^2}. \tag{3.13}$$

The case of separated operators is as follows:

Theorem 6. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If the positive operators satisfy the separation condition*

$$0<\alpha\leq A\leq\beta<\eta\leq B\leq\theta \tag{3.14}$$

and $\delta\geq P\geq\gamma>0$ for some positive constants α, β, γ, δ, η, θ, then

$$\begin{aligned}0&\leq m\left[\frac{\mathcal{P}_f(\theta,\delta)-\mathcal{P}_f(\beta,\delta)-b(\theta-\beta)}{\delta^2(\theta-\beta)}\right]P^2\\&\leq\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)-b(B-A)\\&\leq M\left[\frac{\mathcal{P}_f(\eta,\gamma)-\mathcal{P}_f(\alpha,\gamma)-b(\eta-\alpha)}{\gamma^2(\eta-\alpha)}\right]P^2.\end{aligned} \tag{3.15}$$

Proof. If the positive operators satisfy the separation condition (3.14), then

$$(1-t)\alpha+t\eta+\lambda\gamma\leq(1-t)A+tB+\lambda P\leq(1-t)\beta+t\theta+\lambda\delta$$

for all $\lambda\geq0$ and $t\in[0,1]$.

This implies that

$$((1-t)\beta+t\theta+\lambda\delta)^{-2}\leq((1-t)A+tB+\lambda P)^{-2}\leq((1-t)\alpha+t\eta+\lambda\gamma)^{-2}$$

for all $\lambda\geq0$ and $t\in[0,1]$.

By utilising (3.3) we get

$$\begin{aligned}&mP((1-t)\beta+t\theta+\lambda\delta)^{-2}P\\&\leq P((1-t)A+tB+\lambda P)^{-1}(B-A)((1-t)A+tB+\lambda P)^{-1}P\\&\leq MP((1-t)\alpha+t\eta+\lambda\gamma)^{-2}P.\end{aligned}$$

for all $\lambda\geq0$ and $t\in[0,1]$.

If we integrate and use the identity (2.8) we deduce that

$$\begin{aligned}0&\leq mP\left[\int_0^\infty\lambda^2\left(\int_0^1((1-t)\beta+t\theta+\lambda\delta)^{-2}dt\right)dw(\lambda)\right]P\\&\leq\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)-b(B-A)\\&\leq MP\left[\int_0^\infty\lambda^2\left(\int_0^1((1-t)\alpha+t\eta+\lambda\gamma)^{-2}dt\right)dw(\lambda)\right]P.\end{aligned} \tag{3.16}$$

From the identity (2.8) we have

$$\begin{aligned}
&\mathcal{P}_f(\eta,\gamma)-\mathcal{P}_f(\alpha,\gamma)\\
&=b(\eta-\alpha)+\int_0^\infty \lambda^2\left[\int_0^1 \gamma\left((1-t)\alpha+t\eta+\lambda\gamma\right)^{-1}(\eta-\alpha)\right.\\
&\left.\times\left((1-t)\alpha+t\eta+\lambda\gamma\right)^{-1}\gamma dt\right]dw(\lambda)\\
&=b(\eta-\alpha)+\gamma^2(\eta-\alpha)\int_0^\infty \lambda^2\left(\int_0^1\left((1-t)\alpha+t\eta+\lambda\gamma\right)^{-2}dt\right)dw(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_0^\infty \lambda^2\left(\int_0^1\left((1-t)\alpha+t\eta+\lambda\gamma\right)^{-2}dt\right)dw(\lambda)\\
&=\frac{\mathcal{P}_f(\eta,\gamma)-\mathcal{P}_f(\alpha,\gamma)-b(\eta-\alpha)}{\gamma^2(\eta-\alpha)}.
\end{aligned}$$

We also have by identity (2.8) that

$$\begin{aligned}
&\mathcal{P}_f(\theta,\delta)-\mathcal{P}_f(\beta,\delta)\\
&=b(\theta-\beta)+\delta^2(\theta-\beta)\int_0^\infty \lambda^2\left(\int_0^1\left((1-t)\beta+t\theta+\lambda\delta\right)^{-2}dt\right)dw(\lambda),
\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_0^\infty \lambda^2\left(\int_0^1\left((1-t)\beta+t\theta+\lambda\delta\right)^{-2}dt\right)dw(\lambda)\\
&=\frac{\mathcal{P}_f(\theta,\delta)-\mathcal{P}_f(\beta,\delta)-b(\theta-\beta)}{\delta^2(\theta-\beta)}.
\end{aligned}$$

By making use of (3.16) we deduce the desired result (3.15). □

Corollary 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If the operators A, B, P satisfy the assumptions of Theorem 6, then*

$$\begin{aligned}
0&\le m\left(\frac{\mathcal{P}_f(\theta,\delta)-\mathcal{P}_f(\beta,\delta)}{\theta-\beta}\right)\frac{P^2}{\delta^2}\\
&\le \mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\le M\left(\frac{\mathcal{P}_f(\eta,\gamma)-\mathcal{P}_f(\alpha,\gamma)}{\eta-\alpha}\right)\frac{P^2}{\gamma^2}.
\end{aligned}\tag{3.17}$$

4. Some Examples of Interest

Consider the power function $f_r(t)=t^r$, $t\ge 0$ and $r\in(0,1]$. Then for x, $y>0$ we have

$$\mathcal{P}_{f_r}(x,y)=x^r y^{1-r}.$$

For the logarithmic function $f=\ln$ we have

$$P_{\ln}(x,y):=y\ln\left(xy^{-1}\right)$$

for x, $y>0$.

If $\beta \geq A \geq \alpha > 0$, $B > 0$, $\delta \geq P \geq \gamma > 0$ and $0 < m \leq B - A \leq M$ for some constants α, β, γ, δ, m, M, then by (3.8) for the power function we obtain

$$0 < \frac{m\delta^{r-3}}{M}\left[(M+\beta)^r - \beta^r\right] P^2 \leq P\sharp_r B - P\sharp_r A \tag{4.1}$$
$$\leq \frac{M\gamma^{r-3}}{m}\left[(m+\alpha)^r - \alpha^r\right] P^2$$

while for the logarithmic function we get

$$0 \leq \frac{m\delta^{-1}}{M}\ln\left(\frac{M+\beta}{\beta}\right) P^2 \leq S(P|B) - S(P|A) \leq \frac{M\gamma^{-1}}{m}\ln\left(\frac{m+\gamma}{\gamma}\right) P^2. \tag{4.2}$$

If $\beta \geq A$, $B \geq \alpha > 0$, $\delta \geq P \geq \gamma > 0$ and $0 < m \leq B - A \leq M$ for some constants α, β, γ, δ, m, M, then by (3.13) for the power function we have

$$0 \leq rm\beta^{r-1}\delta^{-r-1}P^2 \leq P\sharp_r B - P\sharp_r A \leq rM\alpha^{r-1}\gamma^{-r-1}P^2 \tag{4.3}$$

and for the logarithmic function,

$$0 \leq m\delta^{-1}\beta^{-1}P^2 \leq S(P|B) - S(P|A) \leq M\alpha^{-1}\gamma^{-1}P^2. \tag{4.4}$$

If the positive operators satisfy the separation condition $0 < \alpha \leq A \leq \beta < \eta \leq B \leq \theta$ and $\delta \geq P \geq \gamma > 0$ for some positive constants α, β, γ, δ, η, θ then by the inequality (3.17) for the power function, we get

$$0 \leq m\delta^{-1-r}\left(\frac{\theta^r - \beta^r}{\theta - \beta}\right) P^2 \leq P\sharp_r B - P\sharp_r A \leq M\gamma^{-1-r}\left(\frac{\eta^r - \alpha^r}{\eta - \alpha}\right) P^2 \tag{4.5}$$

while for the logarithmic function

$$0 \leq m\delta^{-1}\left(\frac{\ln\theta - \ln\beta}{\theta - \beta}\right) P^2 \leq S(P|B) - S(P|A) \tag{4.6}$$
$$\leq M\gamma^{-1}\left(\frac{\ln\eta - \ln\alpha}{\eta - \alpha}\right) P^2.$$

References

[1] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] S. S. Dragomir, Noncommutative perspectives of operator monotone functions in Hilbert spaces, Preprint [https://arxiv.org/abs/2009.00241], Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 118.

[3] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, *Proc. Natl. Acad. Sci. USA*, **108** (2011), no. 18, 7313–7314.

[4] E. G. Effros, A matrix convexity approach to some celebrated quantum inequalities, *Proc. Natl. Acad. Sci. USA* **106** (2009), 1006–1008.

[5] E. G. Effros and F. Hansen, Noncomutative perspectives, *Ann. Funct. Anal.* **5** (2014), no. 2, 74–79.

[6] J. I. Fujii and E. Kamei, Uhlmann's interpolational method for operator means. *Math. Japon.* **34** (1989), no. 4, 541–547.

[7] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory. *Math. Japon.* **34** (1989), no. 3, 341–348.

[8] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[9] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[10] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[11] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[12] M. Nakamura and H. Umegaki, A note on the entropy for operator algebras. *Proc. Japan Acad.* **37** (1961) 149–154.

[13] I. Nikoufar and M. Shamohammadi, The converse of the Loewner–Heinz inequality via perspective, *Lin. & Multilin. Alg.*, **66** (2018), N0. 2, 243–249.

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