UPPER AND LOWER BOUNDS FOR NONCOMMUTATIVE PERSPECTIVES OF OPERATOR MONOTONE FUNCTIONS: THE CASE OF SECOND VARIABLE

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Abstract. Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. We can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$. In this paper we show among others that, if $\sigma\geq C\geq\rho>0$, $D>0$, $\varsigma\geq Q\geq\tau>0$ and $0<n\leq D-C\leq N$ for some constants ρ, σ, ς, τ, n, N, then

$$\begin{aligned}0&\leq\frac{n}{N\varsigma^2}\left[\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)\right]Q^2\\&\leq\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\\&\leq\frac{N}{n\tau^2}\left[\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)\right]Q^2.\end{aligned}$$

Applications for *weighted operator geometric mean* and the perspective

$$\mathcal{P}_{\ln(\cdot+1)}(B,A):=A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}+1\right)A^{1/2},\ A,\ B>0$$

are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq0$) if $\langle Tx,x\rangle\geq0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

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By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.2}$$

for all $t > 0$.

In 1934, K. Löwner [12] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} dw(\lambda) \tag{1.3}$$

where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure w on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} dw(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [11]. The function ln is also operator monotone on $(0, \infty)$.

For other examples of operator monotone functions, see [9] and [10].

Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B, A)$ by setting

$$\mathcal{P}_f(B, A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B, A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f : (0, \infty) \to \mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right), \; x > 0.$$

It is well known that (see for instance [14]), if $f : (0, \infty) \to \mathbb{R}$ is continuous on $(0, \infty)$, then for all A, $B > 0$,

$$\mathcal{P}_{\tilde{f}}(A, B) = \mathcal{P}_f(B, A). \tag{1.4}$$

If f is nonnegative and operator monotone on $(0, \infty)$, then $\tilde{f}$ is operator monotone on $(0, \infty)$, see [14].

The following inequality is of interest, see [14]:

Theorem 2. *Assume that f is nonnegative and operator monotone on $(0, \infty)$. If $A \geq C > 0$ and $B \geq D > 0$, then*

$$\mathcal{P}_f(A, B) \geq \mathcal{P}_f(C, D). \tag{1.5}$$

It is well known that (see [5] and [4] or [6]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A) \mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r : [0,\infty) \to [0,\infty)$, $f_r(t) = t^r$, $r \in [0,1]$, then

$$P_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

We define the *weighted operator arithmetic mean* by

$$A\nabla_r B := (1-r)A + rB,\ r \in [0,1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_r B \le A\nabla_r B$$

for any $r \in [0,1]$.

If we take the function $f = \ln$, then

$$P_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B. Kamei and Fujii [7], [8] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [13].

In the recent paper [3] we established the following representation result for the difference of perspective in the first variable:

Theorem 3. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.3). Then for all* A, B, $P>0$ *we have*

$$\begin{aligned}(1.6)\qquad &\mathcal{P}_f(B,P) - \mathcal{P}_f(A,P)\\ &= b(B-A) + \int_0^\infty \lambda^2\left[\int_0^1 P\left((1-t)A+tB+\lambda P\right)^{-1}(B-A)\right.\\ &\left.\times\left((1-t)A+tB+\lambda P\right)^{-1}Pdt\right]dw(\lambda).\end{aligned}$$

We also obtained the following upper and lower bounds for the difference of perspectives in the first variable:

Theorem 4. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$. *If* $\beta \ge A \ge \alpha > 0$, $B>0$, $\delta \ge P \ge \gamma > 0$ *and* $0 < m \le B-A \le M$ *for some constants* α, β, γ, δ, m, M, *then*

$$\begin{aligned}(1.7)\qquad 0 &\le m\left[\frac{\mathcal{P}_f(M+\beta,\delta)-\mathcal{P}_f(\beta,\delta)}{\delta^2 M}\right]P^2\\ &\le \mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\\ &\le M\left[\frac{\mathcal{P}_f(m+\alpha,\gamma)-\mathcal{P}_f(\alpha,\gamma)}{\gamma^2 m}\right]P^2.\end{aligned}$$

Some bounds for the weighted operator arithmetic mean and the relative operator entropy were also given.

Motivated by the above results, in this paper we show among others that, if $\sigma \geq C \geq \rho > 0$, $D > 0$, $\varsigma \geq Q \geq \tau > 0$ and $0 < n \leq D - C \leq N$ for some constants ρ, σ, ς, τ, n, N, then

$$\begin{aligned} 0 &\leq \frac{n}{N\varsigma^2} \left[\mathcal{P}_f(\varsigma, N+\sigma) - \mathcal{P}_f(\varsigma, \sigma) \right] Q^2 \\ &\leq \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \\ &\leq \frac{N}{n\tau^2} \left[\mathcal{P}_f(\tau, n+\rho) - \mathcal{P}_f(\tau, \rho) \right] Q^2. \end{aligned}$$

Applications for *weighted operator geometric mean* and the perspective

$$\mathcal{P}_{\ln(\cdot+1)}(B, A) := A^{1/2} \ln\left(A^{-1/2} B A^{-1/2} + 1 \right) A^{1/2}, \ A, B > 0.$$

are also provided.

2. Some Preliminary Facts

We start to the following identity of interest for the transpose function $\tilde{f}$ [2]:

Lemma 1. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation (1.3). Then for all U, $V > 0$ we have*

$$\begin{aligned} &\tilde{f}(V) - \tilde{f}(U) \\ &= a(V - U) + \int_0^\infty \lambda \left(\int_0^1 (1 + \lambda [(1-t)U + tV])^{-1} \right. \\ &\left. \times (V - U)(1 + \lambda [(1-t)U + tV])^{-1} dt \right) dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. From (1.3) we have

$$f(t) = a + bt + t \int_0^\infty \frac{\lambda}{t + \lambda} dw(\lambda), \ t > 0.$$

If we put $\frac{1}{t}$ instead of t, then we get

$$\begin{aligned} f\left(\frac{1}{t}\right) &= a + b\frac{1}{t} + \frac{1}{t} \int_0^\infty \frac{\lambda}{\frac{1}{t} + \lambda} dw(\lambda) \\ &= a + b\frac{1}{t} + \frac{1}{t} \int_0^\infty \frac{t\lambda}{1 + t\lambda} dw(\lambda) \end{aligned}$$

and by multiplication with $t > 0$, we obtain

$$\tilde{f}(t) = b + ta + \int_0^\infty \frac{t\lambda}{1 + t\lambda} dw(\lambda) = b + ta + \int_0^\infty \left(1 - \frac{1}{1 + t\lambda} \right) dw(\lambda).$$

Therefore

$$\tilde{f}(V) - \tilde{f}(U) = a(V - U) + \int_0^\infty \left[(1 + U\lambda)^{-1} - (1 + V\lambda)^{-1} \right] dw(\lambda). \tag{2.2}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T + tS) - f(T)}{t} \right] = T^{-1} S T^{-1} \tag{2.3}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for $C,\ D$ selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

From (2.5) we get

$$\begin{aligned}&(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\\&=\int_0^1((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}((1+V\lambda)-(1+U\lambda))\\&\times((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}\,dt\\&=\int_0^1\lambda(1+\lambda[(1-t)U+tV])^{-1}(V-U)(1+\lambda[(1-t)U+tV])^{-1}\,dt.\end{aligned} \tag{2.6}$$

Therefore, by (2.2) we derive

$$\begin{aligned}&\tilde f(V)-\tilde f(U)\\&=a(V-U)+\int_0^\infty\left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda)\\&=a(V-U)+\int_0^\infty\lambda\left(\int_0^1(1+\lambda[(1-t)U+tV])^{-1}\right.\\&\left.\times(V-U)(1+\lambda[(1-t)U+tV])^{-1}\,dt\right)dw(\lambda)\end{aligned} \tag{2.7}$$

and the identity (2.1) is proved. □

The following representation for the difference in the second variable holds [2]:

Theorem 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). Then for all $C,\ D,\ Q>0$ we have*

$$\begin{aligned}&\mathcal{P}_{\tilde f}(D,Q)-\mathcal{P}_{\tilde f}(C,Q)\\&=a(D-C)+\int_0^\infty\lambda\left(\int_0^1Q(Q+\lambda[(1-t)C+tD])^{-1}(D-C)\right.\\&\left.\times[(Q+\lambda(1-t)C+tD)]^{-1}Q\,dt\right)dw(\lambda).\end{aligned} \tag{2.8}$$

Proof. If we take $V = Q^{-1/2}DQ^{-1/2}$ and $U = Q^{-1/2}CQ^{-1/2}$ in (2.1), then we get

$$\begin{aligned}(2.9)\quad & \tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right) - \tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right) \\ &= a\left(Q^{-1/2}DQ^{-1/2} - Q^{-1/2}CQ^{-1/2}\right) \\ &+ \int_0^\infty \lambda \left(\int_0^1 \left(1 + \lambda\left[(1-t)\,Q^{-1/2}CQ^{-1/2} + tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}\right. \\ &\times \left(Q^{-1/2}DQ^{-1/2} - Q^{-1/2}CQ^{-1/2}\right) \\ &\times \left.\left(1 + \lambda\left[(1-t)\,Q^{-1/2}CQ^{-1/2} + tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1} dt\right) dw(\lambda)\end{aligned}$$

$$\begin{aligned}&= aQ^{-1/2}(D-C)\,Q^{-1/2} \\ &+ \int_0^\infty \lambda \left(\int_0^1 \left[Q^{-1/2}(Q + \lambda[(1-t)\,C + tD])\right]^{-1}\right. \\ &\times Q^{-1/2}(D-C)\,Q^{-1/2}\left.\left[Q^{-1/2}(Q + \lambda[(1-t)\,C + tD])\,Q^{-1/2}\right]^{-1} dt\right) dw(\lambda) \\ &= aQ^{-1/2}(D-C)\,Q^{-1/2} \\ &+ \int_0^\infty \lambda \left(\int_0^1 Q^{1/2}\left[(Q + \lambda[(1-t)\,C + tD])\right]^{-1}(D-C)\right. \\ &\times \left.\left[(Q + \lambda[(1-t)\,C + tD])\right]^{-1} Q^{1/2} dt\right) dw(\lambda).\end{aligned}$$

If we multiply both sides by $Q^{1/2}$ we get the desired result (2.8). □

Corollary 1. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation (1.3). Then for all C, D, $Q > 0$ we have*

$$\begin{aligned}(2.10)\quad & \mathcal{P}_f(Q, D) - \mathcal{P}_f(Q, C) \\ &= a(D-C) + \int_0^\infty \lambda \left(\int_0^1 Q\,(Q + \lambda[(1-t)\,C + tD])^{-1}(D-C)\right. \\ &\times \left.(Q + \lambda[(1-t)\,C + tD])^{-1} Q dt\right) dw(\lambda).\end{aligned}$$

We also have identity for the *weighted operator geometric mean:*

Proposition 1. *For all C, D, $Q > 0$ and $r \in (0, 1]$ we have*

$$\begin{aligned}(2.11)\quad & D\sharp_r Q - C\sharp_r Q \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^r \left(\int_0^1 Q\,(Q + \lambda[(1-t)\,C + tD])^{-1}(D-C)\right. \\ &\times \left.(Q + \lambda[(1-t)\,C + tD])^{-1} Q dt\right) d\lambda.\end{aligned}$$

The proof follows by (2.10) and (1.1) for the measure $dw(\lambda) = \frac{\sin(r\pi)}{\pi}\lambda^{r-1}d\lambda$.

3. Upper and Lower Bounds

We have the following upper and lower bounds for the difference of perspectives in the second variable:

Theorem 6. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If $\sigma\geq C\geq\rho>0$, $D>0$, $\varsigma\geq Q\geq\tau>0$ and $0<n\leq D-C\leq N$ for some constants ρ, σ, ς, τ, n, N, then*

$$
\begin{aligned}
(3.1)\qquad 0 &\leq n\left[\frac{\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)-aN}{N\varsigma^2}\right]Q^2\\
&\leq \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-a(D-C)\\
&\leq N\left[\frac{\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)-an}{n\tau^2}\right]Q^2.
\end{aligned}
$$

Proof. Since $0<n\leq D-C\leq N$, then by multiplying both sides by

$$
\begin{aligned}
(3.2)\quad 0 &< n\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}\\
&\leq \left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}(D-C)\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}\\
&\leq N\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}
\end{aligned}
$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we multiply the inequality (3.2) both sides by Q we obtain

$$
\begin{aligned}
(3.3)\quad 0 &< nQ\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}Q\\
&\leq Q\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}(D-C)\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}Q\\
&\leq NQ\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}Q.
\end{aligned}
$$

Now, observe that

$$Q+\lambda\left[(1-t)C+tD\right]=Q+\lambda\left[C+t(D-C)\right]$$

for all $\lambda\geq 0$ and $t\in[0,1]$. Then we get the double inequality

$$\tau+\lambda(\rho+tn)\leq Q+\lambda\left[C+t(D-C)\right]\leq\varsigma+\lambda(\sigma+tN)$$

namely

$$\left[\varsigma+\lambda(\sigma+tN)\right]^{-1}\leq\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}\leq\left[\tau+\lambda(\rho+tn)\right]^{-1},$$

which implies that

$$\left[\varsigma+\lambda(\sigma+tN)\right]^{-2}\leq\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}\leq\left[\tau+\lambda(\rho+tn)\right]^{-2},$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

This implies that

$$nQ\left[\varsigma+\lambda(\sigma+tN)\right]^{-2}Q\leq nQ\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}Q,$$

and

$$NQ\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}Q\leq NQ\left[\tau+\lambda(\rho+tn)\right]^{-2}Q$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

By utilising (3.3) we derive

$$
\begin{aligned}
(3.4)\qquad 0 &< nQ\left[\varsigma+\lambda(\sigma+tN)\right]^{-2}Q\\
&\leq Q\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}(D-C)\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}Q\\
&\leq NQ\left[\tau+\lambda(\rho+tn)\right]^{-2}Q.
\end{aligned}
$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we integrate (3.4) over t on $[0,1]$, multiply by λ and integrate over the measure $w(\lambda)$ on $[0,\infty)$, then we get

$$\begin{aligned}0 &\le n\int_0^\infty \lambda\left(\int_0^1 [\varsigma+\lambda(\sigma+tN)]^{-2}dt\right)dw(\lambda)\,Q^2\\ &\le \int_0^\infty \lambda\left[\int_0^1 Q\left(Q+\lambda[(1-t)C+tD]\right)^{-1}(D-C)\right.\\ &\left.\times\left(Q+\lambda[(1-t)C+tD]\right)^{-1}Qdt\right]dw(\lambda)\\ &\le N\int_0^\infty \lambda\left(\int_0^1 [\tau+\lambda(\rho+tn)]^{-2}dt\right)dw(\lambda)\,Q^2,\end{aligned}$$

namely, by the identity (2.10),

$$\begin{aligned}0 &\le n\int_0^\infty \lambda\left(\int_0^1 [\varsigma+\lambda(\sigma+tN)]^{-2}dt\right)dw(\lambda)\,Q^2\\ &\le \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-a(D-C)\\ &\left.\times\left(Q+\lambda[(1-t)C+tD]\right)^{-1}Qdt\right]dw(\lambda)\\ &\le N\int_0^\infty \lambda\left(\int_0^1 [\tau+\lambda(\rho+tn)]^{-2}dt\right)dw(\lambda)\,Q^2.\end{aligned}\tag{3.5}$$

Now, observe that

$$\begin{aligned}&\int_0^\infty \lambda\left(\int_0^1 [\varsigma+\lambda(\sigma+tN)]^{-2}dt\right)dw(\lambda)\\ &=\int_0^\infty \lambda\left(\int_0^1 [\varsigma+\lambda((1-t)\sigma+t(N+\sigma))]^{-2}dt\right)dw(\lambda).\end{aligned}$$

By the identity (2.10) for $C=\sigma$, $D=N+\sigma$ and $Q=\varsigma$,

$$\begin{aligned}&\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)\\ &=aN+\int_0^\infty \lambda\left(\int_0^1 \varsigma\left(\varsigma+\lambda[(1-t)\sigma+t(N+\sigma)]\right)^{-1}N\right.\\ &\left.\times\left(\varsigma+\lambda[(1-t)\sigma+t(N+\sigma)]\right)^{-1}\varsigma dt\right)dw(\lambda)\\ &=aN+N\varsigma^2\int_0^\infty \lambda\left[\int_0^1 \left(\varsigma+\lambda[(1-t)\sigma+t(N+\sigma)]\right)^{-2}dt\right]dw(\lambda),\end{aligned}$$

which gives

$$\begin{aligned}&\int_0^\infty \lambda\left[\int_0^1 \left(\varsigma+\lambda[(1-t)\sigma+t(N+\sigma)]\right)^{-2}dt\right]dw(\lambda)\\ &=\frac{\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)-aN}{N\varsigma^2}.\end{aligned}\tag{3.6}$$

Observe also that

$$\begin{aligned}&\int_0^\infty \lambda\left(\int_0^1 [\tau+\lambda(\rho+tn)]^{-2}dt\right)dw(\lambda)\\ &=\int_0^\infty \lambda\left(\int_0^1 [\tau+\lambda((1-t)\rho+t(n+\rho))]^{-2}dt\right)dw(\lambda).\end{aligned}$$

By the identity (2.10) for $C=\rho$, $D=n+\rho$ and $Q=\tau$,

$$\begin{aligned}
&\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)\\
&=an+\int_0^\infty \lambda\left(\int_0^1 \tau\left(\tau+\lambda\left[(1-t)\rho+t(n+\sigma)\right]\right)^{-1}n\right.\\
&\left.\times\left(\tau+\lambda\left[(1-t)\rho+t(n+\rho)\right]\right)^{-1}\tau dt\right)dw(\lambda)\\
&=an+n\tau^2\int_0^\infty \lambda\left(\int_0^1 \left(\tau+\lambda\left[(1-t)\rho+t(n+\rho)\right]\right)^{-2}dt\right)dw(\lambda),
\end{aligned}$$

which gives

$$\begin{aligned}
(3.7)\qquad &\int_0^\infty \lambda\left(\int_0^1 \left(\tau+\lambda\left[(1-t)\rho+t(n+\rho)\right]\right)^{-2}dt\right)dw(\lambda)\\
&=\frac{\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)-an}{n\tau^2}.
\end{aligned}$$

By utilising (3.5)-(3.7) we derive (3.1). □

Remark 1. *If the function f is operator monotone on $[0,\infty)$, then we can take $a=f(0)$ and by (3.1) we get*

$$\begin{aligned}
(3.8)\qquad 0&\le n\left[\frac{\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)-f(0)N}{N\varsigma^2}\right]Q^2\\
&\le \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-f(0)(D-C)\\
&\le N\left[\frac{\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)-f(0)n}{n\tau^2}\right]Q^2.
\end{aligned}$$

If the integral representation for the operator monotone function is not available, then we can state the following result as well:

Corollary 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone and nonnegative on $[0,\infty)$. If $\sigma\ge C\ge\rho>0$, $D>0$, $\varsigma\ge Q\ge\tau>0$ and $0<n\le D-C\le N$ for some constants ρ, σ, ς, τ, n, N, then*

$$\begin{aligned}
(3.9)\qquad 0&\le \frac{n}{N\varsigma^2}\left[\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)\right]Q^2\\
&\le \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\\
&\le \frac{N}{n\tau^2}\left[\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)\right]Q^2.
\end{aligned}$$

Proof. From (3.8) we get

$$\begin{aligned}
0&<n\left[\frac{\mathcal{P}_f(\varsigma,N+\sigma)-\mathcal{P}_f(\varsigma,\sigma)-f(0)N}{N\varsigma^2}\right]Q^2+f(0)(D-C)\\
&\le \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\\
&\le N\left[\frac{\mathcal{P}_f(\tau,n+\rho)-\mathcal{P}_f(\tau,\rho)-f(0)n}{n\tau^2}\right]Q^2+f(0)(D-C),
\end{aligned}$$

namely

$$0<\frac{n}{N\varsigma^{2}}\left[\mathcal{P}_{f}\left(\varsigma,N+\sigma\right)-\mathcal{P}_{f}\left(\varsigma,\sigma\right)\right]Q^{2}+f\left(0\right)\left(D-C-\frac{n}{\varsigma^{2}}Q^{2}\right)$$
$$\leq\mathcal{P}_{f}\left(Q,D\right)-\mathcal{P}_{f}\left(Q,C\right)$$
$$\leq\frac{N}{n\tau^{2}}\left[\mathcal{P}_{f}\left(\tau,n+\rho\right)-\mathcal{P}_{f}\left(\tau,\rho\right)\right]Q^{2}+f\left(0\right)\left(D-C-\frac{N}{\tau^{2}}Q^{2}\right). \tag{3.10}$$

Since $f(0)\geq 0$, $n\leq D-C$ and $\frac{Q^2}{\varsigma^2}\leq 1$, hence

$$D-C-\frac{n}{\varsigma^{2}}Q^{2}\geq 0$$

and the first inequality in (3.9) is proved.

Since $f(0)\geq 0$, $N\geq D-C$ and $\frac{Q^2}{\tau^2}\geq 1$, hence

$$D-C-\frac{N}{\tau^{2}}Q^{2}\leq 0$$

and the third inequality in (3.9) is proved. □

Theorem 7. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If $\sigma\geq C, D\geq\rho>0$, $\varsigma\geq Q\geq\tau>0$ and $0<n\leq D-C\leq N$ for some constants ρ, σ, ς, τ, n, N, then*

$$0\leq n\left[\frac{f\left(\frac{\varsigma}{\sigma}\right)-f'\left(\frac{\varsigma}{\sigma}\right)\frac{\varsigma}{\sigma}-a}{\varsigma^{2}}\right]Q^{2}$$
$$\leq\mathcal{P}_{f}\left(Q,D\right)-\mathcal{P}_{f}\left(Q,C\right)-a\left(D-C\right)$$
$$\leq N\left[\frac{f\left(\frac{\tau}{\rho}\right)-f'\left(\frac{\tau}{\rho}\right)\frac{\tau}{\rho}-a}{\tau^{2}}\right]Q^{2}. \tag{3.11}$$

Proof. From the condition $\sigma\geq C$, $D\geq\rho>0$ and $\varsigma\geq Q\geq\tau>0$, we get

$$\tau+\lambda\rho\leq Q+\lambda\left[(1-t)C+tD\right]\leq\varsigma+\lambda\sigma$$

for all $\lambda\geq 0$ and $t\in[0,1]$, which implies that

$$(\varsigma+\lambda\sigma)^{-2}\leq\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-2}\leq(\tau+\lambda\rho)^{-2}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

From (3.3) we get

$$0<nQ\left(\varsigma+\lambda\sigma\right)^{-2}Q$$
$$\leq Q\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}\left(D-C\right)\left(Q+\lambda\left[(1-t)C+tD\right]\right)^{-1}Q$$
$$\leq NQ\left(\tau+\lambda\rho\right)^{-2}Q. \tag{3.12}$$

for all $\lambda\geq 0$ and $t\in[0,1]$.

If we integrate (3.12) over t on $[0,1]$, multiply by λ and integrate over the measure $w(\lambda)$ on $[0,\infty)$, then we get

$$
\begin{aligned}
0 &\leq n\left[\int_0^\infty \lambda(\varsigma+\lambda\sigma)^{-2}\,dw(\lambda)\right]Q^2 \\
&\leq Q\int_0^\infty \lambda\left[\int_0^1 (Q+\lambda[(1-t)C+tD])^{-1}\,dt\,(D-C)\right. \\
&\quad \left.\times (Q+\lambda[(1-t)C+tD])^{-1}\,dtQ\right]dw(\lambda) \\
&\leq N\left[\int_0^\infty \lambda(\tau+\lambda\rho)^{-2}\,dw(\lambda)\right]Q^2,
\end{aligned}
$$

namely, by the identity (2.10),

$$
\begin{aligned}
0 &\leq n\left[\int_0^\infty \lambda(\varsigma+\lambda\sigma)^{-2}\,dw(\lambda)\right]Q^2 \\
&\leq \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-a(D-C) \\
&\leq N\left[\int_0^\infty \lambda(\tau+\lambda\rho)^{-2}\,dw(\lambda)\right]Q^2.
\end{aligned}
\tag{3.13}
$$

For $d,\ c>0$ we consider

$$
J(d,c):=\int_0^\infty \frac{\lambda}{(d\lambda+c)^2}dw(\lambda)=\frac{1}{d^2}\int_0^\infty \frac{\lambda}{\left(\lambda+\frac{c}{d}\right)^2}dw(\lambda).
$$

From (1.3) we have

$$
\frac{f(t)-a}{t}-b=\int_0^\infty \frac{\lambda}{t+\lambda}dw(\lambda),\ t>0.
$$

If we take the derivative over t, then we get

$$
\frac{f'(t)\,t-f(t)+a}{t^2}=-\int_0^\infty \frac{\lambda}{(t+\lambda)^2}dw(\lambda),\ t>0,
$$

namely

$$
\int_0^\infty \frac{\lambda}{(t+\lambda)^2}dw(\lambda)=\frac{f(t)-f'(t)\,t-a}{t^2},\ t>0.
$$

This gives that

$$
\int_0^\infty \frac{\lambda}{\left(\frac{c}{d}+\lambda\right)^2}dw(\lambda)=\frac{f\left(\frac{c}{d}\right)-f'\left(\frac{c}{d}\right)\frac{c}{d}-a}{\left(\frac{c}{d}\right)^2},
$$

namely

$$
J(d,c)=\frac{f\left(\frac{c}{d}\right)-f'\left(\frac{c}{d}\right)\frac{c}{d}-a}{c^2}.
$$

Therefore

$$
\int_0^\infty \lambda(\varsigma+\lambda\sigma)^{-2}\,dw(\lambda)=J(\sigma,\varsigma)=\frac{f\left(\frac{\varsigma}{\sigma}\right)-f'\left(\frac{\varsigma}{\sigma}\right)\frac{\varsigma}{\sigma}-a}{\varsigma^2}
$$

and

$$
\int_0^\infty \lambda(\tau+\lambda\rho)^{-2}\,dw(\lambda)=J(\rho,\tau)=\frac{f\left(\frac{\tau}{\rho}\right)-f'\left(\frac{\tau}{\rho}\right)\frac{\tau}{\rho}-a}{\tau^2}.
$$

By making use of (3.13) we derive (3.11). □

Remark 2. *If the function f is operator monotone on $[0,\infty)$, then we can take $a = f(0)$ and by (3.11) we get*

$$\begin{aligned} 0 &\le n\left[\frac{f\left(\frac{\varsigma}{\sigma}\right) - f'\left(\frac{\varsigma}{\sigma}\right)\frac{\varsigma}{\sigma} - f(0)}{\varsigma^2}\right] Q^2 \\ &\le \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C) \\ &\le N\left[\frac{f\left(\frac{\tau}{\rho}\right) - f'\left(\frac{\tau}{\rho}\right)\frac{\tau}{\rho} - f(0)}{\tau^2}\right] Q^2. \end{aligned} \tag{3.14}$$

Corollary 3. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone and nonnegative on $[0,\infty)$. If $\sigma \ge C, D \ge \rho > 0$, $\varsigma \ge Q \ge \tau > 0$ and $0 < n \le D - C \le N$ for some constants ρ, σ, ς, τ, n, N, then*

$$\begin{aligned} 0 &\le n\left[\frac{f\left(\frac{\varsigma}{\sigma}\right) - f'\left(\frac{\varsigma}{\sigma}\right)\frac{\varsigma}{\sigma}}{\varsigma^2}\right] Q^2 \\ &\le \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) \le N\left[\frac{f\left(\frac{\tau}{\rho}\right) - f'\left(\frac{\tau}{\rho}\right)\frac{\tau}{\rho}}{\tau^2}\right] Q^2. \end{aligned} \tag{3.15}$$

The case of separated operators is as follows:

Theorem 8. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.3). If the positive operators satisfy the separation condition*

$$\sigma \ge D \ge \phi > \upsilon \ge C \ge \rho > 0, \tag{3.16a}$$

and $\varsigma \ge Q \ge \tau > 0$ for some constants ρ, σ, ς, τ, ϕ , υ then

$$\begin{aligned} 0 &\le n\left[\frac{\mathcal{P}_f(\varsigma,\sigma) - \mathcal{P}_f(\varsigma,\upsilon) - a(\sigma-\upsilon)}{(\sigma-\upsilon)\varsigma^2}\right] Q^2 \\ &\le \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C) \\ &\le N\left[\frac{\mathcal{P}_f(\tau,\phi) - \mathcal{P}_f(\tau,\rho) - a(\phi-\rho)}{(\phi-\rho)\tau^2}\right] Q^2. \end{aligned} \tag{3.17}$$

Proof. If the positive operators satisfy the separation condition (3.16a), then

$$\tau + \lambda[(1-t)\rho + t\phi] \le Q + \lambda[(1-t)C + tD] \le \varsigma + \lambda[(1-t)\upsilon + t\sigma]$$

for all $\lambda \ge 0$ and $t \in [0,1]$.

This implies that

$$\begin{aligned} (\varsigma + \lambda[(1-t)\upsilon + t\sigma])^{-2} &\le (Q + \lambda[(1-t)C + tD])^{-2} \\ &\le (\tau + \lambda[(1-t)\rho + t\phi])^{-2} \end{aligned}$$

for all $\lambda \ge 0$ and $t \in [0,1]$.

From (3.3) we get

$$\begin{aligned} 0 &< nQ(\varsigma + \lambda[(1-t)\upsilon + t\sigma])^{-2} Q \\ &\le Q(Q + \lambda[(1-t)C + tD])^{-1}(D-C)(Q + \lambda[(1-t)C + tD])^{-1} Q \\ &\le NQ(\tau + \lambda[(1-t)\rho + t\phi])^{-2} Q. \end{aligned}$$

for all $\lambda \geq 0$ and $t \in [0,1]$.

If we integrate and use the identity (2.10) we deduce that

$$0 < nQ \int_0^\infty \lambda \left(\int_0^1 \left(\varsigma + \lambda \left[(1-t)\upsilon + t\sigma \right] \right)^{-2} dt \right) dw(\lambda) Q \tag{3.18}$$
$$\leq \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - a(D-C)$$
$$\leq NQ \int_0^\infty \lambda \left(\int_0^1 \left(\tau + \lambda \left[(1-t)\rho + t\phi \right] \right)^{-2} dt \right) dw(\lambda) Q.$$

From the identity the identity (2.10) we have

$$\mathcal{P}_f(\varsigma,\sigma) - \mathcal{P}_f(\varsigma,\upsilon)$$
$$= a(\sigma-\upsilon) + \int_0^\infty \lambda \left(\int_0^1 \varsigma \left(\varsigma + \lambda \left[(1-t)\upsilon + t\sigma \right] \right)^{-1} (\sigma - \upsilon) \right.$$
$$\left. \times \left(\varsigma + \lambda \left[(1-t)\upsilon + t\sigma \right] \right)^{-1} \varsigma dt \right) dw(\lambda)$$
$$= a(\sigma-\upsilon) + (\sigma-\upsilon)\varsigma^2 \int_0^\infty \lambda \left(\int_0^1 \left(\varsigma + \lambda \left[(1-t)\upsilon + t\sigma \right] \right)^{-2} dt \right) dw(\lambda),$$

which gives that

$$\int_0^\infty \lambda \left(\int_0^1 \left(\varsigma + \lambda \left[(1-t)\upsilon + t\sigma \right] \right)^{-2} dt \right) dw(\lambda)$$
$$= \frac{\mathcal{P}_f(\varsigma,\sigma) - \mathcal{P}_f(\varsigma,\upsilon) - a(\sigma-\upsilon)}{(\sigma-\upsilon)\varsigma^2}.$$

By the identity (2.10) we also have

$$\mathcal{P}_f(\tau,\phi) - \mathcal{P}_f(\tau,\rho)$$
$$= a(\phi-\rho) + (\phi-\rho)\tau^2 \int_0^\infty \lambda \left(\int_0^1 \left(\tau + \lambda \left[(1-t)\rho + t\phi \right] \right)^{-2} dt \right) dw(\lambda),$$

which gives that

$$\int_0^\infty \lambda \left(\int_0^1 \left(\tau + \lambda \left[(1-t)\rho + t\phi \right] \right)^{-2} dt \right) dw(\lambda)$$
$$= \frac{\mathcal{P}_f(\tau,\phi) - \mathcal{P}_f(\tau,\rho) - a(\phi-\rho)}{(\phi-\rho)\tau^2}.$$

By making use of (3.18) we deduce (3.17). □

Remark 3. *If the function f is operator monotone on $[0,\infty)$, then we can take $a = f(0)$ and by (3.11) we obtain*

$$0 \leq n \left[\frac{\mathcal{P}_f(\varsigma,\sigma) - \mathcal{P}_f(\varsigma,\upsilon) - f(0)(\sigma-\upsilon)}{(\sigma-\upsilon)\varsigma^2} \right] Q^2 \tag{3.19}$$
$$\leq \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - f(0)(D-C)$$
$$\leq N \left[\frac{\mathcal{P}_f(\tau,\phi) - \mathcal{P}_f(\tau,\rho) - f(0)(\phi-\rho)}{(\phi-\rho)\tau^2} \right] Q^2.$$

Corollary 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone and nonnegative on $[0,\infty)$. If the positive operators satisfy the separation condition (3.16a), then*

$$0\le n\left[\frac{\mathcal{P}_f(\varsigma,\sigma)-\mathcal{P}_f(\varsigma,\upsilon)}{(\sigma-\upsilon)\varsigma^2}\right]Q^2 \le \mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\le N\left[\frac{\mathcal{P}_f(\tau,\phi)-\mathcal{P}_f(\tau,\rho)}{(\phi-\rho)\tau^2}\right]Q^2. \tag{3.20}$$

4. Some Examples of Interest

Consider the power function $f_r(t)=t^r$, $t\ge 0$ and $r\in(0,1]$. Then for x, $y>0$ we have

$$\mathcal{P}_{f_r}(x,y)=x^r y^{1-r}.$$

If $\sigma\ge C\ge\rho>0$, $D>0$, $\varsigma\ge Q\ge\tau>0$ and $0<n\le D-C\le N$ for some constants ρ, σ, ς, τ, n, N, then

$$0\le\frac{n}{N}\varsigma^{r-2}\left[(N+\sigma)^{1-r}-\varsigma^r\sigma^{1-r}\right]Q^2\le D\sharp_r Q-C\sharp_r Q \le\frac{N}{n}\tau^{r-2}\left[(n+\rho)^{1-r}-\rho^{1-r}\right]Q^2. \tag{4.1}$$

If $\sigma\ge C, D\ge\rho>0$, $\varsigma\ge Q\ge\tau>0$ and $0<n\le D-C\le N$ for some constants ρ, σ, ς, τ, n, N, then

$$0\le n(1-r)\varsigma^{r-2}\sigma^{-r}Q^2\le D\sharp_r Q-C\sharp_r Q\le N(1-r)\tau^{r-2}\rho^{-r}Q^2. \tag{4.2}$$

If the positive operators satisfy the separation condition $\sigma\ge D\ge\phi>\upsilon\ge C\ge\rho>0$ and $\varsigma\ge Q\ge\tau>0$ for some constants ρ, σ, ς, τ,ϕ , υ then

$$0\le n\varsigma^{r-2}\left(\frac{\sigma^{1-r}-\upsilon^{1-r}}{\sigma-\upsilon}\right)Q^2\le D\sharp_r Q-C\sharp_r Q \le N\tau^{r-2}\left(\frac{\phi^{1-r}-\rho^{1-r}}{\phi-\rho}\right)Q^2. \tag{4.3}$$

Consider the function $f_{\ln(\cdot+1)}(t)=\ln(t+1)$. We have

$$P_{\ln(\cdot+1)}(x,y):=y\ln\left(xy^{-1}+1\right),\ x,y>0$$

and

$$\mathcal{P}_{\ln(\cdot+1)}(B,A):=A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}+1\right)A^{1/2},\ A,B>0.$$

If $\sigma\ge C\ge\rho>0$, $D>0$, $\varsigma\ge Q\ge\tau>0$ and $0<n\le D-C\le N$ for some constants ρ, σ, ς, τ, n, N, then

$$0\le\frac{n}{N\varsigma^2}\left[(N+\sigma)\ln\left(\varsigma(N+\sigma)^{-1}+1\right)-\sigma\ln\left(\varsigma\sigma^{-1}+1\right)\right]Q^2 \le\mathcal{P}_{\ln(\cdot+1)}(Q,D)-\mathcal{P}_{\ln(\cdot+1)}(Q,C) \le\frac{N}{n\tau^2}\left[(n+\rho)\ln\left(\tau(n+\rho)^{-1}+1\right)-\rho\ln\left(\tau\rho^{-1}+1\right)\right]Q^2. \tag{4.4}$$

If $\sigma \geq C, D \geq \rho > 0$, $\varsigma \geq Q \geq \tau > 0$ and $0 < n \leq D - C \leq N$ for some constants ρ, σ, ς, τ, n, N, then

$$0 \leq n \left[\frac{\ln\left(\frac{\varsigma}{\sigma} + 1\right) - \frac{\varsigma}{\sigma+\varsigma}}{\varsigma^2} \right] Q^2 \tag{4.5}$$

$$\leq \mathcal{P}_{\ln(\cdot+1)}(Q, D) - \mathcal{P}_{\ln(\cdot+1)}(Q, C) \leq N \left[\frac{\ln\left(\frac{\rho}{\sigma} + 1\right) - \frac{\rho}{\sigma+\rho}}{\tau^2} \right] Q^2.$$

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