INEQUALITIES FOR THE FORWARD DISTANCE IN METRIC SPACES

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Abstract. In this note we prove among others that

\[ \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2} \sum_{k=1}^{n} p_k (1 - p_k) \sum_{k=1}^{n-1} d(x_k, x_{k+1}) \]

where \((X, d)\) is a metric space, \(x_i \in X, p_i \geq 0, i \in \{1, ..., n\}\) with \(\sum_{i=1}^{n} p_i = 1\) and \(p, q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\).

1. Introduction

Let \(X\) be a nonempty set. A function \(d : X \times X \to [0, \infty)\) is called a distance on \(X\) if the following properties are satisfied:

(d) \(d(x, y) = 0\) if and only if \(x = y\);
(dd) \(d(x, y) = d(y, x)\) for any \(x, y \in X\) (the symmetry of the distance);
(ddd) \(d(x, y) \leq d(x, z) + d(z, y)\) for any \(x, y, z \in X\) (the triangle inequality).

The pair \((X, d)\) is called in the literature a metric space.

Important examples of metric spaces are normed linear spaces. We recall that, a linear space \(E\) over the real or complex number field \(K\) endowed with a function \(|\cdot| : E \to [0, \infty)\), is called a normed space if \(|\cdot|\), the norm, satisfies the properties:

(n) \(|x| = 0\) if and only if \(x = 0\);
(nn) \(|\alpha x| = |\alpha||x|\) for any scalar \(\alpha \in K\) and any vector \(x \in E\);
(nnn) \(|x + y| \leq |x| + |y|\) for each \(x, y \in E\) (the triangle inequality).

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the generalised triangle inequality, or the polygonal inequality which states that: for any points \(x_1, x_2, ..., x_n\) \((n \geq 3)\) in a metric space \((X, d)\), we have the inequality

\[ d(x_1, x_n) \leq d(x_1, x_2) + ... + d(x_{n-1}, x_n). \]

The following result in the general setting of metric spaces holds.

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Theorem 1. Let \((X, d)\) be a metric space and \(x_i \in X, p_i \geq 0, i \in \{1, \ldots, n\}\) with \(\sum_{i=1}^{n} p_i = 1\). Then we have the inequality

\[
\sum_{1 \leq i < j \leq n} p_ipjd(x_i, x_j) \leq \inf_{x \in X} \left[ \sum_{i=1}^{n} p_id(x_i, x) \right].
\]

The inequality is sharp in the sense that the multiplicative constant \(c = 1\) in front of "inf" cannot be replaced by a smaller quantity.

We have:

Corollary 1. Let \((X, d)\) be a metric space and \(x_i \in X, i \in \{1, \ldots, n\}\). If there exists a closed ball of radius \(r > 0\) centered in a point \(x\) containing all the points \(x_i\), i.e., \(x_i \in B(x, r) := \{y \in X : d(x, y) \leq r\}\), then for any \(p_i \geq 0, i \in \{1, \ldots, n\}\) with \(\sum_{i=1}^{n} p_i = 1\) we have the inequality

\[
\sum_{1 \leq i < j \leq n} p_ipjd(x_i, x_j) \leq r.
\]

The inequality (1.2) and its consequences were extended to the case of \(b\)-metric spaces in [4] and for natural powers of the distance in [1].

In the recent paper [2] we obtained the following refinement of the inequality (1.2):

Theorem 2. Let \((X, d)\) be a metric space and \(x_i \in X, p_i \geq 0, i \in \{1, \ldots, n\}\) with \(\sum_{i=1}^{n} p_i = 1\). Then we have the inequality

\[
\sum_{1 \leq i < j \leq n} p_ipjd^s(x_i, x_j) \leq \begin{cases} 
2^{s-1} \inf_{x \in X} \left[ \sum_{k=1}^{n} p_k (1 - p_k) d^s (x_k, x) \right], & s \geq 1 \\
\inf_{x \in X} \left[ \sum_{k=1}^{n} p_k (1 - p_k) d^s (x_k, x) \right], & 0 < s < 1,
\end{cases}
\]

\[
\leq \frac{1}{4} \begin{cases} 
2^{s-1} \inf_{x \in X} \left[ \sum_{k=1}^{n} p_k d^s (x_k, x) \right], & s \geq 1 \\
\inf_{x \in X} \left[ \sum_{k=1}^{n} p_k d^s (x_k, x) \right], & 0 < s < 1.
\end{cases}
\]

In this paper we establish other upper bounds for the sum \(\sum_{1 \leq i < j \leq n} p_ipjd(x_i, x_j)\) in terms of the forward distances \(\sum_{k=1}^{n-1} d(x_k, x_{k+1})\), \(\max_{k=1, n-1} d(x_k, x_{k+1})\) and \(\left[ \sum_{k=1}^{n-1} d^q (x_k, x_{k+1}) \right]^{1/q}\), \(q > 1\).

2. Results

We have the following upper bounds in terms of the forward difference:

Theorem 3. Let \((X, d)\) be a metric space and \(x_i \in X, p_i \geq 0, i \in \{1, \ldots, n\}\) with \(\sum_{i=1}^{n} p_i = 1\). Then we have the inequality

\[
\sum_{1 \leq i < j \leq n} p_ipjd(x_i, x_j) \leq \begin{cases} 
\frac{1}{2} \sum_{k=1}^{n} p_k (1 - p_k) \sum_{k=1}^{n-1} d(x_k, x_{k+1}), \\
\frac{1}{2} \sum_{1 \leq i, j \leq n} p_ipj \max_{k=1, n-1} d(x_k, x_{k+1}), \\
\frac{1}{2} \sum_{1 \leq i, j \leq n} p_ipj |j - i|^{1/p} \left[ \sum_{k=1}^{n-1} d^q (x_k, x_{k+1}) \right]^{1/q},
\end{cases}
\]
for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** By the polygonal inequality we have for $1 \leq i < j \leq n$ that

\[
(2.2) \quad d(x_i, x_j) \leq \sum_{k=i}^{j-1} d(x_k, x_{k+1}) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}).
\]

By Hölder’s inequality we also have for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

\[
(2.3) \quad \sum_{k=i}^{j-1} d(x_k, x_{k+1}) \leq \left\{ (j - i) \max_{k=i,j-1} d(x_k, x_{k+1}) \right\}^{1/p} \left[ \sum_{k=i}^{j-1} d^q(x_k, x_{k+1}) \right]^{1/q} \leq \left\{ (j - i) \max_{k=1,n-1} d(x_k, x_{k+1}) \right\}^{1/p} \left[ \sum_{k=1}^{n-1} d^q(x_k, x_{k+1}) \right]^{1/q}.
\]

From (2.2) we get

\[
(2.4) \quad \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=1}^{n-1} d(x_k, x_{k+1}).
\]

Since

\[
\sum_{1 \leq i < j \leq n} p_i p_j = \frac{1}{2} \left( \sum_{1 \leq i < j \leq n} p_i p_j - \sum_{k=1}^{n} p_k^2 \right) = \frac{1}{2} \left( 1 - \sum_{k=1}^{n} p_k^2 \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} p_k (1 - p_k),
\]

hence by (2.4) we derive the first inequality in (2.1).

By (2.3) we get

\[
\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \sum_{1 \leq i < j \leq n} p_i p_j (j - i) \max_{k=1,n-1} d(x_k, x_{k+1})
\]

\[
= \frac{1}{2} \sum_{1 \leq i < j \leq n} p_i p_j |j - i| \max_{k=1,n-1} d(x_k, x_{k+1})
\]

and the second inequality in (2.1) is proved.

By (2.3) we also get

\[
\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \sum_{1 \leq i < j \leq n} p_i p_j (j - i)^{1/p} \left[ \sum_{k=1}^{n-1} d^q(x_k, x_{k+1}) \right]^{1/q}
\]

\[
= \frac{1}{2} \sum_{1 \leq i < j \leq n} p_i p_j |j - i|^{1/p} \left[ \sum_{k=1}^{n-1} d^q(x_k, x_{k+1}) \right]^{1/q},
\]

which proves the third inequality in (2.1). \qed
Corollary 2. With the assumptions of Theorem 3 we have

\[ \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{8} n \sum_{k=1}^{n-1} d(x_k, x_{k+1}). \]

Proof. Using the elementary inequality

\[ ab \leq \frac{1}{4} (a + b)^2, \quad a, b \geq 0 \]

we get

\[ p_k (1 - p_k) \leq \frac{1}{4} (p_k + 1 - p_k)^2 = \frac{1}{4} \]

for all \( k \in \{1, ..., n\} \).

Therefore

\[ \sum_{k=1}^{n} p_k (1 - p_k) \leq \frac{1}{4} n \]

and by the first inequality in (2.1), we get (2.5). \( \square \)

Corollary 3. With the assumptions of Theorem 3 and if \( p_m := \min_{k \in \{1, ..., n\}} p_k > 0 \), then

\[ \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2} (1 - p_m) \sum_{k=1}^{n-1} d(x_k, x_{k+1}). \]

Proof. Since

\[ 0 \leq 1 - p_k \leq 1 - p_m \]

for all \( k \in \{1, ..., n\} \), hence

\[ \sum_{k=1}^{n} p_k (1 - p_k) \leq \sum_{k=1}^{n} p_k (1 - p_m) = 1 - p_m. \]

By utilising the first inequality in (2.1), we deduce (2.6). \( \square \)

Corollary 4. With the assumptions of Theorem 3 and if \( p_M := \min_{k \in \{1, ..., n\}} p_k < 1 \), then

\[ \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2} (n - 1) p_M \sum_{k=1}^{n-1} d(x_k, x_{k+1}). \]

Proof. We have

\[ \sum_{k=1}^{n} p_k (1 - p_k) \leq p_M \sum_{k=1}^{n} (1 - p_k) = (n - 1) p_M \]

and by the first inequality in (2.1), we deduce (2.7). \( \square \)

Corollary 5. With the assumptions of Corollary 4 we have

\[ \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{n (n^2 - 1)}{6} p_M^2 \sum_{k=1}^{n-1} d(x_k, x_{k+1}). \]
Proof. Observe that
\[
\sum_{1 \leq j < i \leq n} (i - j) = \sum_{1 \leq j \leq 2} (2 - j) + \sum_{1 \leq j \leq 3} (3 - j) + \ldots + \sum_{1 \leq j \leq n} (n - j) = 2 \cdot 2 - (1 + 2) + 3 \cdot 3 - (1 + 2 + 3) + \ldots + n \cdot n - (1 + 2 + \ldots + n) = 1^2 + 2^2 + \ldots + n^2 - 1 - (1 + 2) - (1 + 2 + 3) - \ldots - (1 + 2 + \ldots + n) = \frac{n}{2} \sum_{k=1}^{n} k^2 - \frac{n}{2} \sum_{k=1}^{n} k = \frac{n(n^2 - 1)}{6}.
\]
Since
\[
\sum_{1 \leq i, j \leq n} p_i p_j |j - i| \leq p_M^2 \sum_{1 \leq i, j \leq n} |j - i| = 2p_M^2 \sum_{1 \leq j < i \leq n} (i - j) = 2p_M^2 \frac{n(n^2 - 1)}{6},
\]
hence by the second inequality in (2.1) we get (2.8). \qed

Corollary 6. With the assumptions of Theorem 3 we have
\[
(2.9) \quad \sum_{1 \leq i, j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/p} \left[ \sum_{k=1}^{n-1} d^p(x_k, x_{k+1}) \right]^{1/q}
\]
for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).
In particular, for \( p = q = 2 \), we have
\[
(2.10) \quad \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/2} \left[ \sum_{k=1}^{n-1} d^2(x_k, x_{k+1}) \right]^{1/2}.
\]
Proof. By the concavity of function \( f(t) = t^{1/p}, p > 1 \) and by Jensen’s inequality we have
\[
\frac{\sum_{1 \leq i, j \leq n} p_i p_j |j - i|}{\sum_{1 \leq i, j \leq n} p_i p_j} \leq \left( \frac{\sum_{1 \leq i, j \leq n} p_i p_j |j - i|}{\sum_{1 \leq i, j \leq n} p_i p_j} \right)^{1/p}
\]
and since \( \sum_{1 \leq i, j \leq n} p_i p_j = 1 \), hence
\[
\sum_{1 \leq i, j \leq n} p_i p_j |j - i|^{1/p} \leq \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/p}.
\]
By utilising the third inequality in (2.1) we get (2.10). \qed

Remark 1. If \( p_M := \min_{k \in \{1, \ldots, n\}} p_k < 1 \), then by (2.9) we derive
\[
(2.11) \quad \sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \frac{1}{2 \cdot 3^{1/p}} \left[ n(n^2 - 1) \right]^{1/p} p_M^{2/p} \left[ \sum_{k=1}^{n-1} d^p(x_k, x_{k+1}) \right]^{1/q}.
\]
and from (2.10)

\[
\sum_{1 \leq i < j \leq n} p_ip_j d(x_i, x_j) \leq \frac{1}{2} \cdot 3^{1/2} \left[ n(n^2 - 1) \right]^{1/2} p_M \left[ \sum_{k=1}^{n-1} d^2(x_k, x_{k+1}) \right]^{1/2}.
\]

We also have:

**Corollary 7.** With the assumptions of Theorem 3 we have

\[
\sum_{1 \leq i < j \leq n} p_ip_j d(x_i, x_j)
\leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/p} \left[ \sum_{k=1}^{n} p_k \left( 1 - p_k \right) \sum_{k=1}^{n-1} d^q(x_k, x_{k+1}) \right]^{1/q}.
\]

for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

In particular, for \( p = q = 2 \),

\[
\sum_{1 \leq i < j \leq n} p_ip_j d(x_i, x_j)
\leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/2} \left[ \sum_{k=1}^{n} p_k \left( 1 - p_k \right) \sum_{k=1}^{n-1} d^2(x_k, x_{k+1}) \right]^{1/2}.
\]

**Proof.** By Jensen’s inequality we also have

\[
\frac{\sum_{1 \leq i < j \leq n} p_ip_j |j - i|^{1/p}}{\sum_{1 \leq i < j \leq n} p_ip_j} \leq \left( \frac{\sum_{1 \leq i < j \leq n} p_ip_j |j - i|}{\sum_{1 \leq i < j \leq n} p_ip_j} \right)^{1/p},
\]

namely

\[
\sum_{1 \leq i < j \leq n} p_ip_j |j - i|^{1/p} \leq \left( \sum_{1 \leq i < j \leq n} p_ip_j \right)^{1-1/p} \left( \sum_{1 \leq i < j \leq n} p_ip_j |j - i| \right)^{1/p}
\]

\[
= \left( \sum_{1 \leq i < j \leq n} p_ip_j \right)^{1/q} \left( \sum_{1 \leq i < j \leq n} p_ip_j |j - i| \right)^{1/p}
\]

for \( p > 1 \).

Observe that

\[
\sum_{1 \leq i < j \leq n} p_ip_j = \frac{1}{2} \sum_{k=1}^{n} p_k \left( 1 - p_k \right),
\]

\[
\sum_{1 \leq i < j \leq n} p_ip_j |j - i|^{1/p} = \frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i|^{1/p}
\]

and

\[
\sum_{1 \leq i < j \leq n} p_ip_j |j - i| = \frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i|.
\]
By (2.15) we derive
\[
\frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i|^{1/p} \leq \left( \frac{1}{2} \sum_{k=1}^{n} p_k (1 - p_k) \right)^{1/q} \left( \frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/p} \\
= \left( \frac{1}{2} \right)^{1/q + 1/p} \left( \sum_{k=1}^{n} p_k (1 - p_k) \right)^{1/q} \left( \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/p} \\
= \frac{1}{2} \left( \sum_{k=1}^{n} p_k (1 - p_k) \right)^{1/q} \left( \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/p}
\]

namely
\[
\sum_{1 \leq i, j \leq n} p_ip_j |j - i|^{1/p} \leq \left( \sum_{k=1}^{n} p_k (1 - p_k) \right)^{1/q} \left( \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \right)^{1/p}.
\]

By making use of the third inequality in (2.1), we derive (2.13).

**Remark 2.** If \( p_M := \min_{k \in \{1, ..., n\}} p_k < 1 \), then by (2.13) we derive
\[
(2.16) \quad \sum_{1 \leq i < j \leq n} p_ip_j d(x_i, x_j) \\
\leq \frac{1}{2} \cdot 3^{1/p} (n - 1) [n (n + 1)]^{1/p} p_M^{1+1/p} \left[ \sum_{k=1}^{n-1} d^q (x_k, x_{k+1}) \right]^{1/q}
\]

for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and, in particular,
\[
(2.17) \quad \sum_{1 \leq i < j \leq n} p_ip_j d(x_i, x_j) \\
\leq \frac{1}{2} \cdot 3^{1/p} (n - 1) [n (n + 1)]^{1/2} p_M^{3/2} \left[ \sum_{k=1}^{n-1} d^2 (x_k, x_{k+1}) \right]^{1/2}
\]

### 3. Applications

If \((E, \| \|)\) is a normed linear space and \( x_i \in E, i \in \{1, ..., n\}, p_i \geq 0 (i \in \{1, ..., n\})\) with \( \sum_{i=1}^{n} p_i = 1 \), then by (2.1) we have the inequalities
\[
(3.1) \quad \sum_{1 \leq i < j \leq n} p_ip_j \| x_i - x_j \| \\
\leq \left\{ \begin{array}{l}
\frac{1}{2} \sum_{k=1}^{n} p_k (1 - p_k) \sum_{k=1}^{n-1} \| x_k - x_{k+1} \|, \\
\frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i| \max_{k=1, n-1} \| x_k - x_{k+1} \|, \\
\frac{1}{2} \sum_{1 \leq i, j \leq n} p_ip_j |j - i|^{1/p} \left[ \sum_{k=1}^{n-1} \| x_k - x_{k+1} \|^q \right]^{1/q}.
\end{array} \right.
\]
We also have the uniform bound

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{8} n \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|.
\]

If \( p_m := \min_{k \in \{1, \ldots, n\}} p_k > 0 \), then

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2} (1 - p_m) \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|
\]

while, if \( p_M := \min_{k \in \{1, \ldots, n\}} p_k < 1 \), then

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2} (n - 1) p_M \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|
\]

and

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{n(n^2 - 1)}{6} p_M^2 \max_{k=1, \ldots, n-1} \|x_k - x_{k+1}\|.
\]

For \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/p} \left[ \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q}
\]

and, in particular, for \( p = q = 2 \), we derive

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|
\leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/2} \left[ \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2 \right]^{1/2}.
\]

Moreover, if \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{n(n^2 - 1)^{1/p}}{2 \cdot 3^{1/p}} p_M^{2/p} \left[ \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q}
\]

and the Euclidian case

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| \leq \frac{n(n^2 - 1)^{1/2}}{2 \cdot 3^{1/2}} p_M \left[ \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2 \right]^{1/2}.
\]

Finally, for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
\sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|
\leq \frac{1}{2} \left( \sum_{1 \leq i, j \leq n} p_i p_j |j - i| \right)^{1/p} \left[ \sum_{k=1}^{n-1} p_k (1 - p_k) \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q}.
\]
In particular, for \( p = q = 2 \),

\[
\sum_{1 \leq i < j \leq n} p_i p_j \| x_i - x_j \| \leq \frac{1}{2} \left( \sum_{1 \leq i,j \leq n} p_i p_j | j - i | \right)^{1/2} \left[ \sum_{k=1}^{n} p_k (1 - p_k) \sum_{k=1}^{n-1} \| x_k - x_{k+1} \|^2 \right]^{1/2}.
\]

If \( p_M := \min_{k \in \{1, \ldots, n\}} p_k < 1 \), then

\[
\sum_{1 \leq i < j \leq n} p_i p_j \| x_i - x_j \| \leq \frac{1}{2} \cdot 3^{1/p} (n-1) [n (n+1)]^{1/p} p_{M}^{1+1/p} \left[ \sum_{k=1}^{n-1} \| x_k - x_{k+1} \|^q \right]^{1/q}
\]

for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and, in particular,

\[
\sum_{1 \leq i < j \leq n} p_i p_j \| x_i - x_j \| \leq \frac{1}{2} \cdot 3^{1/p} (n-1) [n (n+1)]^{1/2} p_{M}^{3/2} \left[ \sum_{k=1}^{n-1} \| x_k - x_{k+1} \|^2 \right]^{1/2}.
\]

**References**


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