INEQUALITIES FOR THE FORWARD DISTANCE IN METRIC SPACES

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Abstract. In this note we prove among others that

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \begin{cases} \frac{1}{2}\sum_{k=1}^{n} p_k(1-p_k)\sum_{k=1}^{n-1} d(x_k,x_{k+1}), \\ \frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i| \max_{k=1,n-1} d(x_k,x_{k+1}), \\ \frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|^{1/p}\left[\sum_{k=1}^{n-1} d^q(x_k,x_{k+1})\right]^{1/q}, \end{cases}$$

where (X,d) is a metric space, $x_i\in X$, $p_i\ge 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

1. Introduction

Let X be a nonempty set. A function $d: X\times X\to[0,\infty)$ is called a *distance* on X if the following properties are satisfied:

(d) $d(x,y)=0$ if and only if $x=y$;
(dd) $d(x,y)=d(y,x)$ for any $x,y\in X$ (the *symmetry* of the distance);
(ddd) $d(x,y)\le d(x,z)+d(z,y)$ for any $x,y,z\in X$ (the *triangle inequality*).

The pair (X,d) is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space E over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\|: E\to[0,\infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties:

(n) $\|x\|=0$ if and only if $x=0$;
(nn) $\|\alpha x\|=|\alpha|\,\|x\|$ for any scalar $\alpha\in\mathbb{K}$ and any vector $x\in E$;
(nnn) $\|x+y\|\le\|x\|+\|y\|$ for each $x,y\in E$ (the triangle inequality).

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalised triangle inequality*, or the *polygonal inequality* which states that: for any points $x_1,x_2,...,x_{n-1},x_n$ $(n\ge3)$ in a metric space (X,d), we have the inequality

$$d(x_1,x_n)\le d(x_1,x_2)+...+d(x_{n-1},x_n). \tag{1.1}$$

The following result in the general setting of metric spaces holds.

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Theorem 1. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \inf_{x\in X}\left[\sum_{i=1}^n p_i d(x_i,x)\right]. \tag{1.2}$$

The inequality is sharp in the sense that the multiplicative constant $c=1$ in front of " inf " cannot be replaced by a smaller quantity.

We have:

Corollary 1. *Let (X,d) be a metric space and $x_i \in X$, $i \in \{1,...,n\}$. If there exists a closed ball of radius $r>0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x,r) := \{y \in X : d(x,y) \leq r\}$, then for any $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$ we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq r. \tag{1.3}$$

The inequality (1.2) and its consequences were extended to the case of b-metric spaces in [4] and for natural powers of the distance in [1].

In the recent paper [2] we obtained the following refinement of the inequality (1.2):

Theorem 2. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

(1.4)

$$\sum_{1\leq i<j\leq n} p_i p_j d^s(x_i,x_j) \leq \begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)d^s(x_k,x)\right], & s\geq 1 \\ \\ \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)d^s(x_k,x)\right], & 0<s<1, \end{cases}$$

$$\leq \frac{1}{4}\begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n d^s(x_k,x)\right], & s\geq 1 \\ \\ \inf_{x\in X}\left[\sum_{k=1}^n d^s(x_k,x)\right], & 0<s<1. \end{cases}$$

In this paper we establish other upper bounds for the sum $\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j)$ in terms of the forward distances $\sum_{k=1}^{n-1} d(x_k,x_{k+1})$, $\max_{k=1,n-1} d(x_k,x_{k+1})$ and $\left[\sum_{k=1}^{n-1} d^q(x_k,x_{k+1})\right]^{1/q}$, $q>1$.

2. Results

We have the following upper bounds in terms of the forward difference:

Theorem 3. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \tag{2.1}$$

$$\leq \begin{cases} \frac{1}{2}\sum_{k=1}^n p_k(1-p_k)\sum_{k=1}^{n-1} d(x_k,x_{k+1}), \\ \\ \frac{1}{2}\sum_{1\leq i,j\leq n} p_i p_j |j-i| \max_{k=1,n-1} d(x_k,x_{k+1}), \\ \\ \frac{1}{2}\sum_{1\leq i,j\leq n} p_i p_j |j-i|^{1/p}\left[\sum_{k=1}^{n-1} d^q(x_k,x_{k+1})\right]^{1/q}, \end{cases}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the polygonal inequality we have for $1\leq i<j\leq n$ that

$$d(x_i,x_j)\leq\sum_{k=i}^{j-1}d(x_k,x_{k+1})\leq\sum_{k=1}^{n-1}d(x_k,x_{k+1})\,. \tag{2.2}$$

By Hölder's inequality we also have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\sum_{k=i}^{j-1}d(x_k,x_{k+1}) &\leq\begin{cases}(j-i)\max_{k=i,j-1}d(x_k,x_{k+1})\\ \\ (j-i)^{1/p}\left[\sum_{k=i}^{j-1}d^q(x_k,x_{k+1})\right]^{1/q}\end{cases}\\ &\leq\begin{cases}(j-i)\max_{k=1,n-1}d(x_k,x_{k+1})\\ \\ (j-i)^{1/p}\left[\sum_{k=1}^{n-1}d^q(x_k,x_{k+1})\right]^{1/q}.\end{cases}\end{aligned} \tag{2.3}$$

From (2.2) we get

$$\sum_{1\leq i<j\leq n}p_ip_jd(x_i,x_j)\leq\sum_{1\leq i<j\leq n}p_ip_j\sum_{k=1}^{n-1}d(x_k,x_{k+1})\,. \tag{2.4}$$

Since

$$\begin{aligned}\sum_{1\leq i<j\leq n}p_ip_j&=\frac{1}{2}\left(\sum_{1\leq i,j\leq n}p_ip_j-\sum_{k=1}^{n}p_k^2\right)=\frac{1}{2}\left(1-\sum_{k=1}^{n}p_k^2\right)\\ &=\frac{1}{2}\sum_{k=1}^{n}p_k(1-p_k)\,,\end{aligned}$$

hence by (2.4) we derive the first inequality in (2.1).

By (2.3) we get

$$\begin{aligned}\sum_{1\leq i<j\leq n}p_ip_jd(x_i,x_j)&\leq\sum_{1\leq i<j\leq n}p_ip_j(j-i)\max_{k=1,n-1}d(x_k,x_{k+1})\\ &=\frac{1}{2}\sum_{1\leq i,j\leq n}p_ip_j|j-i|\max_{k=1,n-1}d(x_k,x_{k+1})\end{aligned}$$

and the second inequality in (2.1) is proved.

By (2.3) we also get

$$\begin{aligned}\sum_{1\leq i<j\leq n}p_ip_jd(x_i,x_j)&\leq\sum_{1\leq i<j\leq n}p_ip_j(j-i)^{1/p}\left[\sum_{k=1}^{n-1}d^q(x_k,x_{k+1})\right]^{1/q}\\ &=\frac{1}{2}\sum_{1\leq i,j\leq n}p_ip_j|j-i|^{1/p}\left[\sum_{k=1}^{n-1}d^q(x_k,x_{k+1})\right]^{1/q},\end{aligned}$$

which proves the third inequality in (2.1). □

Corollary 2. *With the assumptions of Theorem 3 we have*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \frac{1}{8} n \sum_{k=1}^{n-1} d(x_k,x_{k+1}). \tag{2.5}$$

Proof. Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2, \ a,b\geq 0$$

we get

$$p_k(1-p_k) \leq \frac{1}{4}(p_k+1-p_k)^2 = \frac{1}{4}$$

for all $k\in\{1,...,n\}$.

Therefore

$$\sum_{k=1}^{n} p_k(1-p_k) \leq \frac{1}{4} n$$

and by the first inequality in (2.1), we get (2.5). □

Corollary 3. *With the assumptions of Theorem 3 and if* $p_m := \min_{k\in\{1,\dots,n\}} p_k > 0$, *then*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \frac{1}{2}(1-p_m) \sum_{k=1}^{n-1} d(x_k,x_{k+1}). \tag{2.6}$$

Proof. Since

$$0 \leq 1-p_k \leq 1-p_m$$

for all $k\in\{1,...,n\}$, hence

$$\sum_{k=1}^{n} p_k(1-p_k) \leq \sum_{k=1}^{n} p_k(1-p_m) = 1-p_m.$$

By utilising the first inequality in (2.1), we deduce (2.6). □

Corollary 4. *With the assumptions of Theorem 3 and if* $p_M := \min_{k\in\{1,\dots,n\}} p_k < 1$, *then*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \frac{1}{2}(n-1) p_M \sum_{k=1}^{n-1} d(x_k,x_{k+1}). \tag{2.7}$$

Proof. We have

$$\sum_{k=1}^{n} p_k(1-p_k) \leq p_M \sum_{k=1}^{n} (1-p_k) = (n-1) p_M$$

and by the first inequality in (2.1), we deduce (2.7). □

Corollary 5. *With the assumptions of Corollary 4 we have*

$$\sum_{1\leq i<j\leq n} p_i p_j d(x_i,x_j) \leq \frac{n\left(n^2-1\right)}{6} p_M^2 \max_{k=1,n-1} d(x_k,x_{k+1}). \tag{2.8}$$

Proof. Observe that

$$
\begin{aligned}
&\sum_{1\le j<i\le n}(i-j)\\
&=\sum_{1\le j\le 2}(2-j)+\sum_{1\le j\le 3}(3-j)+...+\sum_{1\le j\le n}(n-j)\\
&=2\cdot 2-(1+2)+3\cdot 3-(1+2+3)+...+n\cdot n-(1+2+...+n)\\
&=1^2+2^2+...+n^2-1-(1+2)-(1+2+3)-...-(1+2+...+n)\\
&=\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{1}{2}\left(\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k\right)=\frac{n\left(n^2-1\right)}{6}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|\le p_M^2\sum_{1\le i,j\le n}\left|j-i\right|=2p_M^2\sum_{1\le j<i\le n}(i-j)=2p_M^2\frac{n\left(n^2-1\right)}{6}\\
=p_M^2\frac{n\left(n^2-1\right)}{3},
\end{aligned}
$$

hence by the second inequality in (2.1) we get (2.8). □

Corollary 6. *With the assumptions of Theorem 3 we have*

$$
\sum_{1\le i<j\le n}p_ip_jd\left(x_i,x_j\right)\le\frac{1}{2}\left(\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|\right)^{1/p}\left[\sum_{k=1}^{n-1}d^q\left(x_k,x_{k+1}\right)\right]^{1/q}\tag{2.9}
$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular, for $p=q=2$, *we have*

$$
\sum_{1\le i<j\le n}p_ip_jd\left(x_i,x_j\right)\le\frac{1}{2}\left(\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|\right)^{1/2}\left[\sum_{k=1}^{n-1}d^2\left(x_k,x_{k+1}\right)\right]^{1/2}.\tag{2.10}
$$

Proof. By the concavity of function $f(t)=t^{1/p}$, $p>1$ and by Jensen's inequality we have

$$
\frac{\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|^{1/p}}{\sum_{1\le i,j\le n}p_ip_j}\le\left(\frac{\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|}{\sum_{1\le i,j\le n}p_ip_j}\right)^{1/p}
$$

and since $\sum_{1\le i,j\le n}p_ip_j=1$, hence

$$
\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|^{1/p}\le\left(\sum_{1\le i,j\le n}p_ip_j\left|j-i\right|\right)^{1/p}.
$$

By utilising the third inequality in (2.1) we get (2.10). □

Remark 1. *If* $p_M:=\min_{k\in\{1,...,n\}}p_k<1$, *then by (2.9) we derive*

$$
\sum_{1\le i<j\le n}p_ip_jd\left(x_i,x_j\right)\le\frac{1}{2\cdot 3^{1/p}}\left[n\left(n^2-1\right)\right]^{1/p}p_M^{2/p}\left[\sum_{k=1}^{n-1}d^q\left(x_k,x_{k+1}\right)\right]^{1/q}\tag{2.11}
$$

and from (2.10)

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \frac{1}{2\cdot 3^{1/2}}\left[n\left(n^2-1\right)\right]^{1/2} p_M \left[\sum_{k=1}^{n-1} d^2(x_k,x_{k+1})\right]^{1/2}. \tag{2.12}$$

We also have:

Corollary 7. *With the assumptions of Theorem 3 we have*

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \frac{1}{2}\left(\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/p}\left[\sum_{k=1}^{n} p_k(1-p_k)\sum_{k=1}^{n-1} d^q(x_k,x_{k+1})\right]^{1/q} \tag{2.13}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular, for $p=q=2$,

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \frac{1}{2}\left(\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/2}\left[\sum_{k=1}^{n} p_k(1-p_k)\sum_{k=1}^{n-1} d^2(x_k,x_{k+1})\right]^{1/2}. \tag{2.14}$$

Proof. By Jensen's inequality we also have

$$\frac{\sum_{1\le i<j\le n} p_i p_j |j-i|^{1/p}}{\sum_{1\le i<j\le n} p_i p_j} \le \left(\frac{\sum_{1\le i<j\le n} p_i p_j |j-i|}{\sum_{1\le i<j\le n} p_i p_j}\right)^{1/p},$$

namely

$$\begin{aligned}\sum_{1\le i<j\le n} p_i p_j |j-i|^{1/p} &\le \left(\sum_{1\le i<j\le n} p_i p_j\right)^{1-1/p}\left(\sum_{1\le i<j\le n} p_i p_j |j-i|\right)^{1/p} \\ &= \left(\sum_{1\le i<j\le n} p_i p_j\right)^{1/q}\left(\sum_{1\le i<j\le n} p_i p_j |j-i|\right)^{1/p}\end{aligned} \tag{2.15}$$

for $p>1$.

Observe that

$$\sum_{1\le i<j\le n} p_i p_j = \frac{1}{2}\sum_{k=1}^{n} p_k(1-p_k),$$

$$\sum_{1\le i<j\le n} p_i p_j |j-i|^{1/p} = \frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|^{1/p}$$

and

$$\sum_{1\le i<j\le n} p_i p_j |j-i| = \frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|.$$

By (2.15) we derive

$$\begin{aligned}
\frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|^{1/p} &\le \left(\frac{1}{2}\sum_{k=1}^{n} p_k(1-p_k)\right)^{1/q} \left(\frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/p} \\
&= \left(\frac{1}{2}\right)^{1/q+1/p} \left(\sum_{k=1}^{n} p_k(1-p_k)\right)^{1/q} \left(\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/p} \\
&= \frac{1}{2}\left(\sum_{k=1}^{n} p_k(1-p_k)\right)^{1/q} \left(\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/p}
\end{aligned}$$

namely

$$\sum_{1\le i,j\le n} p_i p_j |j-i|^{1/p} \le \left(\sum_{k=1}^{n} p_k(1-p_k)\right)^{1/q} \left(\sum_{1\le i,j\le n} p_i p_j |j-i|\right)^{1/p}.$$

By making use of the third inequality in (2.1), we derive (2.13). □

Remark 2. *If $p_M := \min_{k\in\{1,\dots,n\}} p_k < 1$, then by (2.13) we derive*

$$\begin{aligned}
&\sum_{1\le i<j\le n} p_i p_j d(x_i, x_j) \\
&\le \frac{1}{2\cdot 3^{1/p}}(n-1)[n(n+1)]^{1/p} p_M^{1+1/p} \left[\sum_{k=1}^{n-1} d^q(x_k, x_{k+1})\right]^{1/q}
\end{aligned} \tag{2.16}$$

for $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and, in particular,

$$\begin{aligned}
&\sum_{1\le i<j\le n} p_i p_j d(x_i, x_j) \\
&\le \frac{1}{2\cdot 3^{1/p}}(n-1)[n(n+1)]^{1/2} p_M^{3/2} \left[\sum_{k=1}^{n-1} d^2(x_k, x_{k+1})\right]^{1/2}
\end{aligned} \tag{2.17}$$

3. Applications

If $(E, \|\cdot\|)$ is a normed linear space and $x_i \in E$, $i \in \{1, ..., n\}$, $p_i \ge 0$ $(i \in \{1, ..., n\})$ with $\sum_{i=1}^{n} p_i = 1$, then by (2.1) we have the inequalities

$$\begin{aligned}
&\sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\| \\
&\le \begin{cases}
\frac{1}{2}\sum_{k=1}^{n} p_k(1-p_k)\sum_{k=1}^{n-1}\|x_k - x_{k+1}\|, \\[2mm]
\frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i| \max_{k=1,n-1}\|x_k - x_{k+1}\|, \\[2mm]
\frac{1}{2}\sum_{1\le i,j\le n} p_i p_j |j-i|^{1/p}\left[\sum_{k=1}^{n-1}\|x_k - x_{k+1}\|^q\right]^{1/q}.
\end{cases}
\end{aligned} \tag{3.1}$$

We also have the uniform bound

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{8} n \sum_{k=1}^{n-1} \|x_k - x_{k+1}\| . \tag{3.2}$$

If $p_m := \min_{k\in\{1,\ldots,n\}} p_k > 0$, then

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2} (1 - p_m) \sum_{k=1}^{n-1} \|x_k - x_{k+1}\| , \tag{3.3}$$

while, if $p_M := \min_{k\in\{1,\ldots,n\}} p_k < 1$, then

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2} (n-1) p_M \sum_{k=1}^{n-1} \|x_k - x_{k+1}\| \tag{3.4}$$

and

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{n\left(n^2-1\right)}{6} p_M^2 \max_{k=1,n-1} \|x_k - x_{k+1}\| . \tag{3.5}$$

For p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2} \left(\sum_{1\leq i,j\leq n} p_i p_j |j-i| \right)^{1/p} \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q} \tag{3.6}$$

and, in particular, for $p = q = 2$, we derive

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \\ &\leq \frac{1}{2} \left(\sum_{1\leq i,j\leq n} p_i p_j |j-i| \right)^{1/2} \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2 \right]^{1/2} . \end{aligned} \tag{3.7}$$

Moreover, if p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{\left[n\left(n^2-1\right)\right]^{1/p}}{2\cdot 3^{1/p}} p_M^{2/p} \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q} \tag{3.8}$$

and the Euclidian case

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{\left[n\left(n^2-1\right)\right]^{1/2}}{2\cdot 3^{1/2}} p_M \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2 \right]^{1/2} . \tag{3.9}$$

Finally, for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \\ &\leq \frac{1}{2} \left(\sum_{1\leq i,j\leq n} p_i p_j |j-i| \right)^{1/p} \left[\sum_{k=1}^{n} p_k (1-p_k) \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q \right]^{1/q} . \end{aligned} \tag{3.10}$$

In particular, for $p = q = 2$,

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2}\left(\sum_{1\leq i,j\leq n} p_i p_j |j-i|\right)^{1/2} \left[\sum_{k=1}^{n} p_k (1-p_k) \sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2\right]^{1/2}. \tag{3.11}$$

If $p_M := \min_{k\in\{1,\ldots,n\}} p_k < 1$, then

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2\cdot 3^{1/p}} (n-1) [n(n+1)]^{1/p} p_M^{1+1/p} \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^q\right]^{1/q} \tag{3.12}$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and, in particular,

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \frac{1}{2\cdot 3^{1/p}} (n-1) [n(n+1)]^{1/2} p_M^{3/2} \left[\sum_{k=1}^{n-1} \|x_k - x_{k+1}\|^2\right]^{1/2}. \tag{3.13}$$

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