LIPSCHITZ TYPE INEQUALITIES FOR NONCOMMUTATIVE PERSPECTIVES OF OPERATOR MONOTONE FUNCTIONS IN HILBERT SPACES

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Abstract. Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. We can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$.

Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. We show in this paper among others that

$$\begin{aligned}&\left\|\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\right\|\\ &\le\frac{\|P\|^2\|B-A\|}{p^2}\begin{cases}\left(\frac{\mathcal{P}_f(m_2,p)-\mathcal{P}_f(m_1,p)}{m_2-m_1}\right) & \text{if } m_1\neq m_2,\\ \\ f'\left(\frac{m}{p}\right) & \text{if } m_1=m_2=m\end{cases}\end{aligned}$$

for all $A\ge m_1>0$, $B\ge m_2>0$ and $P\ge p>0$.

If f is nonnegative on $(0,\infty)$, then for all $C\ge n_1>0$, $D\ge n_2>0$, $Q>q>0$ we also have

$$\begin{aligned}&\left\|\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\right\|\\ &\le\frac{\|Q\|^2\|D-C\|}{q^2}\begin{cases}\left[\frac{\mathcal{P}_f(q,n_2)-\mathcal{P}_f(q,n_1)}{n_2-n_1}\right] & \text{if } n_2\neq n_1,\\ \\ \left[f\left(\frac{q}{n}\right)-\frac{q}{n}f'\left(\frac{q}{n}\right)\right] & \text{if } n_2=n_1=n.\end{cases}\end{aligned}$$

Some applications for *weighted operator geometric mean* and *relative operator entropy* are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B||\|\le L\|A-B\|$$

for any A, $B\in\mathcal{B}(H)$.

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However, as shown by Farforovskaya in [10], [11] and Kato in [18], the following inequality holds

$$\| |A| - |B| \| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log \left(\frac{\|A\| + \|B\|}{\|A - B\|} \right) \right) \tag{1.1}$$

for any A, $B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^* C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\| |A| - |B| \|_{HS} \leq \sqrt{2} \|A - B\|_{HS}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\| |A| - |B| \| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3 \right), \tag{1.2}$$

where

$$a_1 = \left\| A^{-1} \right\| \|A\| \text{ and } a_2 = \left\| A^{-1} \right\| + \left\| A^{-1} \right\|^3 \|A\|^2 .$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.3}$$

where f is an *operator monotone function* on $(0, \infty)$ and A, $B \geq aI_H > 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [12] and the references therein.

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [4, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.4}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left(\frac{u + t}{u + 1} \right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.5}$$

for all $t > 0$.

In 1934, K. Löwner [19] had given a definitive characterization of operator monotone functions as follows, see for instance [4, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} dw(\lambda) \tag{1.6}$$

where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure w on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0,1]$, [17]. The function ln is also operator monotone on $(0,\infty)$.

For other examples of operator monotone functions, see [15] and [16].

Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f : (0,\infty) \to \mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right), \ x > 0.$$

It is well known that (see for instance [21]), if $f : (0,\infty) \to \mathbb{R}$ is continuous on $(0,\infty)$, then for all A, $B > 0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A).$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [21].

The following inequality is of interest, see [21]:

Theorem 2. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A \geq C > 0$ and $B \geq D > 0$, then*

$$\mathcal{P}_f(A,B) \geq \mathcal{P}_f(C,D).$$

It is well known that (see [8] and [7] or [9]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A) \mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r : [0,\infty) \to [0,\infty),\ f_r(t) = t^r,\ r \in [0,1]$, then

$$P_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

We define the *weighted operator arithmetic mean* by

$$A\nabla_r B := (1-r)A + rB,\ r \in [0,1].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_r B \le A\nabla_r B$$

for any $r \in [0,1]$.

If we take the function $f = \ln$, then

$$P_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [13], [14] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [20].

Motivated by the above results, we show in this paper among others that

$$\begin{aligned}&\|\mathcal{P}_f(B,P) - \mathcal{P}_f(A,P)\| \\ &\le \frac{\|P\|^2\|B-A\|}{p^2}\begin{cases}\left(\frac{\mathcal{P}_f(m_2,p)-\mathcal{P}_f(m_1,p)}{m_2-m_1}\right) & \text{if } m_1 \ne m_2,\\ \\ f'\left(\frac{m}{p}\right) & \text{if } m_1 = m_2 = m\end{cases}\end{aligned}$$

for all $A \ge m_1 > 0$, $B \ge m_2 > 0$ and $P \ge p > 0$.

If f is nonnegative on $(0,\infty)$, then for all $C \ge n_1 > 0$, $D \ge n_2 > 0$, $Q > q > 0$ we also have

$$\begin{aligned}&\|\mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C)\| \\ &\le \frac{\|Q\|^2\|D-C\|}{q^2}\begin{cases}\left[\frac{\mathcal{P}_f(q,n_2)-\mathcal{P}_f(q,n_1)}{n_2-n_1}\right] & \text{if } n_2 \ne n_1,\\ \\ \left[f\left(\frac{q}{n}\right) - \frac{q}{n}f'\left(\frac{q}{n}\right)\right] & \text{if } n_2 = n_1 = n.\end{cases}\end{aligned}$$

Some applications for *weighted operator geometric mean* and *relative operator entropy* are also given.

2. Some Preliminary Facts

We start to the following identity of interest [6]:

Lemma 1. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all U, $V > 0$ we have*

$$\begin{aligned}f(V) - f(U) &= b(V-U) \\ &+ \int_0^\infty \lambda^2\left[\int_0^1 ((1-t)U + tV + \lambda)^{-1}\right. \\ &\left.\times (V-U)((1-t)U + tV + \lambda)^{-1}dt\right]dw(\lambda).\end{aligned} \tag{2.1}$$

Proof. Since the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6), then for $U,\ V>0$ we have the representation

$$f(V)-f(U)=b(V-U)+\int_0^\infty \lambda\left[V(V+\lambda)^{-1}-U(U+\lambda)^{-1}\right]dw(\lambda). \tag{2.2}$$

Observe that for $\lambda>0$

$$\begin{aligned}
&V(V+\lambda)^{-1}-U(U+\lambda)^{-1}\\
&=(V+\lambda-\lambda)(V+\lambda)^{-1}-(U+\lambda-\lambda)(U+\lambda)^{-1}\\
&=(V+\lambda)(V+\lambda)^{-1}-\lambda(V+\lambda)^{-1}-(U+\lambda)(U+\lambda)^{-1}+\lambda(U+\lambda)^{-1}\\
&=\lambda\left[(U+\lambda)^{-1}-(V+\lambda)^{-1}\right].
\end{aligned}$$

Therefore, (2.2) becomes, see also [16]

$$f(V)-f(U)=b(V-U)+\int_0^\infty \lambda^2\left[(U+\lambda)^{-1}-(V+\lambda)^{-1}\right]dw(\lambda). \tag{2.3}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for $C,\ D$ selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we replace in (2.6) $C=U+\lambda$ and $D=V+\lambda$ for $\lambda>0$, then

$$\begin{aligned}
&(U+\lambda)^{-1}-(V+\lambda)^{-1}\\
&=\int_0^1((1-t)U+tV+\lambda)^{-1}(V-U)((1-t)U+tV+\lambda)^{-1}\,dt.
\end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

We have the following identity for the difference of perspectives in the first variable [6]:

Theorem 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all A, B, $P>0$ we have*

$$\begin{aligned}
&\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P) \\
&=b(B-A)+\int_0^\infty \lambda^2\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right. \\
&\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda).
\end{aligned}\tag{2.8}$$

Proof. If we take $V=P^{-1/2}BP^{-1/2}$ and $U=P^{-1/2}AP^{-1/2}$ in (2.1), then we get

$$\begin{aligned}
&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right) \\
&=b\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right) \\
&+\int_0^\infty \lambda^2\left[\int_0^1\left((1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\right. \\
&\times\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right) \\
&\left.\times\left((1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}dt\right]dw(\lambda).
\end{aligned}\tag{2.9}$$

Observe that

$$P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}=P^{-1/2}(B-A)P^{-1/2},$$

and

$$(1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda=P^{-1/2}((1-t)A+tB+\lambda P)P^{-1/2},$$

which gives

$$\left((1-t)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}=P^{1/2}((1-t)A+tB+\lambda P)^{-1}P^{1/2}$$

and by (2.9),

$$\begin{aligned}
&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right) \\
&=bP^{-1/2}(B-A)P^{-1/2} \\
&+\int_0^\infty \lambda^2\left[\int_0^1 P^{1/2}((1-t)A+tB+\lambda P)^{-1}P^{1/2}P^{-1/2}(B-A)P^{-1/2}\right. \\
&\left.\times P^{1/2}((1-t)A+tB+\lambda P)^{-1}P^{1/2}dt\right]dw(\lambda) \\
&=bP^{-1/2}(B-A)P^{-1/2} \\
&+\int_0^\infty \lambda^2\left[\int_0^1 P^{1/2}((1-t)A+tB+\lambda P)^{-1}(B-A)\right. \\
&\left.\times((1-t)A+tB+\lambda P)^{-1}P^{1/2}dt\right]dw(\lambda).
\end{aligned}\tag{2.10}$$

If we multiply both sides of (2.10) by $P^{1/2}$ we obtain the desired identity (2.8). □

Lemma 2. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.6). Then for all* U, $V>0$ *we have*

$$\begin{aligned}\tilde{f}(V)-\tilde{f}(U) &= a(V-U)+\int_0^\infty \lambda\left(\int_0^1 (1+\lambda[(1-t)U+tV])^{-1}\right. \\ &\left.\times (V-U)(1+\lambda[(1-t)U+tV])^{-1}dt\right)dw(\lambda).\end{aligned} \tag{2.11}$$

Proof. From (1.6) we have

$$f(t)=a+bt+t\int_0^\infty \frac{\lambda}{t+\lambda}dw(\lambda),\ t>0.$$

If we put $\frac{1}{t}$ instead of t we get

$$\begin{aligned} f\left(\frac{1}{t}\right) &= a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty \frac{\lambda}{\frac{1}{t}+\lambda}dw(\lambda) \\ &= a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty \frac{t\lambda}{1+t\lambda}dw(\lambda)\end{aligned}$$

and by multiplication with $t>0$, we get

$$\tilde{f}(t)=b+ta+\int_0^\infty \frac{t\lambda}{1+t\lambda}dw(\lambda)=b+ta+\int_0^\infty\left(1-\frac{1}{1+t\lambda}\right)dw(\lambda).$$

Therefore

$$\tilde{f}(V)-\tilde{f}(U)=a(V-U)+\int_0^\infty\left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda). \tag{2.12}$$

From (2.6) we get

$$\begin{aligned}(1+U\lambda)^{-1}-(1+V\lambda)^{-1} &= \int_0^1 ((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}((1+V\lambda)-(1+U\lambda)) \\ &\times((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}dt \\ &= \int_0^1 \lambda(1+\lambda[(1-t)U+tV])^{-1}(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt.\end{aligned} \tag{2.13}$$

Therefore, by (2.12) we get

$$\begin{aligned}\tilde{f}(V)-\tilde{f}(U) &= a(V-U)+\int_0^\infty\left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda) \\ &= a(V-U)+\int_0^\infty \lambda\left(\int_0^1 (1+\lambda[(1-t)U+tV])^{-1}\right. \\ &\left.\times(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt\right)dw(\lambda)\end{aligned} \tag{2.14}$$

and the identity (2.11) is proved. □

The dual identity is as follows [6]:

Theorem 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all C, D, $Q>0$ we have*

$$\begin{aligned}(2.15)\quad & \mathcal{P}_{\tilde{f}}(D,Q)-\mathcal{P}_{\tilde{f}}(C,Q)\\ &= a(D-C)+\int_0^\infty \lambda\left(\int_0^1 Q\left[Q+\lambda\left[(1-t)C+tD\right]\right]^{-1}(D-C)\right.\\ &\left.\times\left[Q+\lambda\left[(1-t)C+tD\right]\right]^{-1}Qdt\right)dw(\lambda).\end{aligned}$$

Proof. If we take $V=Q^{-1/2}DQ^{-1/2}$ and $U=Q^{-1/2}CQ^{-1/2}$ in (2.11), then we get

$$\begin{aligned}(2.16)\quad & \tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)-\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)\\ &= a\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\ &+\int_0^\infty \lambda\left(\int_0^1 \left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}\right.\\ &\times\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\ &\left.\times\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}dt\right)dw(\lambda)\end{aligned}$$

$$\begin{aligned}&= aQ^{-1/2}(D-C)Q^{-1/2}\\ &+\int_0^\infty \lambda\left(\int_0^1 \left[Q^{-1/2}\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}\right.\\ &\left.\times Q^{-1/2}(D-C)Q^{-1/2}\left[Q^{-1/2}\left(Q+\lambda\left[(1-t)C+tD\right]\right)Q^{-1/2}\right]^{-1}dt\right)dw(\lambda)\\ &= aQ^{-1/2}(D-C)Q^{-1/2}\\ &+\int_0^\infty \lambda\left(\int_0^1 Q^{1/2}\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}(D-C)\right.\\ &\left.\times\left[\left(Q+\lambda\left[(1-t)C+tD\right]\right)\right]^{-1}Q^{1/2}dt\right)dw(\lambda).\end{aligned}$$

If we multiply both sides by $Q^{1/2}$ we get the desired result (2.15). □

Corollary 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all C, D, $Q>0$ we have*

$$\begin{aligned}(2.17)\quad & \mathcal{P}_{f}(Q,D)-\mathcal{P}_{f}(Q,C)\\ &= a(D-C)+\int_0^\infty \lambda\left(\int_0^1 Q\left[Q+\lambda\left[(1-t)C+tD\right]\right]^{-1}(D-C)\right.\\ &\left.\times\left[Q+\lambda\left[(1-t)C+tD\right]\right]^{-1}Qdt\right)dw(\lambda).\end{aligned}$$

We also have:

Corollary 2. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.6). Then for all* A, B, C, $D>0$ *we have*

$$\begin{aligned}(2.18)\qquad &\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)\\ &=b(A-C)+a(B-D)\\ &+\int_0^\infty \lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\ &\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\\ &+\int_0^\infty \lambda\left(\int_0^1 C[C+\lambda[(1-t)D+tB]]^{-1}(B-D)\right.\\ &\left.\times[C+\lambda[(1-t)D+tB]]^{-1}Cdt\right)dw(\lambda).\end{aligned}$$

Proof. Observe that

$$(2.19)\quad \mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)=\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)+\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D).$$

Since, by (2.8),

$$\begin{aligned}(2.20)\qquad &\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)\\ &=b(A-C)+\int_0^\infty \lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\ &\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\end{aligned}$$

and by (2.17),

$$\begin{aligned}(2.21)\qquad &\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D)\\ &=a(B-D)+\int_0^\infty \lambda\left(\int_0^1 C[C+\lambda[(1-t)D+tB]]^{-1}(B-D)\right.\\ &\left.\times[(C+\lambda[(1-t)D+tB])]^{-1}Cdt\right)dw(\lambda),\end{aligned}$$

hence by (2.19)-(2.21) we obtain (2.18). □

3. Lipschitz Type Inequalities

We have the following Lipschitz type inequality for the perspective in the first variable:

Theorem 5. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.6). Then for all* $A\geq m_1>0$, $B\geq m_2>0$ *and* $P\geq p>0$ *we have*

$$\begin{aligned}(3.1)\qquad &\|\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)-b(B-A)\|\\ &\leq\frac{\|P\|^2\|B-A\|}{p^2}\begin{cases}\left(\frac{\mathcal{P}_f(m_2,p)-\mathcal{P}_f(m_1,p)}{m_2-m_1}-b\right) & \text{if } m_1\neq m_2,\\[2ex] \left(f'\left(\frac{m}{p}\right)-b\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned}$$

Proof. Assume that $m_1 \neq m_2$. From (2.8), by taking the norm, we get that

$$\begin{aligned} (3.2)\quad & \left\| \mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) - b(B-A) \right\| \\ & \leq \int_0^\infty \lambda^2 \left[\int_0^1 \left\| P\left((1-t)A + tB + \lambda P\right)^{-1}(B-A) \right.\right. \\ & \left.\left. \times \left((1-t)A + tB + \lambda P\right)^{-1} P \right\| dt \right] dw(\lambda) \\ & \leq \|P\|^2 \|B-A\| \int_0^\infty \lambda^2 \left(\int_0^1 \left\| \left((1-t)A + tB + \lambda P\right)^{-1} \right\|^2 dt \right) dw(\lambda) \end{aligned}$$

for $A,\ B,\ P > 0$.

We have

$$(1-t)A + tB + \lambda P \geq (1-t)m_1 + tm_2 + \lambda p,$$

which implies that

$$\left((1-t)A + tB + \lambda P\right)^{-1} \leq \left((1-t)m_1 + tm_2 + \lambda p\right)^{-1}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

By taking the norm, we then get

$$\left\| \left((1-t)A + tB + \lambda P\right)^{-1} \right\| \leq \left((1-t)m_1 + tm_2 + \lambda p\right)^{-1},$$

which implies that

$$\left\| \left((1-t)A + tB + \lambda P\right)^{-1} \right\|^2 \leq \left((1-t)m_1 + tm_2 + \lambda p\right)^{-2},$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

By (3.2) we derive

$$\begin{aligned} (3.3)\quad & \left\| \mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) - b(B-A) \right\| \\ & \leq \|P\|^2 \|B-A\| \int_0^\infty \lambda^2 \left(\int_0^1 \left((1-t)m_1 + tm_2 + \lambda p\right)^{-2} dt \right) dw(\lambda). \end{aligned}$$

From the identity (2.8) for $B = m_2,\ A = m_1$ and $P = p$ we get

$$\begin{aligned} & \mathcal{P}_f(m_2,p) - \mathcal{P}_f(m_1,p) \\ & = b(m_2 - m_1) + \int_0^\infty \lambda^2 \left[\int_0^1 p\left((1-t)m_1 + tm_2 + \lambda p\right)^{-1}(m_2 - m_1) \right. \\ & \left. \times \left((1-t)m_1 + tm_2 + \lambda p\right)^{-1} p dt \right] dw(\lambda) \\ & = b(m_2 - m_1) \\ & + (m_2 - m_1) p^2 \int_0^\infty \lambda^2 \left(\int_0^1 \left((1-t)m_1 + tm_2 + \lambda p\right)^{-2} dt \right) dw(\lambda), \end{aligned}$$

which gives

$$\begin{aligned} & \int_0^\infty \lambda^2 \left(\int_0^1 \left((1-t)m_1 + tm_2 + \lambda p\right)^{-2} dt \right) dw(\lambda) \\ & = \frac{\mathcal{P}_f(m_2,p) - \mathcal{P}_f(m_1,p)}{(m_2 - m_1) p^2} - \frac{b}{p^2} \end{aligned}$$

and the inequality in the first branch of (3.1) is proved.

Let $m_1 = m_2 = m$. Let $\epsilon > 0$. Then $B + \epsilon \geq m + \epsilon > 0$. From the first branch of (3.1) we get

$$\begin{aligned} &\left\|\mathcal{P}_f(B+\epsilon, P) - \mathcal{P}_f(A, P) - b(B+\epsilon - A)\right\| \\ &\leq \|P\|^2 \|B+\epsilon - A\| \left[\frac{\mathcal{P}_f(m+\epsilon, p) - \mathcal{P}_f(m, p)}{\epsilon m p^2} - \frac{b}{p^2}\right]. \end{aligned} \tag{3.4}$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of f,

$$\left\|\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A)\right\| \leq \|P\|^2 \|B - A\| \left(\frac{\partial \mathcal{P}_f(m, p)}{\partial x p^2} - \frac{b}{p^2}\right). \tag{3.5}$$

Since

$$\mathcal{P}_f(x, y) := y f\left(\frac{x}{y}\right),$$

hence

$$\frac{\partial \mathcal{P}_f(x, y)}{\partial x} := f'\left(\frac{x}{y}\right)$$

which give that

$$\frac{\partial \mathcal{P}_f(m, p)}{\partial x} = f'\left(\frac{m}{p}\right)$$

and by (3.5) we deduce the second inequality in (3.1). □

If the parameter $b \geq 0$ is not available, then we can state the following more practical bounds:

Corollary 3. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$. Then for all $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $P \geq p > 0$ we have*

$$\begin{aligned} &\left\|\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P)\right\| \\ &\leq \frac{\|P\|^2 \|B - A\|}{p^2} \begin{cases} \left(\frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1}\right) & \text{if } m_1 \neq m_2, \\ f'\left(\frac{m}{p}\right) & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.6}$$

Proof. By the triangle inequality we get from (3.1) that

$$\begin{aligned} &\left\|\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P)\right\| - b\|B - A\| \\ &\leq \left\|\mathcal{P}_f(B, P) - \mathcal{P}_f(A, P) - b(B - A)\right\| \\ &\leq \begin{cases} \frac{\|P\|^2\|B-A\|}{p^2} \left[\frac{\mathcal{P}_f(m_2, p) - \mathcal{P}_f(m_1, p)}{m_2 - m_1} - b\right] & \text{if } m_1 \neq m_2 \\ \frac{\|P\|^2\|B-A\|}{p^2} \left[f'\left(\frac{m}{p}\right) - b\right] & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

which implies that

$$\|\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\| \tag{3.7}$$
$$\leq \begin{cases} \frac{\|P\|^2\|B-A\|}{p^2}\left[\frac{\mathcal{P}_f(m_2,p)-\mathcal{P}_f(m_1,p)}{m_2-m_1}-b\right]+b\|B-A\| \\ \\ \frac{\|P\|^2\|B-A\|}{p^2}\left[f'\left(\frac{m}{p}\right)-b\right]+b\|B-A\| \end{cases}$$
$$= \begin{cases} \frac{\|P\|^2\|B-A\|}{p^2}\left[\frac{\mathcal{P}_f(m_2,p)-\mathcal{P}_f(m_1,p)}{m_2-m_1}\right]+b\|B-A\|\left(1-\frac{\|P\|^2}{p^2}\right) \\ \\ \frac{\|P\|^2\|B-A\|}{p^2}f'\left(\frac{m}{p}\right)+b\|B-A\|\left(1-\frac{\|P\|^2}{p^2}\right). \end{cases}$$

Observe that $1-\frac{\|P\|^2}{p^2}\leq 0$ and since $b\geq 0$, we get by (3.7) the desired result (3.6). □

Theorem 6. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all $C\geq n_1>0$, $D\geq n_2>0$, $Q>q>0$ we have*

$$\|\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-a(D-C)\| \tag{3.8}$$
$$\leq \frac{\|Q\|^2\|D-C\|}{q^2}\begin{cases} \left[\frac{\mathcal{P}_f(q,n_2)-\mathcal{P}_f(q,n_1)}{n_2-n_1}-a\right] & \text{if } n_2\neq n_1, \\ \\ \left[f\left(\frac{q}{n}\right)-\frac{q}{n}f'\left(\frac{q}{n}\right)-a\right] & \text{if } n_2=n_1=n. \end{cases}$$

Proof. From the representation (2.17) we get, by taking the norm, that

$$\|\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-a(D-C)\| \tag{3.9}$$
$$\leq \|Q\|^2\|D-C\|\int_0^\infty\lambda\left(\int_0^1\left\|[Q+\lambda[(1-t)C+tD]]^{-1}\right\|^2dt\right)dw(\lambda).$$

Since $C\geq n_1>0$, $D\geq n_2>0$, $Q>q>0$,

$$Q+\lambda[(1-t)C+tD]\geq q+\lambda[(1-t)n_1+tn_2],$$

namely

$$(Q+\lambda[(1-t)C+tD])^{-1}\leq(q+\lambda[(1-t)n_1+tn_2])^{-1},$$

which implies that

$$\left\|(Q+\lambda[(1-t)C+tD])^{-1}\right\|\leq(q+\lambda[(1-t)n_1+tn_2])^{-1}.$$

Therefore

$$\left\|(Q+\lambda[(1-t)C+tD])^{-1}\right\|^2\leq(q+\lambda[(1-t)n_1+tn_2])^{-2}$$

and by integration,

$$\int_0^\infty\lambda\left(\int_0^1\left\|[Q+\lambda[(1-t)C+tD]]^{-1}\right\|^2dt\right)dw(\lambda) \tag{3.10}$$
$$\leq\int_0^\infty\lambda\left(\int_0^1(q+\lambda[(1-t)n_1+tn_2])^{-2}dt\right)dw(\lambda).$$

By utilising (3.9) and (3.10) we obtain

$$\begin{aligned} &\left\| \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - a(D-C) \right\| \\ &\leq \|Q\|^2 \|D-C\| \int_0^\infty \lambda \left(\int_0^1 (q + \lambda[(1-t)n_1 + tn_2])^{-2} dt \right) dw(\lambda). \end{aligned} \tag{3.11}$$

If in the identity (2.17) we choose $D = n_2$, $C = n_1$ and $Q = q$ then we get

$$\begin{aligned} &\mathcal{P}_f(q,n_2) - \mathcal{P}_f(q,n_1) \\ &= a(n_2 - n_1) + \int_0^\infty \lambda \left(\int_0^1 q[q + \lambda[(1-t)n_1 + tn_2]]^{-1}(n_2 - n_1) \right. \\ &\left. \times [q + \lambda[(1-t)n_1 + tn_2]]^{-1} q dt \right) dw(\lambda) \\ &= a(n_2 - n_1) \\ &+ q^2(n_2 - n_1) \int_0^\infty \lambda \left(\int_0^1 [q + \lambda[(1-t)n_1 + tn_2]]^{-2} dt \right) dw(\lambda). \end{aligned} \tag{3.12}$$

If $n_2 \neq n_1$, then by (3.12) we get

$$\begin{aligned} &\int_0^\infty \lambda \left(\int_0^1 [q + \lambda[(1-t)n_1 + tn_2]]^{-2} dt \right) dw(\lambda) \\ &= \frac{1}{q^2} \left[\frac{\mathcal{P}_f(q,n_2) - \mathcal{P}_f(q,n_1)}{n_2 - n_1} - a \right]. \end{aligned} \tag{3.13}$$

By making use of (3.11) and (3.13) we derive the first branch in (3.8).

Let $n_1 = n_2 = n$. Let $\epsilon > 0$. Then $D + \epsilon \geq n + \epsilon > 0$. From the first branch of (3.8) we get

$$\begin{aligned} &\left\| \mathcal{P}_f(Q, D+\epsilon) - \mathcal{P}_f(Q,C) - a(D+\epsilon-C) \right\| \\ &\leq \frac{\|Q\|^2 \|D+\epsilon-C\|}{q^2} \left[\frac{\mathcal{P}_f(q,n+\epsilon) - \mathcal{P}_f(q,n)}{\epsilon} - a \right] \end{aligned}$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of f,

$$\begin{aligned} &\left\| \mathcal{P}_f(Q,D) - \mathcal{P}_f(Q,C) - a(D-C) \right\| \\ &\leq \frac{\|Q\|^2 \|D-C\|}{q^2} \left[\frac{\partial \mathcal{P}_f(q,n)}{\partial y} - a \right]. \end{aligned} \tag{3.14}$$

Since

$$\mathcal{P}_f(x,y) := y f\left(\frac{x}{y}\right),$$

hence

$$\begin{aligned} \frac{\partial \mathcal{P}_f(x,y)}{\partial y} &:= f\left(\frac{x}{y}\right) + y f'\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)' = f\left(\frac{x}{y}\right) - \frac{yx}{y^2} f'\left(\frac{x}{y}\right) \\ &= f\left(\frac{x}{y}\right) - \frac{x}{y} f'\left(\frac{x}{y}\right), \end{aligned}$$

which give that

$$\frac{\partial \mathcal{P}_f(q,n)}{\partial y} = f\left(\frac{q}{n}\right) - \frac{q}{n} f'\left(\frac{q}{n}\right),$$

and the second branch of (3.8) is also proved. □

Remark 1. *If $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then $a=f(0)$ and we have*

$$\begin{aligned}(3.15)\qquad &\left\|\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)-f(0)(D-C)\right\|\\ &\le\frac{\|Q\|^2\|D-C\|}{q^2}\begin{cases}\left[\frac{\mathcal{P}_f(q,n_2)-\mathcal{P}_f(q,n_1)}{n_2-n_1}-f(0)\right] & \text{if } n_2\ne n_1,\\ \\ \left[f\left(\frac{q}{n}\right)-\frac{q}{n}f'\left(\frac{q}{n}\right)-f(0)\right] & \text{if } n_2=n_1=n\end{cases}\end{aligned}$$

provided $C\ge n_1>0$, $D\ge n_2>0$, $Q>q>0$.

Corollary 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone and nonnegative in $(0,\infty)$. Then for all $C\ge n_1>0$, $D\ge n_2>0$, $Q>q>0$ we have*

$$\begin{aligned}(3.16)\qquad &\left\|\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\right\|\\ &\le\frac{\|Q\|^2\|D-C\|}{q^2}\begin{cases}\left[\frac{\mathcal{P}_f(q,n_2)-\mathcal{P}_f(q,n_1)}{n_2-n_1}\right] & \text{if } n_2\ne n_1,\\ \\ \left[f\left(\frac{q}{n}\right)-\frac{q}{n}f'\left(\frac{q}{n}\right)\right] & \text{if } n_2=n_1=n.\end{cases}\end{aligned}$$

The proof is similar to the one provided in the proof of Corollary 3.

Corollary 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.6). Then for all $A\ge m_1>0$, $B\ge m_2>0$, $C\ge n_1>0$, $D\ge n_2>0$,*

$$\begin{aligned}(3.17)\qquad &\left\|\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)-b(A-C)-a(B-D)\right\|\\ &\le\frac{\|B\|^2}{m_2}\|A-C\|\begin{cases}\left[\frac{\mathcal{P}_f(m_1,m_2)-\mathcal{P}_f(n_1,m_2)}{m_1-n_1}-b\right], & m_1\ne n_1\\ \\ \left[f'\left(\frac{m_1}{m_2}\right)-b\right], & m_1=n_1\end{cases}\\ &+\frac{\|C\|^2}{n_1}\|B-D\|\begin{cases}\left[\frac{\mathcal{P}_f(n_1,m_2)-\mathcal{P}_f(n_1,n_2)}{m_2-n_2}-a\right], & m_2\ne n_2\\ \\ \left[f\left(\frac{n_1}{n_2}\right)-\frac{n_1}{n_2}f'\left(\frac{n_1}{n_2}\right)-a\right], & m_2=n_2.\end{cases}\end{aligned}$$

Proof. From Theorems 5 and 6 we have

$$\begin{aligned}&\left\|\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)-b(A-C)-a(B-D)\right\|\\ &=\left\|\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)+\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D)-b(A-C)-a(B-D)\right\|\\ &\le\left\|\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)-b(A-C)\right\|\\ &+\left\|\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D)-a(B-D)\right\|\\ &\le\frac{\|B\|^2}{m_2}\|A-C\|\begin{cases}\left[\frac{\mathcal{P}_f(m_1,m_2)-\mathcal{P}_f(n_1,m_2)}{m_1-n_1}-b\right], & m_1\ne n_1\\ \\ \left[f'\left(\frac{m_1}{m_2}\right)-b\right], & m_1=n_1\end{cases}\\ &+\frac{\|C\|^2}{n_1}\|B-D\|\begin{cases}\left[\frac{\mathcal{P}_f(n_1,m_2)-\mathcal{P}_f(n_1,n_2)}{m_2-n_2}-a\right], & m_2\ne n_2\\ \\ \left[f\left(\frac{n_1}{n_2}\right)-\frac{n_1}{n_2}f'\left(\frac{n_1}{n_2}\right)-a\right], & m_2=n_2,\end{cases}\end{aligned}$$

which proves (3.17). □

4. Some Examples

If $f_r : [0,\infty) \to [0,\infty),\ f_r(t) = t^r,\ r \in [0,1]$, then

$$P_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

Observe also that

$$P_{f_r}(x,y) = y^{1/2}\left(y^{-1/2}xy^{-1/2}\right)^r y^{1/2} = x^r y^{1-r},\ x,\ y > 0.$$

From (3.6) we get for the power function

$$\|P\sharp_r B - P\sharp_r A\| \le \frac{\|P\|^2\|B-A\|}{p^{r+1}}\begin{cases}\left(\frac{m_2^r - m_1^r}{m_2 - m_1}\right) & \text{if } m_1 \ne m_2,\\ \\ rm^{r-1} & \text{if } m_1 = m_2 = m\end{cases} \tag{4.1}$$

for all $A \ge m_1 > 0$, $B \ge m_2 > 0$ and $P \ge p > 0$.

From (3.16) we obtain

$$\|D\sharp_r Q - C\sharp_r Q\| \le \frac{\|Q\|^2\|D-C\|}{q^{2-r}}\begin{cases}\left(\frac{n_2^{1-r} - n_1^{1-r}}{n_2 - n_1}\right) & \text{if } n_2 \ne n_1,\\ \\ \frac{(1-r)}{n^r} & \text{if } n_2 = n_1 = n\end{cases} \tag{4.2}$$

for all $C \ge n_1 > 0$, $D \ge n_2 > 0$, $Q > q > 0$.

If we take the function $f = \ln$, then

$$P_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Observe also that

$$P_{\ln}(x,y) := y^{1/2}\ln\left(y^{-1/2}xy^{-1/2}\right)y^{1/2} = y\ln\left(\frac{x}{y}\right),\ x,\ y > 0.$$

If we use the inequality (3.6) for the logarithmic function, then we obtain

$$\|S(P|B) - S(P|A)\| \le \frac{\|P\|^2\|B-A\|}{p}\begin{cases}\left(\frac{\ln m_2 - \ln m_1}{m_2 - m_1}\right) & \text{if } m_1 \ne m_2,\\ \\ \frac{1}{m} & \text{if } m_1 = m_2 = m,\end{cases} \tag{4.3}$$

where $A \ge m_1 > 0$, $B \ge m_2 > 0$ and $P \ge p > 0$.

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