BOUNDS ON PARTIAL CONCAVITY OF NONCOMMUTATIVE PERSPECTIVES FOR OPERATOR MONOTONE FUNCTIONS IN HILBERT SPACES

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Abstract. Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}dw(\lambda),$$

where $a\in\mathbb{R}$, $b\geq 0$ and w a positive measure on $(0,\infty)$. We can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$.

In this paper we show among others that, if A, B, $P>0$,

$$\begin{aligned}0&\leq\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]\\&=\int_0^\infty\lambda^2\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda)\end{aligned}$$

and if C, D, $Q>0$,

$$\begin{aligned}0&\leq\mathcal{P}_f\left(Q,\frac{C+D}{2}\right)-\frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]\\&=\int_0^\infty\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda),\end{aligned}$$

where $\mathcal{P}_{f_{-1}}(B,A):=AB^{-1}A$. Several upper and lower bounds for these partial differences are given. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

1. Introduction

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

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If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [14]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A). \tag{1.1}$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [14].

The following inequality is of interest, see [14]:

Theorem 1. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_f(A,B) \geq \mathcal{P}_f(C,D). \tag{1.2}$$

It is well known that (see [4] and [3] or [5]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r:[0,\infty)\to[0,\infty)$, $f_r(t)=t^r$, $r\in[0,1]$, then

$$\mathcal{P}_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

If we take the function $f=\ln$, then

$$\mathcal{P}_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: \mathcal{S}(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [6], [7] defined the *relative operator entropy* $\mathcal{S}(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [13].

If $f_{-1}:(0,\infty)\to(0,\infty)$, $f_{-1}(x)=x^{-1}$, then $\tilde{f}_{-1}:(0,\infty)\to(0,\infty)$ is given by $\tilde{f}_{-1}(x)=x^2=f_2(x)$ and

$$\mathcal{P}_{f_{-1}}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2} = AB^{-1}A = \mathcal{P}_{f_2}(A,B)$$

for $A,\ B>0$.

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.3}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.4}$$

for all $t>0$.

In 1934, K. Löwner [12] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}dw(\lambda) \tag{1.5}$$

where $a\in\mathbb{R}$ and $b\geq0$ and a positive measure w on $(0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}dw(\lambda)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [11]. The function ln is also operator monotone on $(0,\infty)$.

For other examples of operator monotone functions, see [8] and [9].

In this paper we show among others that, if A, B, $P>0$,

$$\begin{aligned}0&\leq\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]\\&=\int_0^\infty\lambda^2\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda)\end{aligned}$$

and if C, D, $Q>0$,

$$\begin{aligned}0&\leq\mathcal{P}_f\left(Q,\frac{C+D}{2}\right)-\frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]\\&=\int_0^\infty\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda),\end{aligned}$$

where $\mathcal{P}_{f_{-1}}(B,A) := AB^{-1}A$. Several upper and lower bounds for these partial differences are given. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

2. Some Preliminary Facts

We start with the following elementary identity that give a simple proof for the fact that the function $f(t) = t^{-1}$ is operator convex on $(0,\infty)$, see for instance [10, p. 8]:

Lemma 1. *For any C, $D > 0$ we have*

$$\frac{C^{-1}+D^{-1}}{2} - \left(\frac{C+D}{2}\right)^{-1} = \frac{\left(C^{-1}-D^{-1}\right)\left(C^{-1}+D^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)}{2} \geq 0. \tag{2.1}$$

If more assumptions are made for the operators C and D, then one can obtain the following lower and upper bounds:

Corollary 1. *Assume that $0 < \alpha \leq C \leq \beta$ and $0 < \gamma \leq D \leq \delta$ for some constants α, β, γ, δ. Then*

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2} \leq \frac{C^{-1}+D^{-1}}{2} - \left(\frac{C+D}{2}\right)^{-1} \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}. \tag{2.2}$$

Proof. We have $\beta^{-1} \leq C^{-1} \leq \alpha^{-1}$ and $\delta^{-1} \leq D^{-1} \leq \gamma^{-1}$, which gives

$$\beta^{-1}+\delta^{-1} \leq C^{-1}+D^{-1} \leq \alpha^{-1}+\gamma^{-1}$$

namely

$$\left(\alpha^{-1}+\gamma^{-1}\right)^{-1} \leq \left(C^{-1}+D^{-1}\right)^{-1} \leq \left(\beta^{-1}+\delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(C^{-1}-D^{-1}\right)$ and dividing by 2, we get

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2} \leq \frac{\left(C^{-1}-D^{-1}\right)\left(C^{-1}+D^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)}{2} \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}.$$

□

A continuous function $g : \mathcal{S}\mathrm{C}_I(H) \to \mathrm{D}(H)$ is said to be *Gâteaux differentiable* in $C \in \mathcal{S}\mathrm{C}_I(H)$, the class of selfadjoint operators on I, along the direction $D \in \mathrm{B}(H)$ if the following limit exists in the strong topology of $\mathrm{B}(H)$

$$\nabla g_C(D) := \lim_{s\to 0} \frac{g(C+sD)-g(C)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $D \in \mathrm{B}(H)$, then we say that g is *Gâteaux differentiable* in C and we can write $g \in \mathcal{G}(C)$. If this is true for any C in an open set $\mathcal{S}$ from $\mathcal{S}\mathrm{C}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $C, D \in \mathcal{S}\mathrm{C}_I(H)$ we consider the segment of selfadjoint operators

$$[C, D] := \{(1-t)C + tD \mid t \in [0,1]\}.$$

We observe that $C, D \in [C, D]$ and $[C, D] \subset \mathcal{S}\mathrm{C}_I(H)$.

We have the following gradient inequalities, see for instance [2]:

Lemma 2. *Let f be an operator convex function on I and $C, D \in \mathcal{S}C_I(H)$, with $C \neq D$. If $f \in \mathcal{G}([C, D])$, then*

$$\nabla_D f(D-C) \geq f(D) - f(C) \geq \nabla_C f(D-C). \tag{2.4}$$

Let $T, S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the *Gâteaux derivative* is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = -T^{-1}ST^{-1} \tag{2.5}$$

for $T, S > 0$.

Using (2.5) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1} \tag{2.6}$$

for all $C, D > 0$.

If

$$m \leq D - C \leq M$$

for some constants m, M, then

$$mD^{-2} \leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1} \leq MC^{-2}$$

and by (2.6) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.7}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2}. \tag{2.8}$$

Corollary 2. *Assume that $0 < \alpha \leq C \leq \beta$, $0 < \gamma \leq D \leq \delta$ and $0 < m \leq D - C \leq M$ for some constants $\alpha, \beta, \gamma, \delta, m, M$. Then*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} \frac{m^2}{\delta^4} \\ &\leq \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} \left(C^{-1} - D^{-1}\right)^2 \leq \frac{C^{-1} + D^{-1}}{2} - \left(\frac{C+D}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1} \left(C^{-1} - D^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1} \frac{M^2}{\alpha^4}. \end{aligned} \tag{2.9}$$

Proof. From (2.8) we have

$$0 < \frac{m}{\delta^2} \le C^{-1} - D^{-1} \le \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \le \left(C^{-1} - D^{-1}\right)^2 \le \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.9). □

Remark 1. *If the positive operators C, D are separated, namely $0 < \alpha \le C \le \beta < \gamma \le D \le \delta$ for some constants α, β, γ, δ, then obviously $0 < \gamma - \beta \le D - C \le \delta - \alpha$ and by (2.9) for $m = \gamma - \beta$ and $M = \delta - \alpha$, we get*

$$\begin{aligned} (2.10)\qquad 0 &< \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} \frac{(\gamma - \beta)^2}{\delta^4} \le \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(C^{-1} - D^{-1}\right)^2 \\ &\le \frac{C^{-1} + D^{-1}}{2} - \left(\frac{C + D}{2}\right)^{-1} \\ &\le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(C^{-1} - D^{-1}\right)^2 \le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1} \frac{(\delta - \alpha)^2}{\alpha^4}. \end{aligned}$$

If we put together the above results for the noncommutative perspective $\mathcal{P}_{f_{-1}}(A, P)$, then we can state the following result:

Theorem 3. *Let A, B, $P > 0$.*

(i) *We have the representations*

$$\begin{aligned} (2.11)\qquad 0 &\le \frac{\mathcal{P}_{f_{-1}}(A, P) + \mathcal{P}_{f_{-1}}(B, P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A + B}{2}, P\right) \\ &= \frac{1}{2} P\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right) P \\ &= \frac{1}{2}\mathcal{P}_{f_{-1}}\left(\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)\left(A^{-1} - B^{-1}\right), P\right) \\ &= \frac{1}{2}\left(\mathcal{P}_{f_{-1}}(A, P) - \mathcal{P}_{f_{-1}}(B, P)\right)\left(\mathcal{P}_{f_{-1}}(A, P) + \mathcal{P}_{f_{-1}}(B, P)\right)^{-1} \\ &\times \left(\mathcal{P}_{f_{-1}}(A, P) - \mathcal{P}_{f_{-1}}(B, P)\right). \end{aligned}$$

(ii) *Assume that $0 < \alpha P \le A \le \beta P$ and $0 < \gamma P \le B \le \delta P$ for some constants α, β, γ, δ, then*

$$\begin{aligned} (2.12)\qquad 0 &\le \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} P\left(A^{-1} - B^{-1}\right) P\left(A^{-1} - B^{-1}\right) P \\ &\le \frac{\mathcal{P}_{f_{-1}}(A, P) + \mathcal{P}_{f_{-1}}(B, P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A + B}{2}, P\right) \\ &\le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1} P\left(A^{-1} - B^{-1}\right) P\left(A^{-1} - B^{-1}\right) P. \end{aligned}$$

(iii) *Assume that* $0 < \alpha P \le A \le \beta P$, $0 < \gamma P \le B \le \delta P$ *and* $0 < mP \le B - A \le MP$ *for some constants* α, β, γ, δ, m, M, *then*

$$0 < \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4}P \le \frac{\mathcal{P}_{f_{-1}}(A,P) + \mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}, P\right) \le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}P. \tag{2.13}$$

(iv) *Assume that* $0 < \alpha P \le A \le \beta P < \gamma P \le B \le \delta P$ *for some constants* α, β, γ, δ, *then*

$$0 < \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\frac{(\gamma - \beta)^2}{\delta^4}P \le \frac{\mathcal{P}_{f_{-1}}(A,P) + \mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}, P\right) \le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\frac{(\delta - \alpha)^2}{\alpha^4}P. \tag{2.14}$$

Proof. (i) If we take in (2.1) $C = P^{-1/2}AP^{-1/2}$ and $D = P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned} 0 \le{}& \frac{\left(P^{-1/2}AP^{-1/2}\right)^{-1} + \left(P^{-1/2}BP^{-1/2}\right)^{-1}}{2} - \left(\frac{P^{-1/2}AP^{-1/2} + P^{-1/2}BP^{-1/2}}{2}\right)^{-1} \\ ={}& \frac{1}{2}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1} - \left(P^{-1/2}BP^{-1/2}\right)^{-1}\right) \\ &\times \left(\left(P^{-1/2}AP^{-1/2}\right)^{-1} + \left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^{-1} \\ &\times \left(\left(P^{-1/2}AP^{-1/2}\right)^{-1} - \left(P^{-1/2}BP^{-1/2}\right)^{-1}\right), \end{aligned} \tag{2.15}$$

namely

$$\begin{aligned} 0 \le{}& P^{1/2}\frac{A^{-1} + B^{-1}}{2}P^{1/2} - P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\ ={}& \frac{1}{2}P^{1/2}\left(A^{-1} - B^{-1}\right)P^{1/2} \\ &\times P^{-1/2}\left(A^{-1} + B^{-1}\right)^{-1}P^{-1/2}P^{1/2}\left(A^{-1} - B^{-1}\right)P^{1/2}, \end{aligned}$$

that is equivalent to

$$\begin{aligned} 0 \le{}& P^{1/2}\frac{A^{-1} + B^{-1}}{2}P^{1/2} - P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\ ={}& \frac{1}{2}P^{1/2}\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)P^{1/2}. \end{aligned}$$

If we multiply this inequality both sides by $P^{1/2}$ we get

$$
\begin{aligned}
0 &\leq \frac{PA^{-1}P + PB^{-1}P}{2} - P\left(\frac{A+B}{2}\right)^{-1} P \\
&= \frac{1}{2}P\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)P \\
&= \frac{1}{2}P\left[\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)\left(A^{-1} - B^{-1}\right)\right]^{-1}P
\end{aligned}
$$

which is the first part of (2.11).

Observe also that

$$
\begin{aligned}
&\frac{1}{2}P\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)P \\
&= \frac{1}{2}\left(PA^{-1}P - PB^{-1}P\right)P^{-1}\left(A^{-1} + B^{-1}\right)^{-1}P^{-1}\left(PA^{-1}P - PB^{-1}P\right) \\
&= \frac{1}{2}\left(PA^{-1}P - PB^{-1}P\right)\left(PA^{-1}P + PB^{-1}P\right)^{-1}\left(PA^{-1}P - PB^{-1}P\right),
\end{aligned}
$$

which proves the last part of (2.11).

(ii) From (2.1) we also have by multiplying both sides by P, that

$$
\begin{aligned}
&P\left[\frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1}\right]P \\
&= \frac{PA^{-1}P + PB^{-1}P}{2} - P\left(\frac{A+B}{2}\right)^{-1}P \\
&= \frac{1}{2}P\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)P \geq 0,
\end{aligned} \tag{2.16}
$$

$A, B, P > 0$.

If $0 < \alpha P \leq A \leq P\beta$ and $0 < \gamma P \leq B \leq \delta P$ for some constants α, β, γ, δ, then by multiplying both sides with $P^{-1/2}$ we get $0 < \alpha \leq P^{-1/2}AP^{-1/2} \leq \beta$ and $0 < \gamma \leq P^{-1/2}BP^{-1/2} \leq \delta$. If we use inequality (2.2) for $C = P^{-1/2}AP^{-1/2}$ and $D = P^{-1/2}BP^{-1/2}$, then we get

$$
\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1} - \left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^2 \\
&\leq \frac{\left(P^{-1/2}AP^{-1/2}\right)^{-1} + \left(P^{-1/2}BP^{-1/2}\right)^{-1}}{2} \\
&- \left(\frac{P^{-1/2}AP^{-1/2} + P^{-1/2}BP^{-1/2}}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1} - \left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^2,
\end{aligned}
$$

namely

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2}\right)^{2} \\
&\leq \frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2}\right)^{2}
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}P^{1/2}\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P^{1/2} \\
&\leq \frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}P^{1/2}\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P^{1/2}.
\end{aligned}$$

If we multiply both sides of this inequality by $P^{1/2}$ we get

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P \\
&\leq \frac{PA^{-1}P+PB^{-1}P}{2}-P\left(\frac{A+B}{2}\right)^{-1}P \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P.
\end{aligned}$$

By making use of (2.11) and (2.16), we get (2.12).

(iii) From (2.9) we have the bounds

$$\begin{aligned}
0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^{2}}{\delta^{4}} \\
&\leq \frac{C^{-1}+D^{-1}}{2}-\left(\frac{C+D}{2}\right)^{-1}\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^{2}}{\alpha^{4}}
\end{aligned}$$

and by inserting in this inequality $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned}
0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^{2}}{\delta^{4}} \\
&\leq \frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^{2}}{\alpha^{4}}.
\end{aligned}$$

Multiplying both sides of this inequality by $P^{1/2}$, we get (2.13). □

3. Identities for Operator Monotone Functions

We start with the following representation of the Jensen's difference for the first variable in the noncommutative perspective:

Theorem 4. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.5). For all A, B, $P>0$, we have*

$$\begin{aligned}(3.1)\qquad 0 &\le \mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]\\ &=\int_0^\infty \lambda^2\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\ &\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda).\end{aligned}$$

Proof. Observe that, by (1.5),

$$(3.2)\qquad f(t)=a+bt+\int_0^\infty\left[\lambda-\lambda^2(t+\lambda)^{-1}\right]dw(\lambda),\ t>0.$$

We have for C, $D>0$ that

$$\begin{aligned}(3.3)\qquad & f\left(\frac{C+D}{2}\right)-\frac{1}{2}\left[f(C)+f(D)\right]\\ &=a+b\left(\frac{C+D}{2}\right)+\int_0^\infty\left[\lambda-\lambda^2\left(\frac{C+D}{2}+\lambda\right)^{-1}\right]dw(\lambda)\\ &-\frac{1}{2}\left[a+bC+\int_0^\infty\left[\lambda-\lambda^2(C+\lambda)^{-1}\right]dw(\lambda)\right]\\ &-\frac{1}{2}\left[a+bD+\int_0^\infty\left[\lambda-\lambda^2(D+\lambda)^{-1}\right]dw(\lambda)\right]\\ &=\int_0^\infty\lambda^2\left[\frac{(C+\lambda)^{-1}+(D+\lambda)^{-1}}{2}-\left(\frac{C+D}{2}+\lambda\right)^{-1}\right]dw(\lambda).\end{aligned}$$

Let A, B, $C>0$. If we take in (3.3) $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned}& f\left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}\right)\\ &-\frac{1}{2}\left[f\left(P^{-1/2}AP^{-1/2}\right)+f\left(P^{-1/2}BP^{-1/2}\right)\right]\\ &=\int_0^\infty\lambda^2\left[\frac{\left(P^{-1/2}AP^{-1/2}+\lambda\right)^{-1}+\left(P^{-1/2}BP^{-1/2}+\lambda\right)^{-1}}{2}\right.\\ &\left.-\left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}+\lambda\right)^{-1}\right]dw(\lambda),\end{aligned}$$

which is equivalent to

$$\begin{aligned}& f\left(P^{-1/2}\left(\frac{A+B}{2}\right)P^{-1/2}\right)-\frac{1}{2}\left[f\left(P^{-1/2}AP^{-1/2}\right)+f\left(P^{-1/2}BP^{-1/2}\right)\right]\\ &=\int_0^\infty\lambda^2\left[\frac{P^{1/2}(A+\lambda P)^{-1}P^{1/2}+P^{1/2}(B+\lambda P)^{-1}P^{1/2}}{2}\right.\\ &\left.-P^{1/2}\left(\frac{A+B}{2}+\lambda P\right)^{-1}P^{1/2}\right]dw(\lambda).\end{aligned}$$

If we multiply this identity both sides with $P^{1/2}$, then we get

$$\begin{aligned}(3.4)\quad & P^{1/2}f\left(P^{-1/2}\left(\frac{A+B}{2}\right)P^{-1/2}\right)P^{1/2} \\ & -\frac{1}{2}\left[P^{1/2}f\left(P^{-1/2}AP^{-1/2}\right)P^{1/2}+P^{1/2}f\left(P^{-1/2}BP^{-1/2}\right)P^{1/2}\right] \\ & =\int_0^\infty \lambda^2\left[\frac{P\left(A+\lambda P\right)^{-1}P+P\left(B+\lambda P\right)^{-1}P}{2}\right. \\ & \left.-P\left(\frac{A+B}{2}+\lambda P\right)^{-1}P\right]dw\left(\lambda\right),\end{aligned}$$

which proves the equality in (3.1).

Since the function $g\left(t\right)=t^{-1}$ is operator convex on $(0,\infty)$, hence for A, B, $P>0$ and $\lambda\geq 0$,

$$\frac{\left(A+\lambda P\right)^{-1}+\left(B+\lambda P\right)^{-1}}{2}\geq\left(\frac{A+B}{2}+\lambda P\right)^{-1}.$$

If we multiply this inequality both sides by $P>0$, then we get

$$\frac{P\left(A+\lambda P\right)^{-1}P+P\left(B+\lambda P\right)^{-1}P}{2}-P\left(\frac{A+B}{2}+\lambda P\right)^{-1}P\geq 0$$

for $\lambda\geq 0$.

Finally, if we multiply this inequality by λ^2 and integrate on $[0,\infty)$ over $w\left(\lambda\right)$, then we get the positivity of the integral in (3.4). □

The case of transpose function $\tilde{f}$ is as follows:

Theorem 5. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.5). For all C, D, $Q>0$, we have*

$$\begin{aligned}(3.5)\quad 0 & \leq\mathcal{P}_{\tilde{f}}\left(\frac{C+D}{2},Q\right)-\frac{1}{2}\left[\mathcal{P}_{\tilde{f}}\left(C,Q\right)+\mathcal{P}_{\tilde{f}}\left(D,Q\right)\right] \\ & =\int_0^\infty\left[\frac{\mathcal{P}_{f_{-1}}\left(Q+\lambda C,Q\right)+\mathcal{P}_{f_{-1}}\left(Q+\lambda D,Q\right)}{2}\right. \\ & \left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw\left(\lambda\right).\end{aligned}$$

Proof. From (1.5) we have

$$f\left(t\right)=a+bt+t\int_0^\infty\frac{\lambda}{t+\lambda}dw\left(\lambda\right),\ t>0.$$

If we put $\frac{1}{t}$ instead of t, then we get

$$f\left(\frac{1}{t}\right)=a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty\frac{\lambda}{\frac{1}{t}+\lambda}dw\left(\lambda\right)=a+b\frac{1}{t}+\frac{1}{t}\int_0^\infty\frac{t\lambda}{1+t\lambda}dw\left(\lambda\right)$$

and by multiplication with $t>0$, we obtain

$$(3.6)\quad \tilde{f}\left(t\right)=b+ta+\int_0^\infty\frac{t\lambda}{1+t\lambda}dw\left(\lambda\right)=b+ta+\int_0^\infty\left(1-\left(1+t\lambda\right)^{-1}\right)dw\left(\lambda\right).$$

We have for U, $V>0$ that

$$\begin{aligned}
&\tilde{f}\left(\frac{U+V}{2}\right)-\frac{1}{2}\left[\tilde{f}(U)+\tilde{f}(V)\right] \\
&=b+a\frac{U+V}{2}+\int_0^\infty\left(1-\left(1+\lambda\frac{U+V}{2}\right)^{-1}\right)dw(\lambda) \\
&-\frac{1}{2}\left[b+aU+\int_0^\infty\left(1-(1+\lambda U)^{-1}\right)dw(\lambda)\right] \\
&-\frac{1}{2}\left[b+aV+\int_0^\infty\left(1-(1+\lambda V)^{-1}\right)dw(\lambda)\right] \\
&=\int_0^\infty\left[\frac{(1+\lambda U)^{-1}+(1+\lambda V)^{-1}}{2}-\left(1+\lambda\frac{U+V}{2}\right)^{-1}\right]dw(\lambda).
\end{aligned}\tag{3.7}$$

If we take $U=Q^{-1/2}CQ^{-1/2}$, $V=Q^{-1/2}DQ^{-1/2}$ in (3.7), then we get

$$\begin{aligned}
&\tilde{f}\left(\frac{Q^{-1/2}CQ^{-1/2}+Q^{-1/2}DQ^{-1/2}}{2}\right) \\
&-\frac{1}{2}\left[\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)+\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)\right] \\
&=\int_0^\infty\left[\frac{\left(1+\lambda Q^{-1/2}CQ^{-1/2}\right)^{-1}+\left(1+\lambda Q^{-1/2}DQ^{-1/2}\right)^{-1}}{2}\right. \\
&\left.-\left(1+\lambda\frac{Q^{-1/2}CQ^{-1/2}+Q^{-1/2}DQ^{-1/2}}{2}\right)^{-1}\right]dw(\lambda),
\end{aligned}$$

namely

$$\begin{aligned}
&\tilde{f}\left(Q^{-1/2}\frac{C+D}{2}Q^{-1/2}\right) \\
&-\frac{1}{2}\left[\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)+\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)\right] \\
&=\int_0^\infty\left[\frac{Q^{1/2}(Q+\lambda C)^{-1}Q^{1/2}+Q^{1/2}(Q+\lambda D)^{-1}Q^{1/2}}{2}\right. \\
&\left.-Q^{1/2}\left(Q+\lambda\frac{C+D}{2}\right)^{-1}Q^{1/2}\right]dw(\lambda),
\end{aligned}$$

for C, D, $Q>0$.

If we multiply this equality both sides by $Q^{1/2}$, then we obtain

$$Q^{1/2}\tilde{f}\left(Q^{-1/2}\frac{C+D}{2}Q^{-1/2}\right)Q^{1/2}$$
$$-\frac{1}{2}\left[Q^{1/2}\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)Q^{1/2}+Q^{1/2}\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)Q^{1/2}\right]$$
$$=\int_0^\infty\left[\frac{Q\left(Q+\lambda C\right)^{-1}Q+Q\left(Q+\lambda D\right)^{-1}Q}{2}\right.$$
$$\left.-Q\left(Q+\lambda\frac{C+D}{2}\right)^{-1}Q\right]dw\left(\lambda\right),$$

and the identity (3.5) is proved.

The inequality follows by the operator convexity of $g\left(t\right)=t^{-1}$ on $\left(0,\infty\right)$. □

Corollary 3. *Assume that function $f:\left(0,\infty\right)\to\mathbb{R}$ is operator monotone in $\left(0,\infty\right)$ and has the representation (1.5). For all C, D, $Q>0$, we have*

$$0\leq\mathcal{P}_f\left(Q,\frac{C+D}{2}\right)-\frac{1}{2}\left[\mathcal{P}_f\left(Q,C\right)+\mathcal{P}_f\left(Q,D\right)\right] \tag{3.8}$$
$$=\int_0^\infty\left[\frac{\mathcal{P}_{f_{-1}}\left(Q+\lambda C,Q\right)+\mathcal{P}_{f_{-1}}\left(Q+\lambda D,Q\right)}{2}\right.$$
$$\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw\left(\lambda\right).$$

4. Upper and Lower Bounds

We have:

Theorem 6. *Assume that function $f:\left(0,\infty\right)\to\mathbb{R}$ is operator monotone in $\left(0,\infty\right)$ and has the representation (1.5). If $0<\alpha P\leq A\leq\beta P$, $0<\gamma P\leq B\leq\delta P$ and $0<mP\leq B-A\leq MP$ for some constants α, β, γ, δ, m, M, then*

$$0<\frac{1}{2}m^2PF_w\left(\delta,\gamma,\alpha\right) \tag{4.1}$$
$$\leq\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f\left(A,P\right)+\mathcal{P}_f\left(B,P\right)\right]$$
$$\leq\frac{1}{2}M^2PF_w\left(\alpha,\beta,\delta\right),$$

where

$$F_w\left(h,k,l\right):=\int_0^\infty\frac{\left(k+\lambda\right)\left(l+\lambda\right)}{l+k+2\lambda}\frac{\lambda^2}{\left(h+\lambda\right)^4}dw\left(\lambda\right)$$

for h, k, $l>0$.

Proof. Since $0<\alpha P\leq A\leq\beta P$, $0<\gamma P\leq B\leq\delta P$ and $0<mP\leq B-A\leq MP$ for some constants α, β, γ, δ, m, M then

$$0<\left(\alpha+\lambda\right)P\leq A+\lambda P\leq\left(\beta+\lambda\right)P,$$
$$0<\left(\gamma+\lambda\right)P\leq B+\lambda P\leq\left(\delta+\lambda\right)P$$

and

$$0<mP\leq B+\lambda P-A-\lambda P\leq MP$$

for all $\lambda\geq0$.

By employing the inequality (2.13) we get

$$0<\frac{1}{2}\left((\alpha+\lambda)^{-1}+(\gamma+\lambda)^{-1}\right)^{-1}\frac{m^2}{(\delta+\lambda)^4}P$$
$$\leq\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}-\mathcal{P}_{f_{-1}}\left(\frac{A+\lambda P+B+\lambda P}{2},P\right)$$
$$\leq\frac{1}{2}\left((\beta+\lambda)^{-1}+(\delta+\lambda)^{-1}\right)^{-1}\frac{M^2}{(\alpha+\lambda)^4}P.$$

By multiplying with λ^2 and integrating, we get

$$0<\frac{1}{2}m^2P\int_0^\infty\left((\alpha+\lambda)^{-1}+(\gamma+\lambda)^{-1}\right)^{-1}\frac{\lambda^2}{(\delta+\lambda)^4}dw(\lambda) \tag{4.2}$$
$$\leq\int_0^\infty\lambda^2\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.$$
$$\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+\lambda P+B+\lambda P}{2},P\right)\right]dw(\lambda)$$
$$\leq\frac{1}{2}M^2P\int_0^\infty\left((\beta+\lambda)^{-1}+(\delta+\lambda)^{-1}\right)^{-1}\frac{\lambda^2}{(\alpha+\lambda)^4}dw(\lambda).$$

This is equivalent, via Theorem 4 to

$$0<\frac{1}{2}m^2P\int_0^\infty\frac{(\gamma+\lambda)(\alpha+\lambda)}{\alpha+\gamma+2\lambda}\frac{\lambda^2}{(\delta+\lambda)^4}dw(\lambda) \tag{4.3}$$
$$\leq\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]$$
$$\leq\frac{1}{2}M^2P\int_0^\infty\frac{(\beta+\lambda)(\delta+\lambda)}{\delta+\beta+2\lambda}\frac{\lambda^2}{(\alpha+\lambda)^4}dw(\lambda).$$

□

Consider the function

$$g_{k,l}(\lambda):=\frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda},\ \ \lambda\geq 0.$$

Observe that

$$g_{k,l}'(\lambda):=\left(\frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda}\right)'=\frac{(l+k+2\lambda)^2-2(k+\lambda)(l+\lambda)}{(l+k+2\lambda)^2}$$
$$=\frac{(k+\lambda)^2+(l+\lambda)^2}{(l+k+2\lambda)^2}>0.$$

This gives that

$$\frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda}=g_{k,l}(\lambda)\geq g_{k,l}(0)=\frac{kl}{l+k}$$

for all $\lambda\geq 0$.

Also

$$\frac{(k+\lambda)(l+\lambda)}{l+k}\geq\frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda}=g_{k,l}(\lambda)$$

for all $\lambda\geq 0$.

The following result contains more practical bounds that are in terms of the derivatives of operator monotone function f :

Corollary 4. *With the assumptions of Theorem 6, we have*

$$\begin{aligned}
(4.4)\qquad & 0<\frac{1}{2}\frac{\gamma\alpha m^{2}P}{\alpha+\gamma} \\
& \times\left[\frac{1}{6}\left(\gamma-\delta\right)\left(\alpha-\delta\right)f'''\left(\delta\right)-\left(\frac{\alpha+\gamma}{2}-\delta\right)f''\left(\delta\right)+f'\left(\delta\right)-b\right] \\
& \leq\mathcal{P}_{f}\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_{f}\left(A,P\right)+\mathcal{P}_{f}\left(B,P\right)\right] \\
& \leq\frac{1}{2}\frac{M^{2}P}{\delta+\beta} \\
& \times\left[\frac{1}{6}\left(\beta-\alpha\right)\left(\delta-\alpha\right)f'''\left(\alpha\right)-\left(\frac{\delta+\beta}{2}-\alpha\right)f''\left(\alpha\right)+f'\left(\alpha\right)-b\right].
\end{aligned}$$

Proof. Using the above remarks, we get from (4.1) that

$$\begin{aligned}
(4.5)\qquad & 0<\frac{1}{2}\frac{\gamma\alpha m^{2}P}{\alpha+\gamma}\int_{0}^{\infty}\frac{\left(\gamma+\lambda\right)\left(\alpha+\lambda\right)\lambda^{2}}{\left(\delta+\lambda\right)^{4}}dw\left(\lambda\right) \\
& \leq\mathcal{P}_{f}\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_{f}\left(A,P\right)+\mathcal{P}_{f}\left(B,P\right)\right] \\
& \leq\frac{1}{2}\frac{M^{2}P}{\delta+\beta}\int_{0}^{\infty}\frac{\left(\beta+\lambda\right)\left(\delta+\lambda\right)\lambda^{2}}{\left(\alpha+\lambda\right)^{4}}dw\left(\lambda\right).
\end{aligned}$$

From (3.2), by taking the derivative over t and using the properties of the integral, we get

$$f'\left(t\right)-b=\int_{0}^{\infty}\frac{\lambda^{2}}{\left(t+\lambda\right)^{2}}dw\left(\lambda\right),\ t>0,$$

$$-\frac{1}{2}f''\left(t\right)=\int_{0}^{\infty}\frac{\lambda^{2}}{\left(t+\lambda\right)^{3}}dw\left(\lambda\right),\ t>0,$$

and

$$\frac{1}{6}f'''\left(t\right)=\int_{0}^{\infty}\frac{\lambda^{2}}{\left(t+\lambda\right)^{4}}dw\left(\lambda\right),\ t>0.$$

Observe that

$$\begin{aligned}
& \frac{\left(\beta+\lambda\right)\left(\delta+\lambda\right)}{\left(\alpha+\lambda\right)^{4}} \\
& =\frac{\left(\beta-\alpha+\lambda+\alpha\right)\left(\delta-\alpha+\lambda+\alpha\right)}{\left(\alpha+\lambda\right)^{4}} \\
& =\left(\beta-\alpha\right)\left(\delta-\alpha\right)\frac{1}{\left(\alpha+\lambda\right)^{4}}+\left(\delta+\beta-2\alpha\right)\frac{1}{\left(\alpha+\lambda\right)^{3}}+\frac{1}{\left(\alpha+\lambda\right)^{2}}.
\end{aligned}$$

Therefore

$$\int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)\lambda^2}{(\alpha+\lambda)^4}dw(\lambda)$$
$$=(\beta-\alpha)(\delta-\alpha)\int_0^\infty \frac{\lambda^2}{(\alpha+\lambda)^4}dw(\lambda)$$
$$+(\delta+\beta-2\alpha)\int_0^\infty \frac{\lambda^2}{(\alpha+\lambda)^3}dw(\lambda)+\int_0^\infty \frac{\lambda^2}{(\alpha+\lambda)^2}dw(\lambda)$$
$$=\frac{1}{6}(\beta-\alpha)(\delta-\alpha)f'''(\alpha)-\left(\frac{\delta+\beta}{2}-\alpha\right)f''(\alpha)+f'(\alpha)-b.$$

Similarly

$$\int_0^\infty \frac{(\gamma+\lambda)(\alpha+\lambda)\lambda^2}{(\delta+\lambda)^4}dw(\lambda)$$
$$=\frac{1}{6}(\gamma-\delta)(\alpha-\delta)f'''(\alpha)-\left(\frac{\alpha+\gamma}{2}-\delta\right)f''(\delta)+f'(\delta)-b$$

and by (4.5) we obtain (4.4). □

Remark 2. *Since $b\geq 0$, then from (4.4) we get the upper bound*

$$(0\leq)\,\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right] \tag{4.6}$$
$$\leq\frac{1}{2}\frac{M^2P}{\delta+\beta}\left[\frac{1}{6}(\beta-\alpha)(\delta-\alpha)f'''(\alpha)-\left(\frac{\delta+\beta}{2}-\alpha\right)f''(\alpha)+f'(\alpha)\right]$$

for any operator monotone function $f:(0,\infty)\to\mathbb{R}$.

We also have:

Theorem 7. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.5). If $0<\kappa Q\leq C\leq\mu Q$, $0<\nu Q\leq D\leq\xi Q$ and $0<nQ\leq D-C\leq NQ$ for some constants α, β, γ, δ, m, M, then*

$$0<\frac{1}{2}n^2G_w(\xi,\kappa,\nu)Q \tag{4.7}$$
$$\leq\mathcal{P}_f\left(Q,\frac{C+D}{2}\right)-\frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]$$
$$\leq\frac{1}{2}N^2G_w(\kappa,\mu,\xi)Q,$$

where

$$G_w(h,k,l):=\int_0^\infty \frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda}\frac{\lambda^2}{(1+h\lambda)^4}dw(\lambda)$$

for h, k, $l>0$.

Proof. Since $0<\kappa Q\leq C\leq\mu Q$, $0<\nu Q\leq D\leq\xi Q$ and $0<nQ\leq D-C\leq NQ$ for some constants α, β, γ, δ, m, M, then

$$(1+\kappa\lambda)Q\leq Q+\lambda C\leq(1+\mu\lambda)Q,$$
$$(1+\nu\lambda)Q\leq Q+\lambda D\leq(1+\xi\lambda)Q$$

and

$$n\lambda Q\leq Q+\lambda D-Q-\lambda C\leq\lambda NQ$$

for $\lambda > 0$.

By employing the inequality (2.13) we get

$$0 < \frac{1}{2}\left((1+\kappa\lambda)^{-1} + (1+\nu\lambda)^{-1}\right)^{-1} \frac{n^2\lambda^2}{(1+\xi\lambda)^4} Q \tag{4.8}$$
$$\leq \frac{\mathcal{P}_{f_{-1}}(Q+\lambda C, Q) + \mathcal{P}_{f_{-1}}(Q+\lambda D, Q)}{2}$$
$$- \mathcal{P}_{f_{-1}}\left(Q + \lambda\frac{C+D}{2}, Q\right)$$
$$\leq \frac{1}{2}\left((1+\mu\lambda)^{-1} + (1+\xi\lambda)^{-1}\right)^{-1} \frac{\lambda^2 N^2}{(1+\kappa\lambda)^4} Q.$$

for $\lambda > 0$.

If we integrate (4.8) we get

$$0 < \frac{1}{2}n^2\left(\int_0^\infty \left((1+\kappa\lambda)^{-1} + (1+\nu\lambda)^{-1}\right)^{-1} \frac{\lambda^2}{(1+\xi\lambda)^4} dw(\lambda)\right) Q$$
$$\leq \int_0^\infty \left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C, Q) + \mathcal{P}_{f_{-1}}(Q+\lambda D, Q)}{2}\right.$$
$$\left. -\mathcal{P}_{f_{-1}}\left(Q + \lambda\frac{C+D}{2}, Q\right)\right] dw(\lambda)$$
$$\leq \frac{1}{2}N^2\left(\int_0^\infty \left((1+\mu\lambda)^{-1} + (1+\xi\lambda)^{-1}\right)^{-1} \frac{\lambda^2}{(1+\kappa\lambda)^4} dw(\lambda)\right) Q$$

and by (3.8)

$$0 < \frac{1}{2}n^2\left(\int_0^\infty \frac{(1+\kappa\lambda)(1+\nu\lambda)}{2+(\kappa+\nu)\lambda} \frac{\lambda^2}{(1+\xi\lambda)^4} dw(\lambda)\right) Q$$
$$\leq \mathcal{P}_f\left(Q, \frac{C+D}{2}\right) - \frac{1}{2}\left[\mathcal{P}_f(Q,C) + \mathcal{P}_f(Q,D)\right]$$
$$\leq \frac{1}{2}N^2\left(\int_0^\infty \frac{(1+\mu\lambda)(1+\xi\lambda)}{2+(\mu+\xi)\lambda} \frac{\lambda^2}{(1+\kappa\lambda)^4} dw(\lambda)\right) Q$$

and the inequality (4.7) is obtained. □

Corollary 5. *With the assumptions of Theorem 7, we have*

$$(0 \leq)\, \mathcal{P}_f\left(Q, \frac{C+D}{2}\right) - \frac{1}{2}\left[\mathcal{P}_f(Q,C) + \mathcal{P}_f(Q,D)\right] \tag{4.9}$$
$$\leq \frac{1}{8}N^2\left[\frac{1}{6}(\mu+\xi)\tilde{f}'''(\kappa) - \tilde{f}''(\kappa)\right] Q.$$

Proof. Using the elementary inequality

$$ab \leq \frac{1}{4}(a+b)^2,\ a,b \geq 0$$

we get

$$\frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda} \leq \frac{1}{4}\left[2+(k+l)\lambda\right],$$

therefore

$$G_w(h,k,l)=\int_0^\infty \frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda}\frac{\lambda^2}{(1+h\lambda)^4}dw(\lambda) \tag{4.10}$$
$$\le \frac{1}{4}\int_0^\infty [2+(k+l)\lambda]\frac{\lambda^2}{(1+h\lambda)^4}dw(\lambda)$$
$$=\frac{1}{4}\left[2\int_0^\infty \frac{\lambda^2}{(1+h\lambda)^4}dw(\lambda)+(k+l)\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4}dw(\lambda)\right]$$
$$\le \frac{1}{4}\left[2\int_0^\infty \frac{\lambda^2}{(1+h\lambda)^3}dw(\lambda)+(k+l)\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4}dw(\lambda)\right].$$

From (3.6) we have

$$\tilde{f}'(t)=a+\int_0^\infty \frac{\lambda}{(1+t\lambda)^2}dw(\lambda),$$

$$-\frac{1}{2}\tilde{f}''(t)=\int_0^\infty \frac{\lambda^2}{(1+t\lambda)^3}dw(\lambda)$$

and

$$\frac{1}{6}\tilde{f}'''(t)=\int_0^\infty \frac{\lambda^3}{(1+t\lambda)^4}dw(\lambda).$$

Therefore

$$2\int_0^\infty \frac{\lambda^2}{(1+h\lambda)^3}dw(\lambda)+(k+l)\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4}dw(\lambda) \tag{4.11}$$
$$=\frac{1}{6}(k+l)\tilde{f}'''(h)-\tilde{f}''(h).$$

By making use of (4.10), (4.11) and Theorem 7, we obtain (4.9). □

5. Some Examples

Consider the function $f_r:[0,\infty)\to[0,\infty)$, $f_r(t)=t^r$, $r\in(0,1)$, then

$$f'(t)=rt^{r-1},\ f''(t)=r(r-1)t^{r-2} \text{ and } f'''(t)=r(r-1)(r-2)t^{r-3},\ t>0.$$

We have

$$\frac{1}{6}(\beta-\alpha)(\delta-\alpha)f'''(\alpha)-\left(\frac{\delta+\beta}{2}-\alpha\right)f''(\alpha)+f'(\alpha)$$
$$=\frac{1}{6}(\beta-\alpha)(\delta-\alpha)r(r-1)(r-2)\alpha^{r-3}-\left(\frac{\delta+\beta}{2}-\alpha\right)r(r-1)\alpha^{r-2}$$
$$+r\alpha^{r-1}$$
$$=r\alpha^{r-3}\left[\frac{1}{6}\left(\beta\delta-(\beta+\delta)\alpha+\alpha^2\right)\left(r^2-3r+2\right)-(r-1)\left(\frac{\delta+\beta}{2}\alpha-\alpha^2\right)+\alpha^2\right]$$

$$= r\alpha^{r-3}\left\{\left(\frac{r^2-3r+2}{6}+r-1+1\right)\alpha^2\right.$$
$$\left.+\left[-(\beta+\delta)\frac{r^2-3r+2}{6}-(r-1)\frac{\delta+\beta}{2}\right]\alpha+\frac{\beta\delta}{6}\left(r^2-3r+2\right)\right\}$$
$$=\frac{1}{6}r\alpha^{r-3}\left[(r+1)(r+2)\alpha^2+(1-r)(r+1)\alpha(\delta+\beta)+(1-r)(2-r)\beta\delta\right]$$

and, similarly

$$\frac{1}{6}(\gamma-\delta)(\alpha-\delta)f'''(\delta)-\left(\frac{\alpha+\gamma}{2}-\delta\right)f''(\delta)+f'(\delta)$$
$$=\frac{1}{6}r\delta^{r-3}\left[(r+1)(r+2)\delta^2+(1-r)(r+1)\delta(\alpha+\gamma)+(1-r)(2-r)\gamma\alpha\right].$$

We can then state the following result:

Proposition 1. *If $0<\alpha P\le A\le\beta P$, $0<\gamma P\le B\le\delta P$ and $0<mP\le B-A\le MP$ for some constants α, β, γ, δ, m, M, then*

$$0<\frac{1}{12}\frac{r\gamma\alpha\delta^{r-3}m^2P}{\alpha+\gamma}\left[(r+1)(r+2)\delta^2+(1-r)(r+1)\delta(\alpha+\gamma)+(1-r)(2-r)\gamma\alpha\right]$$
$$\le P\sharp_r\left(\frac{A+B}{2}\right)-\frac{1}{2}\left(P\sharp_rA+P\sharp_rB\right)$$
$$\le\frac{1}{12}\frac{r\alpha^{r-3}M^2P}{\delta+\beta}\times\left[(r+1)(r+2)\alpha^2+(1-r)(r+1)\alpha(\delta+\beta)+(1-r)(2-r)\beta\delta\right].\tag{5.1}$$

Consider the function $f(t)=\ln t$, $t>0$, then

$$f'(t)=t^{-1},\ f''(t)=-t^{-2}\text{ and }f'''(t)=2t^{-3},\ t>0.$$

Therefore

$$\frac{1}{6}(\beta-\alpha)(\delta-\alpha)f'''(\alpha)-\left(\frac{\delta+\beta}{2}-\alpha\right)f''(\alpha)+f'(\alpha)$$
$$=\frac{1}{3}(\beta-\alpha)(\delta-\alpha)\alpha^{-3}+\left(\frac{\delta+\beta}{2}-\alpha\right)\alpha^{-2}+\alpha^{-1}$$
$$=\alpha^{-3}\left[\frac{1}{3}\left(\beta\delta-(\delta+\beta)\alpha+\alpha^2\right)+\frac{\delta+\beta}{2}\alpha\right]$$
$$=\frac{1}{6}\alpha^{-3}\left[2\beta\delta+(\delta+\beta)\alpha+2\alpha^2\right].$$

Proposition 2. *If $0<\alpha P\le A\le\beta P$, $0<B\le\delta P$ and $0\le B-A\le MP$ for some constants α, β, δ, M, then*

$$(0\le)\,\mathcal{S}\left(P|\frac{A+B}{2}\right)-\frac{1}{2}\left[\mathcal{S}(P|A)+\mathcal{S}(P|B)\right]\le\frac{1}{12}\frac{\alpha^{-3}M^2P}{\delta+\beta}\left[2\beta\delta+(\delta+\beta)\alpha+2\alpha^2\right].\tag{5.2}$$

The proof follows for the operator monotone function $f(t)=\ln t$ and the inequality (5.2) is proved.

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