BOUNDS ON PARTIAL CONVEXITY OF NONCOMMUTATIVE PERSPECTIVES FOR OPERATOR CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}dw(\lambda),$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and w a positive measure on $(0,\infty)$. We can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$.

In this paper we show among others that, if A, B, $P>0$,

$$\begin{aligned}0&\leq\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]-\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A)\\&=\int_0^\infty\lambda^3\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda)\end{aligned}$$

and if C, D, $Q>0$,

$$\begin{aligned}0&\leq\frac{1}{2}\left[\mathcal{P}_{\tilde{f}}(C,Q)+\mathcal{P}_{\tilde{f}}(D,Q)\right]-\mathcal{P}_{\tilde{f}}\left(\frac{C+D}{2},Q\right)\\&-c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right)\\&=\int_0^\infty\lambda\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda).\end{aligned}$$

where $\mathcal{P}_{f_{-1}}(B,A):=AB^{-1}A$. Several upper and lower bounds for these partial differences are given. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

1. Introduction

Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous

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functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A) := A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = A f\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f : (0,\infty) \to \mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = x f\left(x^{-1}\right), \; x > 0.$$

It is well known that (see for instance [14]), if $f : (0,\infty) \to \mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B > 0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A). \tag{1.1}$$

It is well known that (see [4] and [3] or [5]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A) \mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r : [0,\infty) \to [0,\infty)$, $f_r(t) = t^r$, $r \in [0,1]$, then

$$\mathcal{P}_{f_r}(B,A) := A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^r A^{1/2} =: A \sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

If we take the function $f = \ln$, then

$$\mathcal{P}_{\ln}(B,A) := A^{1/2} \ln\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [13].

If $f_{-1} : (0,\infty) \to (0,\infty)$, $f_{-1}(x) = x^{-1}$, then $\tilde{f}_{-1} : (0,\infty) \to (0,\infty)$ is given by $\tilde{f}_{-1}(x) = x^2 = f_2(x)$ and

$$\mathcal{P}_{f_{-1}}(B,A) := A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{-1} A^{1/2} = AB^{-1}A = \mathcal{P}_{f_2}(A,B)$$

for $A,\ B > 0$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)\,(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}dw(\lambda), \tag{1.2}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and w a positive measure on $(0,\infty)$. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_{+}(0)$, the right derivative, in (1.2).

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [10] and the references therein.

As examples of such functions, we note that $f(t)=t^{r}$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^{r}$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^{t}$ is neither operator convex nor operator monotone.

In this paper we show among others that, if A, B, $P>0$,

$$\begin{aligned}
0&\leq\frac{1}{2}\left[\mathcal{P}_{f}(A,P)+\mathcal{P}_{f}(B,P)\right]-\mathcal{P}_{f}\left(\frac{A+B}{2},P\right)-\frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A)\\
&=\int_{0}^{\infty}\lambda^{3}\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\
&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda)
\end{aligned}$$

and if C, D, $Q>0$,

$$\begin{aligned}
0&\leq\frac{1}{2}\left[\mathcal{P}_{\tilde{f}}(C,Q)+\mathcal{P}_{\tilde{f}}(D,Q)\right]-\mathcal{P}_{\tilde{f}}\left(\frac{C+D}{2},Q\right)\\
&-c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right)\\
&=\int_{0}^{\infty}\lambda\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right.\\
&\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda),
\end{aligned}$$

where $\mathcal{P}_{f_{-1}}(B,A):=AB^{-1}A$. Several upper and lower bounds for these partial differences are given. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

2. Some Preliminary Facts

We start with the following elementary identity that give a simple proof for the fact that the function $f(t)=t^{-1}$ is operator convex on $(0,\infty)$, see for instance [10, p. 8]:

Lemma 1. *For any C, $D>0$ we have*

$$\begin{aligned}
&\frac{C^{-1}+D^{-1}}{2}-\left(\frac{C+D}{2}\right)^{-1} \\
&=\frac{\left(C^{-1}-D^{-1}\right)\left(C^{-1}+D^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)}{2}\geq 0.
\end{aligned} \tag{2.1}$$

If more assumptions are made for the operators C and D, then one can obtain the following lower and upper bounds:

Corollary 1. *Assume that* $0<\alpha\leq C\leq\beta$ *and* $0<\gamma\leq D\leq\delta$ *for some constants* α, β, γ, δ. *Then*

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}\leq\frac{C^{-1}+D^{-1}}{2}-\left(\frac{C+D}{2}\right)^{-1} \leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}. \tag{2.2}$$

Proof. We have $\beta^{-1}\leq C^{-1}\leq\alpha^{-1}$ and $\delta^{-1}\leq D^{-1}\leq\gamma^{-1}$, which gives

$$\beta^{-1}+\delta^{-1}\leq C^{-1}+D^{-1}\leq\alpha^{-1}+\gamma^{-1}$$

namely

$$\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\leq\left(C^{-1}+D^{-1}\right)^{-1}\leq\left(\beta^{-1}+\delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(C^{-1}-D^{-1}\right)$ and dividing by 2, we get

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}\leq\frac{\left(C^{-1}-D^{-1}\right)\left(C^{-1}+D^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)}{2} \leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^{2}.$$

□

A continuous function $g:\mathcal{S}\mathrm{C}_{I}\left(H\right)\to\mathrm{B}(H)$ is said to be *Gâteaux differentiable* in $C\in\mathcal{S}\mathrm{C}_{I}\left(H\right)$, the class of selfadjoint operators on I, along the direction $D\in\mathrm{B}(H)$ if the following limit exists in the strong topology of $\mathrm{B}(H)$

$$\nabla g_{C}\left(D\right):=\lim_{s\to0}\frac{g\left(C+sD\right)-g\left(C\right)}{s}\in\mathcal{B}\left(H\right). \tag{2.3}$$

If the limit (2.3) exists for all $D\in\mathrm{B}(H)$, then we say that g is *Gâteaux differentiable* in C and we can write $g\in\mathcal{G}\left(C\right)$. If this is true for any C in an open set $\mathcal{S}$ from $\mathcal{S}\mathrm{C}_{I}\left(H\right)$ we write that $g\in\mathcal{G}\left(\mathcal{S}\right)$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $C,D\in\mathcal{S}\mathrm{C}_{I}\left(H\right)$ we consider the segment of selfadjoint operators

$$[C,D]:=\left\{(1-t)\,C+tD\mid t\in[0,1]\right\}.$$

We observe that C, $D\in[C,D]$ and $[C,D]\subset\mathcal{S}\mathrm{C}_{I}\left(H\right)$.

We have the following gradient inequalities, see for instance [2]:

Lemma 2. *Let* f *be an operator convex function on* I *and* C, $D\in\mathcal{S}C_{I}\left(H\right)$, *with* $C\neq D$. *If* $f\in\mathcal{G}\left([C,D]\right)$, *then*

$$\nabla_{D}f\left(D-C\right)\geq f\left(D\right)-f\left(C\right)\geq\nabla_{C}f\left(D-C\right). \tag{2.4}$$

Let T, $S>0$. The function $f\left(t\right)=t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_{T}\left(S\right):=\lim_{t\to0}\left[\frac{f\left(T+tS\right)-f\left(T\right)}{t}\right]=-T^{-1}ST^{-1} \tag{2.5}$$

for T, $S>0$.

Using (2.5) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1} \tag{2.6}$$

for all C, $D > 0$.

If

$$m \leq D - C \leq M$$

for some constants m, M, then

$$mD^{-2} \leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1} \leq MC^{-2}$$

and by (2.6) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.7}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2}. \tag{2.8}$$

Corollary 2. *Assume that* $0 < \alpha \leq C \leq \beta$, $0 < \gamma \leq D \leq \delta$ *and* $0 < m \leq D - C \leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4} \\ &\leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^2 \leq \frac{C^{-1}+D^{-1}}{2} - \left(\frac{C+D}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}. \end{aligned} \tag{2.9}$$

Proof. From (2.8) we have

$$0 < \frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \leq \left(C^{-1} - D^{-1}\right)^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.9). □

Remark 1. *If the positive operators* C, D *are separated, namely* $0 < \alpha \leq C \leq \beta < \gamma \leq D \leq \delta$ *for some constants* α, β, γ, δ, *then obviously* $0 < \gamma - \beta \leq D - C \leq \delta - \alpha$

and by (2.9) for $m = \gamma - \beta$ *and* $M = \delta - \alpha$, *we get*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4} \le \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^2 \\ &\le \frac{C^{-1}+D^{-1}}{2} - \left(\frac{C+D}{2}\right)^{-1} \\ &\le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)^2 \le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}. \end{aligned} \tag{2.10}$$

If we put together the above results for the noncommutative perspective $\mathcal{P}_{f_{-1}}(A,P)$, then we can state the following result:

Theorem 2. *Let* $A, B, P > 0$.

(i) *We have the representations*

$$\begin{aligned} 0 &\le \frac{\mathcal{P}_{f_{-1}}(A,P)+\mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2},P\right) \\ &= \frac{1}{2}P\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)P \\ &= \frac{1}{2}\mathcal{P}_{f_{-1}}\left(\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)\left(A^{-1}-B^{-1}\right),P\right) \\ &= \frac{1}{2}\left(\mathcal{P}_{f_{-1}}(A,P)-\mathcal{P}_{f_{-1}}(B,P)\right)\left(\mathcal{P}_{f_{-1}}(A,P)+\mathcal{P}_{f_{-1}}(B,P)\right)^{-1} \\ &\times\left(\mathcal{P}_{f_{-1}}(A,P)-\mathcal{P}_{f_{-1}}(B,P)\right). \end{aligned} \tag{2.11}$$

(ii) *Assume that* $0 < \alpha P \le A \le \beta P$ *and* $0 < \gamma P \le B \le \delta P$ *for some constants* α, β, γ, δ, *then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P \\ &\le \frac{\mathcal{P}_{f_{-1}}(A,P)+\mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2},P\right) \\ &\le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P. \end{aligned} \tag{2.12}$$

(iii) *Assume that* $0 < \alpha P \le A \le \beta P$, $0 < \gamma P \le B \le \delta P$ *and* $0 < mP \le B - A \le MP$ *for some constants* α, β, γ, δ, m, M, *then*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4}P \\ &\le \frac{\mathcal{P}_{f_{-1}}(A,P)+\mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2},P\right) \\ &\le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}P. \end{aligned} \tag{2.13}$$

(iv) *Assume that* $0 < \alpha P \le A \le \beta P < \gamma P \le B \le \delta P$ *for some constants* α, β, γ, δ, *then*

$$0 < \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4}P \le \frac{\mathcal{P}_{f_{-1}}(A,P)+\mathcal{P}_{f_{-1}}(B,P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+B}{2},P\right) \le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}P. \tag{2.14}$$

Proof. (i) If we take in (2.1) $C = P^{-1/2}AP^{-1/2}$ and $D = P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned} 0 \le & \frac{\left(P^{-1/2}AP^{-1/2}\right)^{-1}+\left(P^{-1/2}BP^{-1/2}\right)^{-1}}{2} \\ & - \left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}\right)^{-1} \\ = & \frac{1}{2}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1}-\left(P^{-1/2}BP^{-1/2}\right)^{-1}\right) \\ & \times\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1}+\left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^{-1} \\ & \times\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1}-\left(P^{-1/2}BP^{-1/2}\right)^{-1}\right), \end{aligned} \tag{2.15}$$

namely

$$\begin{aligned} 0 \le & P^{1/2}\frac{A^{-1}+B^{-1}}{2}P^{1/2} - P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\ = & \frac{1}{2}P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2} \\ & \times P^{-1/2}\left(A^{-1}+B^{-1}\right)^{-1}P^{-1/2}P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2}, \end{aligned}$$

that is equivalent to

$$\begin{aligned} 0 \le & P^{1/2}\frac{A^{-1}+B^{-1}}{2}P^{1/2} - P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2} \\ = & \frac{1}{2}P^{1/2}\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)P^{1/2}. \end{aligned}$$

If we multiply this inequality both sides by $P^{1/2}$ we get

$$\begin{aligned} 0 \le & \frac{PA^{-1}P+PB^{-1}P}{2} - P\left(\frac{A+B}{2}\right)^{-1}P \\ = & \frac{1}{2}P\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)P \\ = & \frac{1}{2}P\left[\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)\left(A^{-1}-B^{-1}\right)\right]^{-1}P, \end{aligned}$$

which is the first part of (2.11).

Observe also that

$$\begin{aligned}
&\frac{1}{2}P\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)P\\
&=\frac{1}{2}\left(PA^{-1}P-PB^{-1}P\right)P^{-1}\left(A^{-1}+B^{-1}\right)^{-1}P^{-1}\left(PA^{-1}P-PB^{-1}P\right)\\
&=\frac{1}{2}\left(PA^{-1}P-PB^{-1}P\right)\left(PA^{-1}P+PB^{-1}P\right)^{-1}\left(PA^{-1}P-PB^{-1}P\right),
\end{aligned}$$

which proves the last part of (2.11).

(ii) From (2.1) we also have by multiplying both sides by P, that

$$\begin{aligned}
(2.16)\qquad &P\left[\frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1}\right]P\\
&=\frac{PA^{-1}P+PB^{-1}P}{2}-P\left(\frac{A+B}{2}\right)^{-1}P\\
&=\frac{1}{2}P\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)P\geq 0,
\end{aligned}$$

for all A, B, $P>0$.

If $0<\alpha P\leq A\leq P\beta$ and $0<\gamma P\leq B\leq\delta P$ for some constants α, β, γ, δ, then by multiplying both sides with $P^{-1/2}$ we get $0<\alpha\leq P^{-1/2}AP^{-1/2}\leq\beta$ and $0<\gamma\leq P^{-1/2}BP^{-1/2}\leq\delta$. If we use inequality (2.2) for $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1}-\left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^{2}\\
&\leq\frac{\left(P^{-1/2}AP^{-1/2}\right)^{-1}+\left(P^{-1/2}BP^{-1/2}\right)^{-1}}{2}\\
&-\left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}\right)^{-1}\\
&\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(\left(P^{-1/2}AP^{-1/2}\right)^{-1}-\left(P^{-1/2}BP^{-1/2}\right)^{-1}\right)^{2},
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2}\right)^{2}\\
&\leq\frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2}\\
&\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(P^{1/2}\left(A^{-1}-B^{-1}\right)P^{1/2}\right)^{2}
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}P^{1/2}\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P^{1/2}\\
&\leq\frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2}\\
&\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}P^{1/2}\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P^{1/2}.
\end{aligned}$$

If we multiply both sides of this inequality by $P^{1/2}$ we get

$$\begin{aligned}&\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P\\&\le\frac{PA^{-1}P+PB^{-1}P}{2}-P\left(\frac{A+B}{2}\right)^{-1}P\\&\le\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}P\left(A^{-1}-B^{-1}\right)P\left(A^{-1}-B^{-1}\right)P.\end{aligned}$$

By making use of (2.11) and (2.16), we get (2.12).

(iii) From (2.9) we have the bounds

$$\begin{aligned}0&<\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4}\\&\le\frac{C^{-1}+D^{-1}}{2}-\left(\frac{C+D}{2}\right)^{-1}\le\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}\end{aligned}$$

and by inserting in this inequality $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned}0&<\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4}\\&\le\frac{P^{1/2}A^{-1}P^{1/2}+P^{1/2}B^{-1}P^{1/2}}{2}-P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2}\\&\le\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}.\end{aligned}$$

Multiplying both sides of this inequality by $P^{1/2}$, we get (2.13). □

3. Identities for Operator Convex Functions

We start with the following representation of the Jensen's difference for the first variable in the noncommutative perspective:

Theorem 3. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.2). For all A, B, $P>0$, we have*

$$\begin{aligned}(3.1)\qquad 0&\le\frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]-\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A)\\&=\int_0^\infty\lambda^3\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P,P)+\mathcal{P}_{f_{-1}}(B+\lambda P,P)}{2}\right.\\&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P,P\right)\right]dw(\lambda).\end{aligned}$$

Proof. From (1.2) we have

$$f(t)=a+bt+ct^2+\int_0^\infty\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]\lambda dw(\lambda)\qquad(3.2)$$

for $t>0$.

By the operator convexity of f, we have for C, $D>0$ that

$$\begin{aligned}
&\frac{1}{2}\left[f\left(C\right)+f\left(D\right)\right]-f\left(\frac{C+D}{2}\right) \\
&=\frac{1}{2}\left[a+bC+cC^{2}+\int_{0}^{\infty}\left[C-\lambda+\lambda^{2}\left(C+\lambda\right)^{-1}\right]\lambda dw\left(\lambda\right)\right] \\
&+\frac{1}{2}\left[a+bD+cD^{2}+\int_{0}^{\infty}\left[D-\lambda+\lambda^{2}\left(D+\lambda\right)^{-1}\right]\lambda dw\left(\lambda\right)\right] \\
&-\left[a+b\left(\frac{C+D}{2}\right)+c\left(\frac{C+D}{2}\right)^{2}\right. \\
&\left.+\int_{0}^{\infty}\left[\frac{C+D}{2}-\lambda+\lambda^{2}\left(\frac{C+D}{2}+\lambda\right)^{-1}\right]\lambda dw\left(\lambda\right)\right] \\
&=\frac{1}{4}c\left(D-C\right)^{2} \\
&+\int_{0}^{\infty}\lambda^{3}\left[\frac{\left(C+\lambda\right)^{-1}+\left(D+\lambda\right)^{-1}}{2}-\left(\frac{C+D}{2}+\lambda\right)^{-1}\right]dw\left(\lambda\right).
\end{aligned}\tag{3.3}$$

Let A, B, $C>0$. If we take in (3.3) $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\begin{aligned}
0&\leq\frac{1}{2}\left[f\left(P^{-1/2}AP^{-1/2}\right)+f\left(P^{-1/2}BP^{-1/2}\right)\right] \\
&-f\left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}\right) \\
&=\frac{1}{4}c\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)^{2} \\
&+\int_{0}^{\infty}\lambda^{3}\left[\frac{\left(P^{-1/2}AP^{-1/2}+\lambda\right)^{-1}+\left(P^{-1/2}BP^{-1/2}+\lambda\right)^{-1}}{2}\right. \\
&\left.-\left(\frac{P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}}{2}+\lambda\right)^{-1}\right]dw\left(\lambda\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\left[f\left(P^{-1/2}AP^{-1/2}\right)+f\left(P^{-1/2}BP^{-1/2}\right)\right] \\
&-f\left(P^{-1/2}\left(\frac{A+B}{2}\right)P^{-1/2}\right) \\
&=\frac{1}{4}c\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)^{2} \\
&+\int_{0}^{\infty}\lambda^{3}\left[\frac{P^{1/2}\left(A+\lambda P\right)^{-1}P^{1/2}+P^{1/2}\left(B+\lambda P\right)^{-1}P^{1/2}}{2}\right. \\
&\left.-P^{1/2}\left(\frac{A+B}{2}+\lambda P\right)^{-1}P^{1/2}\right]dw\left(\lambda\right).
\end{aligned}$$

If we multiply this identity both sides with $P^{1/2}$, then we get

$$\begin{aligned}
&0 \leq \frac{1}{2}\left[P^{1/2} f\left(P^{-1/2} A P^{-1/2}\right) P^{1/2}+P^{1/2} f\left(P^{-1/2} B P^{-1/2}\right) P^{1/2}\right] \\
&-P^{1/2} f\left(P^{-1/2}\left(\frac{A+B}{2}\right) P^{-1/2}\right) P^{1/2} \\
&=\frac{1}{4} c P^{1/2}\left(P^{-1/2} B P^{-1/2}-P^{-1/2} A P^{-1/2}\right)^{2} P^{1/2} \\
&+\int_{0}^{\infty} \lambda^{3}\left[\frac{P(A+\lambda P)^{-1} P+P(B+\lambda P)^{-1} P}{2}\right. \\
&\left.-P\left(\frac{A+B}{2}+\lambda P\right)^{-1} P\right] d w(\lambda) .
\end{aligned}$$

Observe that

$$\begin{aligned}
&0 \leq P^{1/2}\left(P^{-1/2} B P^{-1/2}-P^{-1/2} A P^{-1/2}\right)^{2} P^{1/2} \\
&=P^{1/2}\left(P^{-1/2}(B-A) P^{-1/2}\right)^{2} P^{1/2} \\
&=P^{1/2} P^{-1/2}(B-A) P^{-1/2} P^{-1/2}(B-A) P^{-1/2} P^{1/2} \\
&=(B-A) P^{-1}(B-A)=\mathcal{P}_{f_{-1}}(P, B-A)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{0}^{\infty} \lambda^{3}\left[\frac{P(A+\lambda P)^{-1} P+P(B+\lambda P)^{-1} P}{2}\right. \\
&\left.-P\left(\frac{A+B}{2}+\lambda P\right)^{-1} P\right] d w(\lambda) \\
&=\int_{0}^{\infty} \lambda^{3}\left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P, P)+\mathcal{P}_{f_{-1}}(B+\lambda P, P)}{2}\right. \\
&\left.-\mathcal{P}_{f_{-1}}\left(\frac{A+B}{2}+\lambda P, P\right)\right] d w(\lambda) \\
&\geq 0,
\end{aligned}$$

where the inequality follows by (2.11), which proves the desired representation (3.1). □

Corollary 3. *With the assumptions of Theorem 3, we have*

$$0 \leq \frac{1}{4} c \mathcal{P}_{f_{-1}}(P, B-A) \leq \frac{1}{2}\left[\mathcal{P}_{f}(A, P)+\mathcal{P}_{f}(B, P)\right]-\mathcal{P}_{f}\left(\frac{A+B}{2}, P\right). \tag{3.4}$$

The first inequality is obvious since $c \geq 0$ in 1.2.

The case of transpose function $\tilde{f}$ is as follows:

Theorem 4. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.2). For all C, D, $Q>0$, we have*

$$\begin{aligned}0 &\le \frac{1}{2}\left[\mathcal{P}_{\tilde f}(C,Q)+\mathcal{P}_{\tilde f}(D,Q)\right]-\mathcal{P}_{\tilde f}\left(\frac{C+D}{2},Q\right)\\ &-c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right)\\ &=\int_0^\infty \lambda\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right.\\ &\left.-\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda).\end{aligned}\tag{3.5}$$

Proof. We have, from the representation (1.2), that

$$\begin{aligned}f\left(\frac{1}{t}\right)&=a+b\frac{1}{t}+c\frac{1}{t^2}+\frac{1}{t^2}\int_0^\infty\frac{\lambda}{\frac{1}{t}+\lambda}dw(\lambda)\\ &=a+b\frac{1}{t}+c\frac{1}{t^2}+\frac{1}{t^2}\int_0^\infty\frac{t\lambda}{1+t\lambda}dw(\lambda)\\ &=a+b\frac{1}{t}+c\frac{1}{t^2}+\frac{1}{t}\int_0^\infty\frac{\lambda}{1+t\lambda}dw(\lambda)\end{aligned}$$

for $t>0$.

If we multiply this identity by t, then we get

$$\tilde f(t)=tf\left(\frac{1}{t}\right)=at+b+c\frac{1}{t}+\int_0^\infty\lambda(1+t\lambda)^{-1}dw(\lambda)$$

for $t>0$.

We have for U, $V>0$ that

$$\begin{aligned}&\frac{1}{2}\left[\tilde f(U)+\tilde f(V)\right]-\tilde f\left(\frac{U+V}{2}\right)\\ &=\frac{1}{2}\left[aU+b+cU^{-1}+\int_0^\infty\lambda(1+U\lambda)^{-1}dw(\lambda)\right]\\ &+\frac{1}{2}\left[aV+b+cV^{-1}+\int_0^\infty\lambda(1+V\lambda)^{-1}dw(\lambda)\right]\\ &-\left[a\frac{U+V}{2}+b+c\left(\frac{U+V}{2}\right)^{-1}+\int_0^\infty\lambda\left(1+\frac{U+V}{2}\lambda\right)^{-1}dw(\lambda)\right]\\ &=c\left[\frac{U^{-1}+V^{-1}}{2}-\left(\frac{U+V}{2}\right)^{-1}\right]\\ &+\int_0^\infty\lambda\left[\frac{(1+U\lambda)^{-1}+(1+V\lambda)^{-1}}{2}-\left(1+\frac{U+V}{2}\lambda\right)^{-1}\right]dw(\lambda).\end{aligned}\tag{3.6}$$

If we take $U=Q^{-1/2}CQ^{-1/2}$, $V=Q^{-1/2}DQ^{-1/2}$ in (3.6), then we get

$$\begin{aligned}
&\frac{1}{2}\left[\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)+\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)\right]\\
&-\tilde{f}\left(\frac{Q^{-1/2}CQ^{-1/2}+Q^{-1/2}DQ^{-1/2}}{2}\right)\\
&=c\left[\frac{\left(Q^{-1/2}CQ^{-1/2}\right)^{-1}+\left(Q^{-1/2}DQ^{-1/2}\right)^{-1}}{2}\right.\\
&\left.-\left(\frac{Q^{-1/2}CQ^{-1/2}+Q^{-1/2}DQ^{-1/2}}{2}\right)^{-1}\right]\\
&=\int_0^\infty \lambda\left[\frac{\left(1+\lambda Q^{-1/2}CQ^{-1/2}\right)^{-1}+\left(1+\lambda Q^{-1/2}DQ^{-1/2}\right)^{-1}}{2}\right.\\
&\left.-\left(1+\lambda\frac{Q^{-1/2}CQ^{-1/2}+Q^{-1/2}DQ^{-1/2}}{2}\right)^{-1}\right]dw\left(\lambda\right),
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}\left[\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)+\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)\right]\\
&-\tilde{f}\left(Q^{-1/2}\frac{C+D}{2}Q^{-1/2}\right)\\
&=c\left[Q^{1/2}\frac{C^{-1}+D^{-1}}{2}Q^{1/2}-Q^{1/2}\left(\frac{C+D}{2}\right)^{-1}Q^{1/2}\right]\\
&+\int_0^\infty \lambda\left[\frac{Q^{1/2}\left(Q+\lambda C\right)^{-1}Q^{1/2}+Q^{1/2}\left(Q+\lambda D\right)^{-1}Q^{1/2}}{2}\right.\\
&\left.-Q^{1/2}\left(Q+\lambda\frac{C+D}{2}\right)^{-1}Q^{1/2}\right]dw\left(\lambda\right),
\end{aligned}$$

for C, D, $Q>0$.

If we multiply this equality both sides by $Q^{1/2}$, then we obtain

$$\begin{aligned}
&\frac{1}{2}\left[Q^{1/2}\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right)Q^{1/2}+Q^{1/2}\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)Q^{1/2}\right]\\
&-Q^{1/2}\tilde{f}\left(Q^{-1/2}\frac{C+D}{2}Q^{-1/2}\right)Q^{1/2}\\
&-c\left[Q\frac{C^{-1}+D^{-1}}{2}Q-Q\left(\frac{C+D}{2}\right)^{-1}Q\right]\\
&=\int_0^\infty \lambda\left[\frac{Q\left(Q+\lambda C\right)^{-1}Q+Q\left(Q+\lambda D\right)^{-1}Q}{2}\right.\\
&\left.-Q\left(Q+\lambda\frac{C+D}{2}\right)^{-1}Q\right]dw\left(\lambda\right)\\
&\geq 0,
\end{aligned}$$

where the inequality follows by (2.11), which proves the desired representation (3.1). □

Corollary 4. *With the assumptions of Theorem 4, we have*

$$0 \le c\left[\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right] \le \frac{1}{2}\left[\mathcal{P}_{\tilde{f}}(C,Q)+\mathcal{P}_{\tilde{f}}(D,Q)\right]-\mathcal{P}_{\tilde{f}}\left(\frac{C+D}{2},Q\right). \tag{3.7}$$

We also have:

Corollary 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]-\mathcal{P}_f\left(Q,\frac{C+D}{2}\right) \\ &- c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right) \\ &= \int_0^\infty \lambda\left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C,Q)+\mathcal{P}_{f_{-1}}(Q+\lambda D,Q)}{2}\right. \\ &\left. -\mathcal{P}_{f_{-1}}\left(Q+\lambda\frac{C+D}{2},Q\right)\right]dw(\lambda) \end{aligned} \tag{3.8}$$

and

$$0 \le c\left[\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right] \le \frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]-\mathcal{P}_f\left(Q,\frac{C+D}{2}\right). \tag{3.9}$$

4. Upper and Lower Bounds

We have:

Theorem 5. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.2). If $0<\alpha P\le A\le\beta P$, $0<\gamma P\le B\le\delta P$ and $0<mP\le B-A\le MP$ for some constants α, β, γ, δ, m, M, then*

$$\begin{aligned} 0 &< \frac{1}{2}m^2PK_w(\delta,\gamma,\alpha) \\ &\le \frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]-\mathcal{P}_f\left(\frac{A+B}{2},P\right)-\frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A) \\ &\le \frac{1}{2}M^2PK_w(\alpha,\beta,\delta), \end{aligned} \tag{4.1}$$

where

$$K_w(h,k,l) := \int_0^\infty \frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda}\frac{\lambda^3}{(h+\lambda)^4}dw(\lambda)$$

for h, k, $l>0$.

Proof. Since $0<\alpha P\le A\le\beta P$, $0<\gamma P\le B\le\delta P$ and $0<mP\le B-A\le MP$ for some constants α, β, γ, δ, m, M then

$$0<(\alpha+\lambda)P\le A+\lambda P\le(\beta+\lambda)P,$$
$$0<(\gamma+\lambda)P\le B+\lambda P\le(\delta+\lambda)P$$

and

$$0 < mP \le B + \lambda P - A - \lambda P \le MP$$

for all $\lambda \ge 0$.

By employing the inequality (2.13) we get

$$\begin{aligned} 0 &< \frac{1}{2}\left((\alpha+\lambda)^{-1} + (\gamma+\lambda)^{-1}\right)^{-1} \frac{m^2}{(\delta+\lambda)^4} P \\ &\le \frac{\mathcal{P}_{f_{-1}}(A+\lambda P, P) + \mathcal{P}_{f_{-1}}(B+\lambda P, P)}{2} - \mathcal{P}_{f_{-1}}\left(\frac{A+\lambda P + B + \lambda P}{2}, P\right) \\ &\le \frac{1}{2}\left((\beta+\lambda)^{-1} + (\delta+\lambda)^{-1}\right)^{-1} \frac{M^2}{(\alpha+\lambda)^4} P. \end{aligned}$$

By multiplying with λ^3 and integrating, we get

$$\begin{aligned} 0 &< \frac{1}{2} m^2 P \int_0^\infty \left((\alpha+\lambda)^{-1} + (\gamma+\lambda)^{-1}\right)^{-1} \frac{\lambda^2}{(\delta+\lambda)^4} dw(\lambda) \\ &\le \int_0^\infty \lambda^3 \left[\frac{\mathcal{P}_{f_{-1}}(A+\lambda P, P) + \mathcal{P}_{f_{-1}}(B+\lambda P, P)}{2}\right. \\ &\left. -\mathcal{P}_{f_{-1}}\left(\frac{A+\lambda P + B + \lambda P}{2}, P\right)\right] dw(\lambda) \\ &\le \frac{1}{2} M^2 P \int_0^\infty \left((\beta+\lambda)^{-1} + (\delta+\lambda)^{-1}\right)^{-1} \frac{\lambda^2}{(\alpha+\lambda)^4} dw(\lambda). \end{aligned} \tag{4.2}$$

This is equivalent, via Theorem 3 to

$$\begin{aligned} 0 &< \frac{1}{2} m^2 P \int_0^\infty \frac{(\gamma+\lambda)(\alpha+\lambda)}{\alpha+\gamma+2\lambda} \frac{\lambda^3}{(\delta+\lambda)^4} dw(\lambda) \\ &\le \frac{1}{2}\left[\mathcal{P}_f(A,P) + \mathcal{P}_f(B,P)\right] - \mathcal{P}_f\left(\frac{A+B}{2}, P\right) - \frac{1}{4} c \mathcal{P}_{f_{-1}}(P, B-A) \\ &\le \frac{1}{2} M^2 P \int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)}{\delta+\beta+2\lambda} \frac{\lambda^3}{(\alpha+\lambda)^4} dw(\lambda), \end{aligned} \tag{4.3}$$

and the theorem is proved. □

By taking the derivative over t in (3.2), we get

$$f'(t) = b + 2ct + \int_0^\infty \left[1 - \lambda^2 (t+\lambda)^{-2}\right] \lambda dw(\lambda),$$

$$f''(t) = 2c + 2\int_0^\infty \lambda^3 (t+\lambda)^{-3} dw(\lambda) > 0$$

and

$$f'''(t) = -6\int_0^\infty \lambda^3 (t+\lambda)^{-4} dw(\lambda) < 0,$$

which gives that

$$\int_0^\infty \frac{\lambda^3}{(t+\lambda)^3} dw(\lambda) = \frac{1}{2} f''(t) - c > 0, \ \ t > 0 \tag{4.4}$$

and

$$\int_0^\infty \frac{\lambda^3}{(t+\lambda)^4} dw(\lambda) = -\frac{1}{6} f'''(t) > 0, \; t > 0. \tag{4.5}$$

Corollary 6. *With the assumptions of Theorem 5, we have the upper bound*

$$\begin{aligned} 0 &\le \frac{1}{2}[\mathcal{P}_f(A,P) + \mathcal{P}_f(B,P)] - \mathcal{P}_f\left(\frac{A+B}{2}, P\right) - \frac{1}{4} c \mathcal{P}_{f_{-1}}(P, B-A) \\ &\le \frac{1}{4} M^2 P \left[\frac{1}{6}\left(\alpha - \frac{\delta+\beta}{2}\right) f'''(\alpha) + \frac{1}{2} f''(\alpha) - c\right] \\ &\le \frac{1}{8} M^2 P \left[\frac{1}{3}\left(\alpha - \frac{\delta+\beta}{2}\right) f'''(\alpha) + f''(\alpha)\right]. \end{aligned} \tag{4.6}$$

Proof. Using the elementary inequality

$$ab \le \frac{1}{4}(a+b)^2, \; a, b \ge 0$$

we get

$$\frac{(k+\lambda)(l+\lambda)}{l+k+2\lambda} \le \frac{1}{4}(l+k+2\lambda)$$

for $\lambda \ge 0$.

From this inequality we get

$$\begin{aligned} \frac{(\beta+\lambda)(\delta+\lambda)}{\delta+\beta+2\lambda} \frac{\lambda^3}{(\alpha+\lambda)^4} &\le \frac{1}{4}(\delta+\beta+2\lambda) \frac{\lambda^3}{(\alpha+\lambda)^4} \\ &= \frac{1}{4}(\delta+\beta) \frac{\lambda^3}{(\alpha+\lambda)^4} + \frac{1}{2} \frac{\lambda^4}{(\alpha+\lambda)^4}. \end{aligned} \tag{4.7}$$

Observe also that

$$\frac{\lambda^4}{(\alpha+\lambda)^4} = \frac{\lambda^4 + \alpha\lambda^3 - \alpha\lambda^3}{(\alpha+\lambda)^4} = \frac{\lambda^3}{(\alpha+\lambda)^3} - \alpha \frac{\lambda^3}{(\alpha+\lambda)^4}$$

and

$$\begin{aligned} &\frac{1}{4}(\delta+\beta) \frac{\lambda^3}{(\alpha+\lambda)^4} + \frac{1}{2} \frac{\lambda^4}{(\alpha+\lambda)^4} \\ &= \frac{1}{4}(\delta+\beta) \frac{\lambda^3}{(\alpha+\lambda)^4} + \frac{1}{2}\left(\frac{\lambda^3}{(\alpha+\lambda)^3} - \alpha \frac{\lambda^3}{(\alpha+\lambda)^4}\right) \\ &= \frac{1}{2}\left(\frac{\delta+\beta}{2} - \alpha\right) \frac{\lambda^3}{(\alpha+\lambda)^4} + \frac{1}{2} \frac{\lambda^3}{(\alpha+\lambda)^3}. \end{aligned}$$

Therefore, by integrating (4.7) we derive

$$\begin{aligned} &\int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)}{\delta+\beta+2\lambda} \frac{\lambda^3}{(\alpha+\lambda)^4} dw(\lambda) \\ &\le \frac{1}{12}\left(\alpha - \frac{\delta+\beta}{2}\right) f'''(\alpha) + \frac{1}{2}\left[\frac{1}{2} f''(\alpha) - c\right] \end{aligned}$$

and by (4.1) we derive (4.6). □

Since the nonnegative parameter c in the representation (1.2) is not always available, then we can state the following result as well:

Corollary 7. *Assume that $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$. If $0<\alpha P\leq A\leq\beta P$, $0\leq B\leq\delta P$ and $0\leq B-A\leq MP$ for some constants α, β, γ, δ, m, M, then*

$$
\begin{aligned}
(4.8)\qquad 0 &\leq \frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]-\mathcal{P}_f\left(\frac{A+B}{2},P\right)\\
&\leq \frac{1}{8}M^2P\left[\frac{1}{3}\left(\alpha-\frac{\delta+\beta}{2}\right)f'''(\alpha)+f''(\alpha)\right].
\end{aligned}
$$

Proof. From (4.6) we derive

$$
\begin{aligned}
(4.9)\qquad 0 &\leq \frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A)\\
&\leq \frac{1}{2}\left[\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P)\right]-\mathcal{P}_f\left(\frac{A+B}{2},P\right)\\
&\leq \frac{1}{4}M^2P\left[\frac{1}{6}\left(\alpha-\frac{\delta+\beta}{2}\right)f'''(\alpha)+\frac{1}{2}f''(\alpha)-c\right]+\frac{1}{4}c\mathcal{P}_{f_{-1}}(P,B-A)\\
&= \frac{1}{8}M^2P\left[\frac{1}{3}\left(\alpha-\frac{\delta+\beta}{2}\right)f'''(\alpha)+f''(\alpha)\right]\\
&\quad+\frac{1}{4}c\left[\mathcal{P}_{f_{-1}}(P,B-A)-M^2P\right].
\end{aligned}
$$

Since $0\leq B-A\leq MP$, then $0\leq P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\leq M$, which implies that

$$
0\leq\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)^2\leq M^2
$$

and by multiplying both sides by $P^{1/2}$, we get

$$
0\leq P^{1/2}\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)^2P^{1/2}\leq M^2P.
$$

Also, using the fact that

$$
\mathcal{P}_{f_{-1}}(P,B-A)=P^{1/2}\left(P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}\right)^2P^{1/2},
$$

we conclude that $\mathcal{P}_{f_{-1}}(P,B-A)-M^2P\leq 0$ and by (4.9) we deduce (4.8). □

We also have:

Theorem 6. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.2). If $0<\kappa Q\leq C\leq\mu Q$, $0<\nu Q\leq D\leq\xi Q$ and $0<nQ\leq D-C\leq NQ$ for some constants α, β, γ, δ, m, M, then*

$$
\begin{aligned}
(4.10)\qquad 0 &< \frac{1}{2}n^2L_w(\xi,\kappa,\nu)\,Q,\\
&\leq \mathcal{P}_f\left(Q,\frac{C+D}{2}\right)-\frac{1}{2}\left[\mathcal{P}_f(Q,C)+\mathcal{P}_f(Q,D)\right]\\
&\quad-c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q)+\mathcal{P}_{f_{-1}}(D,Q)\right]-\mathcal{P}_{f_{-1}}\left(\frac{C+D}{2},Q\right)\right)\\
&\leq \frac{1}{2}N^2L_w(\kappa,\mu,\xi)\,Q,
\end{aligned}
$$

where

$$L_w(h,k,l) := \int_0^\infty \frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda} \frac{\lambda^3}{(1+h\lambda)^4} dw(\lambda)$$

for $h,\ k,\ l > 0$.

Proof. Since $0 < \kappa Q \le C \le \mu Q$, $0 < \nu Q \le D \le \xi Q$ and $0 < nQ \le D - C \le NQ$ for some constants α, β, γ, δ, m, M, then

$$(1+\kappa\lambda) Q \le Q + \lambda C \le (1+\mu\lambda) Q,$$
$$(1+\nu\lambda) Q \le Q + \lambda D \le (1+\xi\lambda) Q$$

and

$$n\lambda Q \le Q + \lambda D - Q - \lambda C \le \lambda N Q$$

for $\lambda > 0$.

By employing the inequality (2.13) we get

$$\begin{aligned} 0 &< \frac{1}{2}\left((1+\kappa\lambda)^{-1} + (1+\nu\lambda)^{-1}\right)^{-1} \frac{n^2\lambda^2}{(1+\xi\lambda)^4} Q \\ &\le \frac{\mathcal{P}_{f_{-1}}(Q+\lambda C, Q) + \mathcal{P}_{f_{-1}}(Q+\lambda D, Q)}{2} \\ &- \mathcal{P}_{f_{-1}}\left(Q + \lambda\frac{C+D}{2}, Q\right) \\ &\le \frac{1}{2}\left((1+\mu\lambda)^{-1} + (1+\xi\lambda)^{-1}\right)^{-1} \frac{\lambda^2 N^2}{(1+\kappa\lambda)^4} Q \end{aligned} \tag{4.11}$$

for $\lambda > 0$.

If we multiply by $\lambda \ge 0$ and integrate (4.11), then we get

$$\begin{aligned} 0 &< \frac{1}{2}n^2 \left(\int_0^\infty \left((1+\kappa\lambda)^{-1} + (1+\nu\lambda)^{-1}\right)^{-1} \frac{\lambda^3}{(1+\xi\lambda)^4} dw(\lambda)\right) Q \\ &\le \int_0^\infty \lambda \left[\frac{\mathcal{P}_{f_{-1}}(Q+\lambda C, Q) + \mathcal{P}_{f_{-1}}(Q+\lambda D, Q)}{2}\right. \\ &\left. -\mathcal{P}_{f_{-1}}\left(Q + \lambda\frac{C+D}{2}, Q\right)\right] dw(\lambda) \\ &\le \frac{1}{2}N^2 \left(\int_0^\infty \left((1+\mu\lambda)^{-1} + (1+\xi\lambda)^{-1}\right)^{-1} \frac{\lambda^3}{(1+\kappa\lambda)^4} dw(\lambda)\right) Q \end{aligned}$$

and by (3.6)

$$\begin{aligned} 0 &< \frac{1}{2}n^2 \left(\int_0^\infty \frac{(1+\kappa\lambda)(1+\nu\lambda)}{2+(\kappa+\nu)\lambda} \frac{\lambda^3}{(1+\xi\lambda)^4} dw(\lambda)\right) Q \\ &\le \mathcal{P}_f\left(Q, \frac{C+D}{2}\right) - \frac{1}{2}\left[\mathcal{P}_f(Q,C) + \mathcal{P}_f(Q,D)\right] \\ &- c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q) + \mathcal{P}_{f_{-1}}(D,Q)\right] - \mathcal{P}_{f_{-1}}\left(\frac{C+D}{2}, Q\right)\right) \\ &\le \frac{1}{2}N^2 \left(\int_0^\infty \frac{(1+\mu\lambda)(1+\xi\lambda)}{2+(\mu+\xi)\lambda} \frac{\lambda^3}{(1+\kappa\lambda)^4} dw(\lambda)\right) Q \end{aligned}$$

and the inequality (4.10) is obtained. □

Corollary 8. *With the assumptions of Theorem 6, we have*

$$0 \le \mathcal{P}_f\left(Q, \frac{C+D}{2}\right) - \frac{1}{2}\left[\mathcal{P}_f(Q,C) + \mathcal{P}_f(Q,D)\right] \tag{4.12}$$
$$- c\left(\frac{1}{2}\left[\mathcal{P}_{f_{-1}}(C,Q) + \mathcal{P}_{f_{-1}}(D,Q)\right] - \mathcal{P}_{f_{-1}}\left(\frac{C+D}{2}, Q\right)\right)$$
$$\le \frac{1}{4\kappa^3}\left[\left(1 + \frac{\mu+\xi}{2\kappa}\right)\left(f''(1/\kappa) - c\right) + \frac{1}{6}\frac{(\mu+\xi) f'''(1/\kappa)}{\kappa^2}\right].$$

Proof. Using the elementary inequality

$$ab \le \frac{1}{4}(a+b)^2, \ a, b \ge 0$$

we get

$$\frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda} \le \frac{1}{4}\left[2 + (k+l)\lambda\right],$$

therefore

$$G_w(h,k,l) = \int_0^\infty \frac{(1+k\lambda)(1+l\lambda)}{2+(k+l)\lambda} \frac{\lambda^3}{(1+h\lambda)^4} dw(\lambda) \tag{4.13}$$
$$\le \frac{1}{4}\int_0^\infty \left[2+(k+l)\lambda\right] \frac{\lambda^3}{(1+h\lambda)^4} dw(\lambda)$$
$$= \frac{1}{4}\left[2\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4} dw(\lambda) + (k+l)\int_0^\infty \frac{\lambda^4}{(1+h\lambda)^4} dw(\lambda)\right]$$
$$\le \frac{1}{4}\left[2\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^3} dw(\lambda) + (k+l)\int_0^\infty \frac{\lambda^4}{(1+h\lambda)^4} dw(\lambda)\right].$$

From (4.4) and (4.5) we have

$$\int_0^\infty \frac{\lambda^3}{(1/h+\lambda)^3} dw(\lambda) = \frac{1}{2} f''(1/h) - c > 0, \ h > 0$$

and

$$\int_0^\infty \frac{\lambda^3}{(1/h+\lambda)^4} dw(\lambda) = -\frac{1}{6} f'''(1/h) > 0, \ t > 0,$$

namely

$$\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^3} dw(\lambda) = \frac{1}{2}\left[\frac{f''(1/h) - c}{h^3}\right] > 0, \ h > 0 \tag{4.14}$$

and

$$\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4} dw(\lambda) = -\frac{1}{6}\frac{f'''(1/h)}{h^4} > 0, \ h > 0. \tag{4.15}$$

Observe that

$$\begin{aligned}
&\int_0^\infty \frac{\lambda^4}{(1+h\lambda)^4}dw(\lambda) \\
&= \frac{1}{h}\int_0^\infty \frac{h\lambda^4+\lambda^3-\lambda^3}{(1+h\lambda)^4}dw(\lambda) = \frac{1}{h}\int_0^\infty \frac{(h\lambda+1)\lambda^3-\lambda^3}{(1+h\lambda)^4}dw(\lambda) \\
&= \frac{1}{h}\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^3}dw(\lambda) - \frac{1}{h}\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^4}dw(\lambda) \\
&= \frac{f''(1/h)-c}{2h^4} + \frac{1}{6}\frac{f'''(1/h)}{h^5}.
\end{aligned}$$

Then

$$\begin{aligned}
&2\int_0^\infty \frac{\lambda^3}{(1+h\lambda)^3}dw(\lambda) + (k+l)\int_0^\infty \frac{\lambda^4}{(1+h\lambda)^4}dw(\lambda) \\
&= \frac{f''(1/h)-c}{h^3} + \frac{(k+l)(f''(1/h)-c)}{2h^4} + \frac{1}{6}\frac{(k+l)f'''(1/h)}{h^5}
\end{aligned}$$

and by the second inequality in (4.9) we obtain (4.12). □

5. Some Examples

We extend the *weighted geometric mean* for $r \in [-1,0) \cup [1,2]$ and A, $B > 0$ as

$$\mathcal{P}_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

where $f_r(t) = t^r$, is operator convex on $(0,\infty)$.

If $0 < \alpha P \le A \le \beta P$, $0 \le B \le \delta P$ and $0 \le B - A \le MP$ for some constants α, β, γ, δ, m, M, then by (4.8)

$$\begin{aligned}
(5.1)\qquad 0 &\le \frac{1}{2}(P\sharp_r A + P\sharp_r B) - P\sharp_r\left(\frac{A+B}{2}\right) \\
&\le \frac{1}{8}r(r-1)M^2P\alpha^{r-3}\left[\frac{1}{3}\left(\alpha - \frac{\delta+\beta}{2}\right)(r-2)+\alpha\right].
\end{aligned}$$

The function $f(t) = -\ln t$ is operator convex on $(0,\infty)$. Consider

$$\mathcal{P}_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

the *relative operator entropy* $S(A|B)$ defined in Introduction. Then by (4.8)

$$(5.2)\qquad 0 \le \mathcal{S}\left(P|\frac{A+B}{2}\right) - \frac{1}{2}\left[\mathcal{S}(P|A) + \mathcal{S}(P|B)\right] \le \frac{1}{24}\frac{M^2P}{\alpha^3}(\alpha+\beta+\delta)$$

if $0 < \alpha P \le A \le \beta P$, $0 \le B \le \delta P$ and $0 \le B - A \le MP$.

Consider the operator convex function $g(t) = t\ln t$, $t > 0$. We have

$$f'(t) = \ln t + 1,\ f''(t) = \frac{1}{t},\ f'''(t) = -\frac{1}{t^2},\ t > 0$$

and

$$\mathcal{P}_g(B,A) = \mathcal{S}(B|A)$$

since $g(t) = t\ln t = -t\ln\left(\frac{1}{t}\right) = \tilde{f}(t)$, $t > 0$.

From (4.8) we get for $0 < \alpha P \leq A \leq \beta P$, $0 \leq B \leq \delta P$ and $0 \leq B - A \leq MP$ that

$$0 \leq \frac{1}{2}\left[\mathcal{S}\left(A|P\right)+\mathcal{S}\left(B|P\right)\right]-\mathcal{S}\left(\frac{A+B}{2},P\right) \leq \frac{1}{48}\frac{M^2 P}{\alpha^2}\left(2\alpha+\delta+\beta\right). \tag{5.3}$$

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