LOWER BOUNDS ON PARTIAL SUPERADDITIVITY OF NONCOMMUTATIVE PERSPECTIVES FOR OPERATORS IN HILBERT SPACES: THE CASE OF FIRST VARIABLE

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Abstract. Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. We define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $A,\ B>0$. In this paper we show among others that, if $A,\ B,\ P>0$ with $BP^{-1}A+AP^{-1}B\geq kP$, then

$$\mathcal{P}_{\ell f}(A+B,P)-\mathcal{P}_{\ell f}(A,P)-\mathcal{P}_{\ell f}(B,P)\geq k\mathcal{P}_{f'}(A+B,P).$$

If the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and $BP^{-1}A+AP^{-1}B\geq 0$, then

$$\mathcal{P}_f(A+B,P)+f(0)P\geq\mathcal{P}_f(A,P)+\mathcal{P}_f(B,P).$$

Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

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for all $t > 0$.

A real valued continuous function f on $[0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [11], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.4}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda) A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0) t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.5}$$

where $c \geq 0$ and a positive measure μ on $[0, \infty)$ such that (1.2) holds.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone. For other examples, see [7], [8], [9], [13] and the references therein.

Assume that A, $B \geq 0$. In the recent paper [12], Moslehian and Najafi showed that $AB + BA$ is positive if and only if the following *operator subadditivity property* holds

$$f(A+B) \leq f(A) + f(B) \tag{1.6}$$

for all nonnegative operator monotone functions f on $[0, \infty)$. For some interesting consequences of this result see [12].

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B, A)$ by setting

$$\mathcal{P}_f(B, A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [15]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A). \tag{1.7}$$

It is well known that (see [3] and [2] or [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r:[0,\infty)\to[0,\infty)$, $f_r(t)=t^r$, $r\in[0,1]$, then

$$\mathcal{P}_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

If we take the function $f=\ln$, then

$$\mathcal{P}_{\ln}(B,A) := A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} =: S(A|B),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [5], [6] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [14].

In this paper we show among others that, if $A,\ B,\ P>0$ with $BP^{-1}A+AP^{-1}B\geq kP$, then

$$\mathcal{P}_{\ell f}(A+B,P) - \mathcal{P}_{\ell f}(A,P) - \mathcal{P}_{\ell f}(B,P) \geq k\mathcal{P}_{f'}(A+B,P),$$

where $\ell(t)=t$ and f is operator monotone in $[0,\infty)$.

If the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and $BP^{-1}A+AP^{-1}B\geq 0$, then

$$\mathcal{P}_f(A+B,P) + f(0)P - \mathcal{P}_f(A,P) - \mathcal{P}_f(B,P) \geq 0.$$

Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

2. General Results

For a positive measure μ on $(0,\infty)$ we define $g_\mu:(0,\infty)\to(0,\infty)$ by

$$g_\mu(t) := t^2\int_0^\infty \frac{\lambda}{t+\lambda}d\mu(\lambda), \tag{2.1}$$

provided the integral is convergent for all $t\in(0,\infty)$.

We start with the following result regarding the operator quasi-superadditivity property of g_μ :

Lemma 1. *For all* C, $D>0$ *we have*

$$g_{\mu}(C+D)-g_{\mu}(C)-g_{\mu}(D) \geq \int_{0}^{\infty}\lambda^{2}(C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1}d\mu(\lambda). \tag{2.2}$$

If $DC+CD\geq 0$, *then*

$$g_{\mu}(C+D)\geq g_{\mu}(C)+g_{\mu}(D). \tag{2.3}$$

Proof. Assume that C, $D>0$. Define

$$K_{\lambda}(C,D):=(C+\lambda)^{-1}+(D+\lambda)^{-1}-(C+D+\lambda)^{-1},$$

where $\lambda\geq 0$.

Therefore

$$\begin{aligned}
&(C+D+\lambda)K_{\lambda}(C,D)(C+D+\lambda)\\
&=(C+D+\lambda)(C+\lambda)^{-1}(C+D+\lambda)\\
&+(C+D+\lambda)(D+\lambda)^{-1}(C+D+\lambda)-C-D-\lambda\\
&=\left(1+D(C+\lambda)^{-1}\right)(C+\lambda+D)\\
&+\left(C(D+\lambda)^{-1}+1\right)(C+D+\lambda)-C-D-\lambda\\
&=C+\lambda+D+D+D(C+\lambda)^{-1}D\\
&+C(D+\lambda)^{-1}C+C+C+D+\lambda-C-D-\lambda\\
&=D(C+\lambda)^{-1}D+C(D+\lambda)^{-1}C+2(C+D)+\lambda=:L_{\lambda}(C,D).
\end{aligned} \tag{2.4}$$

If C, D, $\lambda>0$, then $L_{\lambda}(C,D)\geq 0$, and by multiplying both sides of (2.4) with $(C+D+\lambda)^{-1}$ we get

$$K_{\lambda}(C,D)=(C+D+\lambda)^{-1}L_{\lambda}(C,D)(C+D+\lambda)^{-1}.$$

Further, define for $\lambda>0$

$$W_{\lambda}(C,D):=1-\lambda K_{\lambda}(C,D).$$

Then

$$\begin{aligned}
&(C+D+\lambda)W_{\lambda}(C,D)(C+D+\lambda)\\
&=(C+D+\lambda)(1-\lambda K_{\lambda}(C,D))(C+D+\lambda)\\
&=(C+D+\lambda)^{2}-\lambda(C+D+\lambda)K_{\lambda}(C,D)(C+D+\lambda)\\
&=(C+D+\lambda)(C+D+\lambda)\\
&-\lambda\left[D(C+\lambda)^{-1}D+C(D+\lambda)^{-1}C+2(C+D)+\lambda\right]
\end{aligned}$$

$$= C^2 + DC + \lambda C + CD + D^2 + \lambda D + \lambda C + \lambda D + \lambda^2$$
$$- \lambda D (C+\lambda)^{-1} D - \lambda C (D+\lambda)^{-1} C - 2\lambda (C+D) - \lambda^2$$
$$= C^2 + D^2 + DC + CD - \lambda D (C+\lambda)^{-1} D - \lambda C (D+\lambda)^{-1} C$$
$$= C (D+\lambda)^{-1} (D+\lambda) C - \lambda C (D+\lambda)^{-1} C$$
$$+ D (C+\lambda)^{-1} (C+\lambda) D - \lambda D (C+\lambda)^{-1} D$$
$$+ DC + CD = C (D+\lambda)^{-1} DC + D (C+\lambda)^{-1} CD + DC + CD,$$

which implies that

$$W_\lambda (C, D)$$
$$= (C+D+\lambda)^{-1} \left[C (D+\lambda)^{-1} DC + D (C+\lambda)^{-1} CD + DC + CD \right]$$
$$\times (C+D+\lambda)^{-1}.$$

For $t > 0$ we have

$$g_\mu (t) := \int_0^\infty \lambda t^2 (t+\lambda)^{-1} d\mu (\lambda) \tag{2.5}$$
$$= \int_0^\infty \lambda (t+\lambda-\lambda)^2 (t+\lambda)^{-1} d\mu (\lambda)$$
$$= \int_0^\infty \lambda \left[(t+\lambda)^2 - 2\lambda (t+\lambda) + \lambda^2 \right] (t+\lambda)^{-1} d\mu (\lambda)$$
$$= \int_0^\infty \lambda \left[(t+\lambda) - 2\lambda + \lambda^2 (t+\lambda)^{-1} \right] d\mu (\lambda)$$
$$= \int_0^\infty \lambda \left[t - \lambda + \lambda^2 (t+\lambda)^{-1} \right] d\mu (\lambda).$$

We also have the representation

$$g_\mu (C) + g_\mu (D) - g_\mu (C+D) \tag{2.6}$$
$$= \int_0^\infty \lambda \left[C - \lambda + \lambda^2 (C+\lambda)^{-1} \right] d\mu (\lambda)$$
$$+ \int_0^\infty \lambda \left[D - \lambda + \lambda^2 (D+\lambda)^{-1} \right] d\mu (\lambda)$$
$$- \int_0^\infty \lambda \left[C + D - \lambda + \lambda^2 (C+D+\lambda)^{-1} \right] d\mu (\lambda)$$
$$= \int_0^\infty \lambda \left[\lambda^2 (C+\lambda)^{-1} + \lambda^2 (D+\lambda)^{-1} - \lambda - \lambda^2 (C+D+\lambda)^{-1} \right] d\mu (\lambda)$$
$$= \int_0^\infty \lambda^3 \left[(C+\lambda)^{-1} + (D+\lambda)^{-1} - \lambda^{-1} - (C+D+\lambda)^{-1} \right] d\mu (\lambda)$$
$$= \int_0^\infty \lambda^3 \left(K_\lambda (C, D) - \lambda^{-1} \right) d\mu (\lambda).$$

Put

$$\begin{aligned} Y_{\lambda}(C,D) &:= K_{\lambda}(C,D) - \lambda^{-1} = \lambda^{-1}(\lambda K_{\lambda}(C,D) - 1) \\ &= -\lambda^{-1}(1 - \lambda K_{\lambda}(C,D)) = -\lambda^{-1} W_{\lambda}(C,D) \\ &= -\lambda^{-1}(C+D+\lambda)^{-1} \\ &\times \left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD + DC + CD\right] \\ &\times (C+D+\lambda)^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} & g_{\mu}(C) + g_{\mu}(D) - g_{\mu}(C+D) \\ &= \int_0^{\infty} \lambda^3 Y_{\lambda} d\mu(\lambda) \\ &= -\int_0^{\infty} \lambda^2 (C+D+\lambda)^{-1} \\ &\times \left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD + DC + CD\right] \\ &\times (C+D+\lambda)^{-1} d\mu(\lambda), \end{aligned}$$

which gives the following identity of interest:

$$\begin{aligned} & g_{\mu}(C+D) - g_{\mu}(C) - g_{\mu}(D) \\ &= \int_0^{\infty} \lambda^2 (C+D+\lambda)^{-1} \\ &\times \left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD + DC + CD\right] \\ &\times (C+D+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^{\infty} \lambda^2 (C+D+\lambda)^{-1} \\ &\times \left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD\right](C+D+\lambda)^{-1} d\mu(\lambda) \\ &+ \int_0^{\infty} \lambda^2 (C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} d\mu(\lambda). \end{aligned} \tag{2.7}$$

Since for all C, $D > 0$,

$$(D+\lambda)^{-1} D > 0, \ \ (C+\lambda)^{-1} C > 0$$

for $\lambda \geq 0$, then

$$C(D+\lambda)^{-1}DC, \ \ D(C+\lambda)^{-1}CD \geq 0$$

that gives

$$C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD \geq 0,$$

which implies that

$$(C+D+\lambda)^{-1}\left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD\right](C+D+\lambda)^{-1} \geq 0$$

for $\lambda \geq 0$.

If we multiply this inequality by $\lambda^2 \geq 0$ and integrate on $[0,\infty)$ over μ, we get

$$\begin{aligned}&\int_0^\infty \lambda^2 (C+D+\lambda)^{-1} \\ &\times \left[C(D+\lambda)^{-1}DC + D(C+\lambda)^{-1}CD\right](C+D+\lambda)^{-1} d\mu(\lambda) \\ &\geq 0\end{aligned}$$

for all C, $D>0$.

Then

$$\begin{aligned}&g_\mu(C+D) - g_\mu(C) - g_\mu(D) \\ &\geq \int_0^\infty \lambda^2 (C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} d\mu(\lambda),\end{aligned}$$

for all C, $D>0$ and the inequality (2.2) is proved.

If $DC+CD \geq 0$, then by multiplying both sides by $(C+D+\lambda)^{-1}$ we get

$$(C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} \geq 0$$

and by multiplying with $\lambda^2 \geq 0$ and integrating on $(0,\infty)$ over $d\mu(\lambda)$ we derive (2.3). □

Corollary 1. *With the assumptions of Lemma 1 and if $DC+CD \geq k$ some some real number k, then*

$$\begin{aligned}&g_\mu(C+D) - g_\mu(C) - g_\mu(D) \\ &\geq k\left[g'_\mu(C+D) - g_\mu(C+D)(C+D)^{-1}\right](C+D)^{-1}.\end{aligned} \tag{2.8}$$

Proof. If we multiply the inequality $DC+CD \geq k$ both sides by $(C+D+\lambda)^{-1}$, then we get

$$(C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} \geq k(C+D+\lambda)^{-2}$$

for $\lambda \geq 0$.

If we multiply this inequality by $\lambda^2 \geq 0$ and integrate, then we obtain

$$\begin{aligned}&\int_0^\infty \lambda^2 (C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} d\mu(\lambda) \\ &\geq k\int_0^\infty \lambda^2 (C+D+\lambda)^{-2} d\mu(\lambda).\end{aligned} \tag{2.9}$$

If we take the derivative in (2.5) we get

$$\begin{aligned}g'_\mu(t) &:= \int_0^\infty \lambda\left[1 - \frac{\lambda^2}{(t+\lambda)^2}\right] d\mu(\lambda) = \int_0^\infty \lambda\left[\frac{(t+\lambda)^2 - \lambda^2}{(t+\lambda)^2}\right] d\mu(\lambda) \\ &= \int_0^\infty \lambda\left[\frac{t^2+2t\lambda}{(t+\lambda)^2}\right] d\mu(\lambda) \\ &= t^2\int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda) + 2t\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda),\end{aligned}$$

which gives that

$$2t\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) = g'_\mu(t) - t^2\int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda)$$

namely

$$\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) = \frac{1}{2t} g_\mu'(t) - \frac{t}{2} \int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda). \tag{2.10}$$

From (2.1) we have

$$\frac{g_\mu(t)}{t^2} = \int_0^\infty \frac{\lambda}{t+\lambda} d\mu(\lambda),$$

which, by derivation, gives that

$$\frac{g_\mu'(t) t^2 - 2t g_\mu(t)}{t^4} = -\int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda),$$

namely

$$\frac{g_\mu'(t) t - 2 g_\mu(t)}{2t^2} = -\frac{t}{2} \int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda),$$

and by (2.10)

$$\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) = \frac{1}{2t} g_\mu'(t) + \frac{g_\mu'(t) t - 2 g_\mu(t)}{2t^2} = \frac{g_\mu'(t)}{t} - \frac{g_\mu(t)}{t^2}. \tag{2.11}$$

Therefore

$$\begin{aligned} 0 &\le \int_0^\infty \lambda^2 (C+D+\lambda)^{-2} d\mu(\lambda) \\ &= \left[g_\mu'(C+D) - g_\mu(C+D)(C+D)^{-1} \right] (C+D)^{-1} \end{aligned}$$

and by (2.2) we derive (2.8). □

Remark 1. *With the assumptions of Lemma 1 and if* $DC + CD \ge k \ge 0$, *then we have the refinement of (2.3)*

$$\begin{aligned} & g_\mu(C+D) - g_\mu(C) - g_\mu(D) \\ & \ge k \left[g_\mu'(C+D) - g_\mu(C+D)(C+D)^{-1} \right] (C+D)^{-1} \ge 0. \end{aligned} \tag{2.12}$$

Remark 2. *The symmetrized product of two operators* A, $B \in B(H)$ *is defined by* $S(A,B) = AB + BA$. *In general, the symmetrized product of two operators* A, B *is not positive (see for instance* [13]*). Also Gustafson* [10] *showed that if* $0 \le m \le A \le M$ *and* $0 \le n \le B \le N$, *then we have the lower bound*

$$S(A,B) \ge 2mn - \frac{1}{4}(M-m)(N-n) =: k, \tag{2.13}$$

which can take positive or negative values depending on the parameters m, M, n, N.

We consider the perspective generated by g_μ,

$$\mathcal{P}_{g_\mu}(B,A) := A^{1/2} g_\mu \left(A^{-1/2} B A^{-1/2} \right) A^{1/2} \tag{2.14}$$

for A, $B > 0$.

Theorem 3. *For all A, B, $P>0$ we have*

$$\begin{aligned}&\mathcal{P}_{g_\mu}(A+B,P)-\mathcal{P}_{g_\mu}(A,P)-\mathcal{P}_{g_\mu}(B,P)\\&\geq\int_0^\infty\lambda^2P(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)\\&\times(A+B+\lambda P)^{-1}Pd\mu(\lambda).\end{aligned}\tag{2.15}$$

If $BP^{-1}A+AP^{-1}B\geq 0$, then

$$\mathcal{P}_{g_\mu}(A+B,P)\geq\mathcal{P}_{g_\mu}(A,P)+\mathcal{P}_{g_\mu}(B,P).\tag{2.16}$$

Proof. From (2.2) we get for $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$ that

$$\begin{aligned}&g_\mu\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}\right)\\&-g_\mu\left(P^{-1/2}AP^{-1/2}\right)-g_\mu\left(P^{-1/2}BP^{-1/2}\right)\\&\geq\int_0^\infty\lambda^2\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\\&\times\left(P^{-1/2}BP^{-1/2}P^{-1/2}AP^{-1/2}+P^{-1/2}AP^{-1/2}P^{-1/2}BP^{-1/2}\right)\\&\times\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}+\lambda\right)^{-1}d\mu(\lambda).\end{aligned}\tag{2.17}$$

Observe that

$$\begin{aligned}\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}+\lambda\right)^{-1}&=\left(P^{-1/2}(A+B+\lambda P)P^{-1/2}\right)^{-1}\\&=P^{1/2}(A+B+\lambda P)^{-1}P^{1/2}\end{aligned}$$

and

$$\begin{aligned}&P^{-1/2}BP^{-1/2}P^{-1/2}AP^{-1/2}+P^{-1/2}AP^{-1/2}P^{-1/2}BP^{-1/2}\\&=P^{-1/2}\left(BP^{-1}A+AP^{-1}B\right)P^{-1/2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^\infty\lambda^2P^{1/2}(A+B+\lambda P)^{-1}P^{1/2}P^{-1/2}\left(BP^{-1}A+AP^{-1}B\right)P^{-1/2}\\&\times P^{1/2}(A+B+\lambda P)^{-1}P^{1/2}d\mu(\lambda)\\&=\int_0^\infty\lambda^2P^{1/2}(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)(A+B+\lambda P)^{-1}P^{1/2}d\mu(\lambda)\end{aligned}$$

and by (2.17),

$$\begin{aligned}&g_\mu\left(P^{-1/2}(A+B)P^{-1/2}\right)-g_\mu\left(P^{-1/2}AP^{-1/2}\right)-g_\mu\left(P^{-1/2}BP^{-1/2}\right)\\&\geq\int_0^\infty\lambda^2P^{1/2}(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)(A+B+\lambda P)^{-1}P^{1/2}d\mu(\lambda).\end{aligned}$$

If we multiply this inequality both sides by $P^{1/2}$, then we get (2.15). □

Corollary 2. *If A, B, $P>0$ is such that $BP^{-1}A+AP^{-1}B\geq kP$ with $k\geq 0$, then*

$$\begin{aligned}\mathcal{P}_{g_\mu}(A+B,P)&-\mathcal{P}_{g_\mu}(A,P)-\mathcal{P}_{g_\mu}(B,P)\\&\geq k\left[\mathcal{P}_{g'_\mu/\ell}(A+B,P)-\mathcal{P}_{g_\mu/\ell^2}(A+B,P)\right]\geq 0,\end{aligned}\tag{2.18}$$

where $\ell(t):=t$, $t>0$.

Proof. Since

$$BP^{-1}A+AP^{-1}B\geq kP\geq 0,$$

then by multiplying both sides by $(A+B+\lambda P)^{-1}$, we get

$$\begin{aligned}&(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)(A+B+\lambda P)^{-1}\\&\geq k(A+B+\lambda P)^{-1}P(A+B+\lambda P)^{-1}\geq 0.\end{aligned}$$

Moreover, if we multiply both sides by P we obtain

$$\begin{aligned}&P(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)(A+B+\lambda P)^{-1}P\\&\geq kP(A+B+\lambda P)^{-1}P(A+B+\lambda P)^{-1}P\geq 0,\end{aligned}$$

which, by multiplication with λ^2 and integration, gives

$$\begin{aligned}&\int_0^\infty\lambda^2P(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right)(A+B+\lambda P)^{-1}Pd\mu(\lambda)\\&\geq kP\left(\int_0^\infty\lambda^2(A+B+\lambda P)^{-1}P(A+B+\lambda P)^{-1}d\mu(\lambda)\right)P\geq 0.\end{aligned}\tag{2.19}$$

Observe that

$$\begin{aligned}&\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}+\lambda\right)^{-2}\\&=\left[P^{-1/2}(A+B+P)P^{-1/2}\right]^{-2}=\left[P^{1/2}(A+B+P)^{-1}P^{1/2}\right]^{2}\\&=P^{1/2}(A+B+P)^{-1}P^{1/2}P^{1/2}(A+B+P)^{-1}P^{1/2}\\&=P^{1/2}(A+B+P)^{-1}P(A+B+P)^{-1}P^{1/2}.\end{aligned}$$

Then by (2.19) we get

$$(2.20)\quad \begin{aligned} 0 &\le \int_0^\infty \lambda^2 P(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right) \\ &\times (A+B+\lambda P)^{-1} P d\mu(\lambda) \\ &\ge kP^{1/2}\left(\int_0^\infty \lambda^2\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}+\lambda\right)^{-2} d\mu(\lambda)\right)P^{1/2} \\ &= kP^{1/2}\left[g_\mu'\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}\right)\right. \\ &- g_\mu\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}\right) \\ &\times \left.\left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}\right)^{-1}\right] \\ &\times \left(P^{-1/2}AP^{-1/2}+P^{-1/2}BP^{-1/2}\right)^{-1} P^{1/2} \\ &= k\left[\mathcal{P}_{g_\mu'/\ell}(A+B,P)-\mathcal{P}_{g_\mu/\ell^2}(A+B,P)\right] \end{aligned}$$

and by (2.15) we derive (2.18). □

3. Results for Operator Monotone and Operator Convex Functions

The case of operator monotone functions is as follows:

Proposition 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3), then for $C,\ D>0$*

$$(3.1)\quad \begin{aligned} &(C+D)f(C+D)-Cf(C)-Df(D)-b(DC+CD) \\ &\ge \int_0^\infty \lambda^2 (C+D+\lambda)^{-1}(DC+CD)(C+D+\lambda)^{-1} d\mu(\lambda). \end{aligned}$$

If $DC+CD\ge 0$, then

$$(3.2)\quad (C+D)f(C+D)\ge Cf(C)+Df(D).$$

Proof. From (1.3) we get

$$(3.3)\quad g_\mu(t)=\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda)=tf(t)-f(0)t-bt^2,\ t>0.$$

Then

$$\begin{aligned} &g_\mu(C+D)-g_\mu(C)-g_\mu(D) \\ &=(C+D)f(C+D)-f(0)(C+D)-b(C+D)^2 \\ &-Cf(C)+f(0)C+bC^2-Df(D)+f(0)D+bD^2 \\ &=(C+D)f(C+D)-Cf(C)-Df(D)-b(DC+CD) \end{aligned}$$

and by (2.2) we obtain the desired result (3.1). □

Corollary 3. *With the assumptions of Proposition 1 and if $DC+CD\ge k$, then*

$$(3.4)\quad \begin{aligned} &(C+D)f(C+D)-Cf(C)-Df(D)-b(DC+CD) \\ &\ge k\left[f'(C+D)-b\right]. \end{aligned}$$

Moreover, we have

$$(C+D)\,f\,(C+D)-Cf\,(C)-Df\,(D)\geq kf'\,(C+D)\,. \tag{3.5}$$

Proof. Observe that, by (3.3)

$$g_{\mu}'\,(t)=tf'\,(t)+f\,(t)-f\,(0)-2bt$$

and

$$\begin{aligned}&\left(g_{\mu}'\,(t)-g_{\mu}\,(t)\,t^{-1}\right)t^{-1}\\&=\left[tf'\,(t)+f\,(t)-f\,(0)-2bt-(f\,(t)-f\,(0)-bt)\right]t^{-1}\\&=(tf'\,(t)-bt)\,t^{-1}=f'\,(t)-b.\end{aligned}$$

By (2.8) we derive (3.4).

From (3.4) we get

$$\begin{aligned}&(C+D)\,f\,(C+D)-Cf\,(C)-Df\,(D)\\&\geq k\,[f'\,(C+D)-b]+b\,(DC+CD)\\&=kf'\,(C+D)+b\,(DC+CD-k)\geq kf'\,(C+D)\end{aligned}$$

since $b\geq 0$ and $DC+CD-k\geq 0$. □

Remark 3. *If $DC+CD\geq k\geq 0$, then we have the following refinement of (3.2)*

$$(C+D)\,f\,(C+D)-Cf\,(C)-Df\,(D)\geq kf'\,(C+D)\geq 0. \tag{3.6}$$

If we consider the operator monotone function $f:[0,\infty)\to[0,\infty)$, $f\,(t)=t^{r}$ with $r\in(0,1]$ and assume that C, $D>0$ with $DC+CD\geq k$, for some real constant k, then by (3.5)

$$(C+D)^{r+1}-C^{r+1}-D^{r+1}\geq rk\,(C+D)^{r-1}\,. \tag{3.7}$$

If $k\geq 0$, then we have the strong superadditive inequality

$$(C+D)^{r+1}-C^{r+1}-D^{r+1}\geq rk\,(C+D)^{r-1}\geq 0, \tag{3.8}$$

provided $DC+CD\geq k\geq 0$.

Let $\varepsilon>0$ and consider the operator convex function $f_{\varepsilon}\,(t)=\ln\,(t+\varepsilon)$ on $[0,\infty)$. If C, $D>0$ with $DC+CD\geq k$, for some real constant k, then by (3.5) we get

$$(C+D)\ln\,(C+D+\varepsilon)-C\ln\,(C+\varepsilon)-D\ln\,(D+\varepsilon)\geq k\,(C+D+\varepsilon)^{-1}\,.$$

By letting $\varepsilon\to 0+$ in this inequality, then we get

$$(C+D)\ln\,(C+D)-C\ln C-D\ln D\geq k\,(C+D)^{-1}\,,$$

provided that C, $D>0$ with $DC+CD\geq k$.

If $k\geq 0$, then we have the strong superadditive inequality

$$(C+D)\ln\,(C+D)-C\ln C-D\ln D\geq k\,(C+D)^{-1}\geq 0,$$

provided $DC+CD\geq k\geq 0$.

Proposition 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.5), then for C, $D>0$*

$$\begin{aligned}&f\,(C+D)+f\,(0)-f\,(C)-f\,(D)-c\,(DC+CD)\\&\geq\int_{0}^{\infty}\lambda^{2}\,(C+D+\lambda)^{-1}\,(DC+CD)\,(C+D+\lambda)^{-1}\,d\mu\,(\lambda)\,.\end{aligned} \tag{3.9}$$

If $DC+CD\geq 0$, *then*

$$f(C+D)+f(0)\geq f(C)+f(D). \tag{3.10}$$

Proof. From (1.5) we have

$$g_\mu(t)=\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda)=f(t)-f(0)-f'_+(0)t-ct^2,\ t>0.$$

Then

$$\begin{aligned}
&g_\mu(C+D)-g_\mu(C)-g_\mu(D)\\
&=f(C+D)-f(0)-f'_+(0)(C+D)-c(C+D)^2\\
&-f(C)+f(0)+f'_+(0)C+cC^2-f(D)+f(0)+f'_+(0)D+cD^2\\
&=f(C+D)+f(0)-f(C)-f(D)-c(DC+CD)
\end{aligned}$$

and by (2.2) we obtain the desired result (3.9). □

Corollary 4. *With the assumptions of Proposition 2 and if* $DC+CD\geq k$, *then*

$$\begin{aligned}
&f(C+D)+f(0)-f(C)-f(D)-c(DC+CD)\\
&\geq k\left[f'(C+D)(C+D)^{-1}-(f(C+D)-f(0))(C+D)^{-2}-c\right].
\end{aligned} \tag{3.11}$$

Moreover, we have

$$\begin{aligned}
&f(C+D)+f(0)-f(C)-f(D)\\
&\geq k\left[f'(C+D)(C+D)^{-1}-(f(C+D)-f(0))(C+D)^{-2}\right].
\end{aligned} \tag{3.12}$$

Proof. Observe that, in this case

$$g'_\mu(t)=f'(t)-f'_+(0)-2ct$$

and

$$\begin{aligned}
&\left(g'_\mu(t)-g_\mu(t)t^{-1}\right)t^{-1}\\
&=\left(f'(t)-f'_+(0)-2ct-\left(f(t)-f(0)-f'_+(0)t-ct^2\right)t^{-1}\right)t^{-1}\\
&=\left(f'(t)-f'_+(0)-2ct-f(t)t^{-1}+f(0)t^{-1}+f'_+(0)+ct\right)t^{-1}\\
&=f'(t)t^{-1}-(f(t)-f(0))t^{-2}-c,
\end{aligned}$$

which proves (3.11).

From (3.11) we get

$$\begin{aligned}
&f(C+D)+f(0)-f(C)-f(D)\\
&\geq k\left[f'(C+D)(C+D)^{-1}-(f(C+D)-f(0))(C+D)^{-2}\right]\\
&+c(DC+CD-k).
\end{aligned} \tag{3.13}$$

Since $c\geq 0$ and $DC+CD-k\geq 0$, hence by (3.13) we derive (3.12). □

Remark 4. *If* $C,\ D>0$ *with* $DC+CD\geq k\geq 0$, *then we have the refinement of (3.10)*

$$\begin{aligned}
&f(C+D)+f(0)-f(C)-f(D)\\
&\geq k\left[f'(C+D)(C+D)^{-1}-(f(C+D)-f(0))(C+D)^{-2}\right]\geq 0.
\end{aligned} \tag{3.14}$$

Let $a>0$. The function $f_a(t)=(t+a)^r$ is operator convex on $[0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$. For $1\le r\le 2$, a can be also taken to be zero.

If $C,\ D>0$ with $DC+CD\ge k\ge 0$, then by (3.14) for f_a we get

$$(C+D+a)^r+a^r-(C+a)^r-(D+a)^r \\ \ge k\left[r(C+D+a)^{r-1}(C+D)^{-1}-((C+D+a)^r-a^r)(C+D)^{-2}\right]\ge 0 \tag{3.15}$$

for $a>0$.

For $r=-1$ we obtain from (3.15) that

$$(C+D+a)^{-1}+a^{-1}-(C+a)^{-1}-(D+a)^{-1} \\ \ge k\left[r(C+D+a)^{-2}(C+D)^{-1}-\left((C+D+a)^{-1}-a^{-1}\right)(C+D)^{-2}\right]\ge 0. \tag{3.16}$$

Observe that

$$(C+D+a)^{-1}-a^{-1}=-a^{-1}(C+D+a)^{-1}(C+D).$$

Then

$$\begin{aligned}&r(C+D+a)^{-2}(C+D)^{-1}-\left((C+D+a)^{-1}-a^{-1}\right)(C+D)^{-2}\\&=r(C+D+a)^{-2}(C+D)^{-1}+a^{-1}(C+D+a)^{-1}(C+D)(C+D)^{-2}\\&=r(C+D+a)^{-2}(C+D)^{-1}+a^{-1}(C+D+a)^{-1}(C+D)^{-1}\\&=(C+D+a)^{-1}(C+D)^{-1}\left[r(C+D+a)^{-1}+a^{-1}\right]\end{aligned}$$

and by (3.16) we derive

$$(C+D+a)^{-1}+a^{-1}-(C+a)^{-1}-(D+a)^{-1} \\ \ge k(C+D+a)^{-1}(C+D)^{-1}\left[r(C+D+a)^{-1}+a^{-1}\right]\ge 0, \tag{3.17}$$

provided $C,\ D>0$ with $DC+CD\ge k\ge 0$ and $a>0$.

4. Inequalities for Perspectives

The case of perspectives for operator monotone functions is as follows:

Proposition 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). For all $A,\ B,\ P>0$ we have*

$$\mathcal{P}_{\ell f}(A+B,P)-\mathcal{P}_{\ell f}(A,P)-\mathcal{P}_{\ell f}(B,P)-b\left(BP^{-1}A+AP^{-1}B\right) \\ \ge\int_0^\infty \lambda^2 P(A+B+\lambda P)^{-1}\left(BP^{-1}A+AP^{-1}B\right) \\ \times(A+B+\lambda P)^{-1}Pd\mu(\lambda). \tag{4.1}$$

If $BP^{-1}A+AP^{-1}B\ge 0$, then

$$\mathcal{P}_{\ell f}(A+B,P)-\mathcal{P}_{\ell f}(A,P)-\mathcal{P}_{\ell f}(B,P)\ge b\left(BP^{-1}A+AP^{-1}B\right)\ge 0. \tag{4.2}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3), then

$$g_\mu(t)=tf(t)-f(0)t-bt^2,\ \ t>0.$$

Therefore

$$\mathcal{P}_{g_\mu}(A+B,P) = \mathcal{P}_{\ell f}(A+B,P) - f(0)\,\mathcal{P}_{\ell}(A+B,P) - b\mathcal{P}_{\ell^2}(A+B,P),$$
$$\mathcal{P}_{g_\mu}(A,P) = \mathcal{P}_{\ell f}(A,P) - f(0)\,\mathcal{P}_{\ell}(A,P) - b\mathcal{P}_{\ell^2}(A,P)$$

and

$$\mathcal{P}_{g_\mu}(B,P) = \mathcal{P}_{\ell f}(B,P) - f(0)\,\mathcal{P}_{\ell}(B,P) - b\mathcal{P}_{\ell^2}(B,P).$$

From these representations we get that

$$\begin{aligned} &\mathcal{P}_{g_\mu}(A+B,P) - \mathcal{P}_{g_\mu}(A,P) - \mathcal{P}_{g_\mu}(B,P) \\ &= \mathcal{P}_{\ell f}(A+B,P) - \mathcal{P}_{\ell f}(A,P) - \mathcal{P}_{\ell f}(B,P) \\ &- f(0)\left[\mathcal{P}_{\ell}(A+B,P) - \mathcal{P}_{\ell}(A,P) - \mathcal{P}_{\ell}(B,P)\right] \\ &- b\left[\mathcal{P}_{\ell^2}(A+B,P) - \mathcal{P}_{\ell^2}(A,P) - \mathcal{P}_{\ell^2}(B,P)\right]. \end{aligned} \tag{4.3}$$

Observe that

$$\begin{aligned} &\mathcal{P}_{\ell}(A+B,P) - \mathcal{P}_{\ell}(A,P) - \mathcal{P}_{\ell}(B,P) \\ &= P^{1/2}\left(P^{-1/2}(A+B)\,P^{-1/2}\right)P^{1/2} \\ &- P^{1/2}\left(P^{-1/2}AP^{-1/2}\right)P^{1/2} - P^{1/2}\left(P^{-1/2}BP^{-1/2}\right)P^{1/2} \\ &= A+B-A-B=0 \end{aligned}$$

and

$$\begin{aligned} &\mathcal{P}_{\ell^2}(A+B,P) - \mathcal{P}_{\ell^2}(A,P) - \mathcal{P}_{\ell^2}(B,P) \\ &= P^{1/2}\left(P^{-1/2}(A+B)\,P^{-1/2}\right)^2P^{1/2} \\ &- P^{1/2}\left(P^{-1/2}AP^{-1/2}\right)^2P^{1/2} - P^{1/2}\left(P^{-1/2}BP^{-1/2}\right)^2P^{1/2} \\ &= P^{1/2}P^{-1/2}(A+B)\,P^{-1/2}P^{-1/2}(A+B)\,P^{-1/2}P^{1/2} \\ &- P^{1/2}P^{-1/2}AP^{-1/2}P^{-1/2}AP^{-1/2}P^{1/2} - P^{1/2}P^{-1/2}BP^{-1/2}P^{-1/2}BP^{-1/2}P^{1/2} \\ &= (A+B)\,P^{-1}(A+B) - AP^{-1}A - BP^{-1}B = BP^{-1}A + AP^{-1}B. \end{aligned}$$

By using (2.15) we derive (4.1). □

Corollary 5. *With the assumptions of Proposition 3 and if* $BP^{-1}A + AP^{-1}B \geq kP$, *then*

$$\begin{aligned} &\mathcal{P}_{\ell f}(A+B,P) - \mathcal{P}_{\ell f}(A,P) - \mathcal{P}_{\ell f}(B,P) - b\left(BP^{-1}A + AP^{-1}B\right) \\ &\geq k\left[\mathcal{P}_{f'}(A+B,P) - b\right]. \end{aligned}$$

Moreover,

$$\mathcal{P}_{\ell f}(A+B,P) - \mathcal{P}_{\ell f}(A,P) - \mathcal{P}_{\ell f}(B,P) \geq k\mathcal{P}_{f'}(A+B,P). \tag{4.4}$$

If $k \geq 0$, *then*

$$\mathcal{P}_{\ell f}(A+B,P) - \mathcal{P}_{\ell f}(A,P) - \mathcal{P}_{\ell f}(B,P) \geq k\mathcal{P}_{f'}(A+B,P) \geq 0. \tag{4.5}$$

We can extend the definition of *weighted operator geometric mean*

$$A\sharp_r B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2},$$

for real numbers r if $A,\ B > 0$.

If we apply (4.4) for the operator monotone function $f(t) = t^r$, $r \in (0,1]$, then we get

$$P\sharp_{r+1}(A+B) - P\sharp_{r+1}A - P\sharp_{r+1}B \geq rkP\sharp_{r-1}(A+B) \tag{4.6}$$

if A, B, $P > 0$ with $BP^{-1}A + AP^{-1}B \geq kP$.

Consider the operator convex function $g(t) = t \ln t$, $t > 0$. We have

$$\mathcal{P}_g(B,A) = \mathcal{S}(B|A)$$

since $g(t) = t \ln t = -t \ln\left(\frac{1}{t}\right) = \tilde{f}(t)$, $t > 0$.

If we apply the inequality (4.4) for $f(t) = \ln t$, then we get the inequality

$$\mathcal{S}(A+B|P) - \mathcal{S}(A,P) - \mathcal{S}(B,P) \geq k\mathcal{P}_{\ell^{-1}}(A+B,P) \tag{4.7}$$

if A, B, $P > 0$ with $BP^{-1}A + AP^{-1}B \geq kP$.

Proposition 4. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.5). For all A, B, $P > 0$ we have*

$$\begin{aligned} &\mathcal{P}_f(A+B,P) + f(0)P - \mathcal{P}_f(A,P) - \mathcal{P}_f(B,P) - c\left(BP^{-1}A + AP^{-1}B\right) \\ &\geq \int_0^\infty \lambda^2 P(A+B+\lambda P)^{-1}\left(BP^{-1}A + AP^{-1}B\right) \\ &\times (A+B+\lambda P)^{-1} P d\mu(\lambda). \end{aligned} \tag{4.8}$$

If $BP^{-1}A + AP^{-1}B \geq 0$, then

$$\mathcal{P}_f(A+B,P) + f(0)P - \mathcal{P}_f(A,P) - \mathcal{P}_f(B,P) \geq c\left(BP^{-1}A + AP^{-1}B\right) \geq 0. \tag{4.9}$$

The proof follows by Theorem 3 applied for operator convex functions.

Corollary 6. *With the assumptions of Proposition 4 and if $BP^{-1}A + AP^{-1}B \geq kP$, then*

$$\begin{aligned} &\mathcal{P}_f(A+B,P) + f(0)P - \mathcal{P}_f(A,P) - \mathcal{P}_f(B,P) - c\left(BP^{-1}A + AP^{-1}B\right) \\ &\geq k\left[\mathcal{P}_{f'/\ell}(A+B,P) - \mathcal{P}_{f/\ell^2}(A+B,P) + f(0)\mathcal{P}_{\ell^{-2}}(A+B,P)\right]. \end{aligned} \tag{4.10}$$

One can obtain from (4.10) some particular inequalities by taking the function $f_a(t) = (t+a)^r$ that is operator convex on $[0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$. For $1 \leq r \leq 2$, a can be also taken to be zero. The details are omitted.

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