LOWER BOUNDS ON PARTIAL SUBADDITIVITY OF NONCOMMUTATIVE PERSPECTIVES FOR OPERATORS IN HILBERT SPACES: THE CASE OF SECOND VARIABLE

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Abstract. Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. We define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where A, $B>0$. In this paper we show among others that, if A, B, $P>0$, then

$$\mathcal{P}_{\ell f}(P,A)+\mathcal{P}_{\ell f}(P,B)\geq\mathcal{P}_{\ell f}(P,A+B),$$

where $\ell(t)=t$. If the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$, then

$$\mathcal{P}_f(P,A)+\mathcal{P}_f(P,B)\geq\mathcal{P}_f(P,A+B)+f'_+(0)P.$$

Applications for *weighted operator geometric mean* are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

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A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [10], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.4}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t)=f(0)+f'_+(0)t+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.5}$$

where $c\geq 0$ and a positive measure μ on $[0,\infty)$ such that (1.2) holds.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone. For other examples, see [5], [6], [7], [12] and the references therein.

Assume that A, $B\geq 0$. In the recent paper [11], Moslehian and Najafi showed that $AB+BA$ is positive if and only if the following *operator subadditivity property* holds

$$f(A+B)\leq f(A)+f(B) \tag{1.6}$$

for all nonnegative operator monotone functions f on $[0,\infty)$. For some interesting consequences of this result see [11].

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [13]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all $A,\ B>0$,

$$\mathcal{P}_{\tilde{f}}(A,B) = \mathcal{P}_f(B,A). \tag{1.7}$$

It is well known that (see [3] and [2] or [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

If $f_r:[0,\infty)\to[0,\infty)$, $f_r(t)=t^r$, $r\in[0,1]$, then

$$\mathcal{P}_{f_r}(B,A) := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2} =: A\sharp_r B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight r.

In this paper we show among others that, if $A,\ B,\ P>0$ and function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$, then

$$\mathcal{P}_{\ell f}(P,A)+\mathcal{P}_{\ell f}(P,B)\geq \mathcal{P}_{\ell f}(P,A+B).$$

If the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$, then

$$\mathcal{P}_f(P,A)+\mathcal{P}_f(P,B)\geq \mathcal{P}_f(P,A+B)+f'_+(0)P.$$

Applications for *weighted operator geometric mean* are also provided.

2. General Results

For a positive measure μ on $(0,\infty)$ we define $g_\mu:(0,\infty)\to(0,\infty)$ by

$$g_\mu(t) := t^2\int_0^\infty \frac{\lambda}{t+\lambda}d\mu(\lambda),$$

provided the integral is convergent for all $t\in(0,\infty)$.

Therefore we have the transpose function

$$\tilde{g}_\mu(t) := tg_\mu\left(\frac{1}{t}\right) = \int_0^\infty \frac{\lambda}{1+t\lambda}d\mu(\lambda),\ t>0.$$

We start with the following result regarding the operator subadditivity property of $\tilde{g}_\mu$:

Lemma 1. *For all $C,\ D>0$ we have*

$$\tilde{g}_\mu(C)+\tilde{g}_\mu(D)\geq \tilde{g}_\mu(C+D). \tag{2.1}$$

Proof. We have

$$\begin{aligned}
(2.2)\qquad & \tilde{g}_\mu(C+D)-\tilde{g}_\mu(C)-\tilde{g}_\mu(D)\\
&=\int_0^\infty \lambda\left[(1+\lambda(C+D))^{-1}-(1+\lambda C)^{-1}-(1+\lambda D)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \lambda\left[\lambda^{-1}\left(\lambda^{-1}+C+D\right)^{-1}\right.\\
&\left.-\lambda^{-1}\left(\lambda^{-1}+C\right)^{-1}-\lambda^{-1}\left(\lambda^{-1}+D\right)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty \left[\left(\lambda^{-1}+C+D\right)^{-1}-\left(\lambda^{-1}+C\right)^{-1}-\left(\lambda^{-1}+D\right)^{-1}\right]d\mu(\lambda)\\
&=-\int_0^\infty K_{\lambda^{-1}}(C,D)\,d\mu(\lambda),
\end{aligned}$$

where

$$K_\lambda(C,D):=(C+\lambda)^{-1}+(D+\lambda)^{-1}-(C+D+\lambda)^{-1},$$

for $\lambda\geq 0$ and $C,\ D>0$.

Observe that

$$\begin{aligned}
(2.3)\qquad & (C+D+\lambda)K_\lambda(C,D)(C+D+\lambda)\\
&=(C+D+\lambda)(C+\lambda)^{-1}(C+D+\lambda)\\
&+(C+D+\lambda)(D+\lambda)^{-1}(C+D+\lambda)-C-D-\lambda\\
&=\left(1+D(C+\lambda)^{-1}\right)(C+\lambda+D)\\
&+\left(C(D+\lambda)^{-1}+1\right)(C+D+\lambda)-C-D-\lambda\\
&=C+\lambda+D+D+D(C+\lambda)^{-1}D\\
&+C(D+\lambda)^{-1}C+C+C+D+\lambda-C-D-\lambda\\
&=D(C+\lambda)^{-1}D+C(D+\lambda)^{-1}C+2(C+D)+\lambda\\
&=:L_\lambda(C,D).
\end{aligned}$$

If $C,\ D,\ \lambda>0$, and since $D(C+\lambda)^{-1}D>0$, $C(D+\lambda)^{-1}C>0$, then $L_\lambda(C,D)\geq 0$, and by multiplying both sides of (2.3) with $(C+D+\lambda)^{-1}$ we get

$$K_\lambda(C,D)=(C+D+\lambda)^{-1}L_\lambda(C,D)(C+D+\lambda)^{-1}\geq 0. \qquad (2.4)$$

Therefore $K_{\lambda^{-1}}(C,D)\geq 0$ and by the representation (2.2) we get the desired result (2.1). □

Corollary 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). For all $C,\ D>0$ we have*

$$\begin{aligned}
(2.5)\qquad & f\left(C^{-1}\right)+f\left(D^{-1}\right)-f\left((C+D)^{-1}\right)-f(0)\\
&\geq b\left[C^{-1}+D^{-1}-(C+D)^{-1}\right]\geq 0.
\end{aligned}$$

Proof. Since f is operator monotone satisfying (1.3), then

$$g_\mu(t)=t^2\int_0^\infty \frac{\lambda}{t+\lambda}d\mu(\lambda)=tf(t)-f(0)\,t-bt^2.$$

This gives that

$$\begin{aligned}\tilde{g}_{\mu}(t) &= tg_{\mu}\left(\frac{1}{t}\right) = t\left[\frac{1}{t}f\left(\frac{1}{t}\right) - f(0)\frac{1}{t} - b\left(\frac{1}{t}\right)^{2}\right] \\ &= f\left(\frac{1}{t}\right) - f(0) - b\frac{1}{t}.\end{aligned}$$

Therefore

$$\begin{aligned}&\tilde{g}_{\mu}(C) + \tilde{g}_{\mu}(D) - \tilde{g}_{\mu}(C+D) \\ &= f\left(C^{-1}\right) - f(0) - bC^{-1} + f\left(D^{-1}\right) - f(0) - bD^{-1} \\ &- f\left((C+D)^{-1}\right) + f(0) + b(C+D)^{-1} \\ &= f\left(C^{-1}\right) + f\left(D^{-1}\right) - f\left((C+D)^{-1}\right) - f(0) \\ &- b\left[C^{-1} + D^{-1} - (C+D)^{-1}\right]\end{aligned}$$

for all $C,\ D > 0$. By (2.1) we then get the first inequality in (2.5).

Put $\lambda = 0$ in (2.3) and (2.4) to get

$$\begin{aligned}K_0(C,D) &:= C^{-1} + D^{-1} - (C+D)^{-1} \\ &= (C+D)^{-1}\left(DC^{-1}D + CD^{-1}C + 2(C+D)\right)(C+D)^{-1} \\ &\geq 0,\end{aligned} \tag{2.6}$$

which proves the second part of (2.5) since $b \geq 0$. □

Remark 1. *Consider the operator monotone function $f(t) = t^r$, $r \in (0,1]$. Then by (2.5) we derive the inequality*

$$C^{-r} + D^{-r} \geq (C+D)^{-r} \tag{2.7}$$

for all $C,\ D > 0$.

For $a > 0$, consider the operator convex function $l_a(t) = \ln(t+a)$, $t \in [0,\infty)$. By (2.5) we get

$$\ln\left(C^{-1} + a\right) + \ln\left(D^{-1} + a\right) \geq \ln\left((C+D)^{-1} + a\right) + \ln a \tag{2.8}$$

for all $C,\ D > 0$.

Corollary 2. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.5). For all $C,\ D > 0$ we have*

$$\tilde{f}(C) + \tilde{f}(D) - \tilde{f}(C+D) - f'_{+}(0) \geq c\left[C^{-1} + D^{-1} - (C+D)^{-1}\right] \geq 0. \tag{2.9}$$

Proof. Since f is operator convex satisfying (1.5), then

$$g_{\mu}(t) = \int_0^{\infty} \frac{t^2\lambda}{t+\lambda} d\mu(\lambda) = f(t) - f(0) - f'_{+}(0)t - ct^2,\ t > 0.$$

This gives that

$$\begin{aligned}\tilde{g}_{\mu}(t) &= tg_{\mu}\left(\frac{1}{t}\right) = t\left[f\left(\frac{1}{t}\right) - f(0) - f'_{+}(0)\frac{1}{t} - c\left(\frac{1}{t}\right)^{2}\right] \\ &= tf\left(\frac{1}{t}\right) - f(0)t - f'_{+}(0) - c\frac{1}{t} = \tilde{f}(t) - f(0)t - f'_{+}(0) - c\frac{1}{t}\end{aligned}$$

for $t>0$.

Therefore

$$\begin{aligned}&\tilde{g}_\mu(C)+\tilde{g}_\mu(D)-\tilde{g}_\mu(C+D)\\&=\tilde{f}(C)-f(0)C-f'_+(0)-cC^{-1}+\tilde{f}(D)-f(0)D-f'_+(0)-cD^{-1}\\&-\left(\tilde{f}(C+D)-f(0)C+D-f'_+(0)-c(C+D)^{-1}\right)\\&=\tilde{f}(C)+\tilde{f}(D)-\tilde{f}(C+D)-f'_+(0)-c\left[C^{-1}+D^{-1}-(C+D)^{-1}\right]\end{aligned}$$

and by (2.1) we get (2.9). □

Remark 2. *Let $a>0$. The function $f_a(t)=(t+a)^r$ is operator convex on $[0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$. We have $f'_{a+}(0)=ra^{r-1}$ and since $\tilde{f}_a(t)=t\left(t^{-1}+a\right)^r=t^{1-r}(1+at)^r$, then by (2.9) we get*

$$C^{1-r}(1+aC)^r+D^{1-r}(1+aD)^r\ge(C+D)^{1-r}(1+a(C+D))^r+ra^{r-1}\tag{2.10}$$

for all C, $D>0$ and $a>0$.

If $r=-1$, then we get from (2.10) that

$$C^2(1+aC)^{-1}+D^2(1+aD)^{-1}\ge(C+D)^2(1+a(C+D))^{-1}-a^{-2}.\tag{2.11}$$

3. Inequalities for Perspectives

We have the following result for perspective:

Theorem 3. *For all A, B, $P>0$ we have*

$$\mathcal{P}_{\tilde{g}_\mu}(A,P)+\mathcal{P}_{\tilde{g}_\mu}(B,P)\ge\mathcal{P}_{\tilde{g}_\mu}(A+B,P)\tag{3.1}$$

or, equivalently,

$$\mathcal{P}_{g_\mu}(P,A)+\mathcal{P}_{g_\mu}(P,B)\ge\mathcal{P}_{g_\mu}(P,A+B).\tag{3.2}$$

Proof. If we write the inequality (2.1) for $C=P^{-1/2}AP^{-1/2}$ and $D=P^{-1/2}BP^{-1/2}$, then we get

$$\tilde{g}_\mu\left(P^{-1/2}AP^{-1/2}\right)+\tilde{g}_\mu\left(P^{-1/2}BP^{-1/2}\right)\ge\tilde{g}_\mu\left(P^{-1/2}(A+B)P^{-1/2}\right).$$

If we multiply this inequality both sides by $P^{1/2}>0$, then we get

$$\begin{aligned}&P^{1/2}\tilde{g}_\mu\left(P^{-1/2}AP^{-1/2}\right)P^{1/2}+P^{1/2}\tilde{g}_\mu\left(P^{-1/2}BP^{-1/2}\right)P^{1/2}\\&\ge P^{1/2}\tilde{g}_\mu\left(P^{-1/2}(A+B)P^{-1/2}\right)P^{1/2}\end{aligned}$$

and the inequality (3.1) is obtained.

Since $\mathcal{P}_{\tilde{g}_\mu}(P,Q)=\mathcal{P}_{g_\mu}(Q,P)$ for P, $Q>0$ the inequality (3.2) is equivalent to (3.1). □

The case of operator monotone functions is as follows:

Corollary 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). For all A, B, $P>0$ we have*

$$\begin{aligned}&\mathcal{P}_{\ell f}(P,A)+\mathcal{P}_{\ell f}(P,B)-\mathcal{P}_{\ell f}(P,A+B)\\&\ge bP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P\ge 0.\end{aligned}\tag{3.3}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3), then

$$g_\mu(t)=tf(t)-f(0)t-bt^2,\ t>0.$$

Therefore

$$\mathcal{P}_{g_\mu}(Q,P)=\mathcal{P}_{\ell f}(Q,P)-f(0)\mathcal{P}_{\ell}(Q,P)-b\mathcal{P}_{\ell^2}(Q,P),$$

for P, $Q>0$, where $\ell(t)=t$.

This gives that

$$\begin{aligned}&\mathcal{P}_{g_\mu}(P,A)+\mathcal{P}_{g_\mu}(P,B)-\mathcal{P}_{g_\mu}(P,A+B)\\&=\mathcal{P}_{\ell f}(P,A)-f(0)\mathcal{P}_{\ell}(P,A)-b\mathcal{P}_{\ell^2}(P,A)\\&+\mathcal{P}_{\ell f}(P,B)-f(0)\mathcal{P}_{\ell}(P,B)-b\mathcal{P}_{\ell^2}(P,B)\\&-\mathcal{P}_{\ell f}(P,)(A+B)+f(0)\mathcal{P}_{\ell}(P,A+B)+b\mathcal{P}_{\ell^2}(P,A+B)\\&=\mathcal{P}_{\ell f}(P,A)+\mathcal{P}_{\ell f}(P,B)-\mathcal{P}_{\ell f}(P,A+B)\\&-b\left(\mathcal{P}_{\ell^2}(P,A)+\mathcal{P}_{\ell^2}(P,B)-\mathcal{P}_{\ell^2}(P,A+B)\right)\end{aligned}$$

for all A, B, $P>0$.

Observe that

$$\begin{aligned}&\mathcal{P}_{\ell^2}(P,A)+\mathcal{P}_{\ell^2}(P,B)-\mathcal{P}_{\ell^2}(P,A+B)\\&=A^{1/2}\left(A^{-1/2}PA^{-1/2}\right)^2A^{1/2}+B^{1/2}\left(B^{-1/2}PB^{-1/2}\right)^2B^{1/2}\\&-(A+B)^{1/2}\left((A+B)^{-1/2}P(A+B)^{-1/2}\right)^2(A+B)^{1/2}\\&=A^{1/2}A^{-1/2}PA^{-1/2}A^{-1/2}PA^{-1/2}A^{1/2}+B^{1/2}B^{-1/2}PB^{-1/2}B^{-1/2}PB^{-1/2}B^{1/2}\\&-(A+B)^{1/2}(A+B)^{-1/2}P(A+B)^{-1/2}(A+B)^{-1/2}P(A+B)^{-1/2}(A+B)^{1/2}\\&=PA^{-1}P+PB^{-1}P-P(A+B)^{-1}P\\&=P\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathcal{P}_{g_\mu}(A,P)+\mathcal{P}_{g_\mu}(B,P)-\mathcal{P}_{g_\mu}(A+B,P)\\&=\mathcal{P}_{\ell f}(A,P)+\mathcal{P}_{\ell f}(B,P)-\mathcal{P}_{\ell f}(A+B,P)\\&-bP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P,\end{aligned}$$

for all A, B, $P>0$.

Since, by (3.2),

$$\mathcal{P}_{g_\mu}(A,P)+\mathcal{P}_{g_\mu}(B,P)-\mathcal{P}_{g_\mu}(A+B,P)\geq 0$$

hence

$$\begin{aligned}&\mathcal{P}_{\ell f}(A,P)+\mathcal{P}_{\ell f}(B,P)-\mathcal{P}_{\ell f}(A+B,P)\\&-bP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P\\&\geq 0,\end{aligned}$$

which proves (3.3).

Since, by (2.6) $A^{-1}+B^{-1}-(A+B)^{-1}\geq 0$, then

$$bP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P\geq 0$$

because $b\geq 0$. □

We can extend the definition of *weighted operator geometric mean*

$$A\sharp_r B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^r A^{1/2},$$

for real numbers r if $A,\ B>0$.

If we apply the inequality (3.3) for the operator monotone function $f(t)=t^r$, $r\in(0,1]$, $t\in[0,\infty)$ for which we can take $b=0$, then we get

$$A\sharp_{r+1}P+B\sharp_{r+1}P\geq (A+B)\sharp_{r+1}P$$

for all $A,\ B,\ P>0$.

Corollary 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.5). For all $A,\ B,\ P>0$ we have*

$$\begin{aligned}(3.4)\qquad &\mathcal{P}_f(P,A)+\mathcal{P}_f(P,B)-\mathcal{P}_f(P,A+B)-f'_+(0)P\\ &\geq cP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P\geq 0.\end{aligned}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.5), then

$$g_\mu(t)=f(t)-f(0)-f'_+(0)t-ct^2, t>0.$$

Therefore

$$\mathcal{P}_{g_\mu}(Q,P)=\mathcal{P}_f(Q,P)-f(0)P-f'_+(0)\mathcal{P}_\ell(Q,P)-c\mathcal{P}_{\ell^2}(Q,P),$$

for $P,\ Q>0$, where $\ell(t)=t$.

This gives that

$$\begin{aligned}&\mathcal{P}_{g_\mu}(P,A)+\mathcal{P}_{g_\mu}(P,B)-\mathcal{P}_{g_\mu}(P,A+B)\\ &=\mathcal{P}_f(P,A)-f(0)A-f'_+(0)\mathcal{P}_\ell(P,A)-c\mathcal{P}_{\ell^2}(P,A)\\ &+\mathcal{P}_f(P,B)-f(0)B-f'_+(0)\mathcal{P}_\ell(P,B)-c\mathcal{P}_{\ell^2}(P,B)\\ &-\left(\mathcal{P}_f(A+B,P)-f(0)(A+B)-f'_+(0)\mathcal{P}_\ell(P,A+B)-c\mathcal{P}_{\ell^2}(P,A+B)\right)\\ &=\mathcal{P}_f(P,A)+\mathcal{P}_f(P,B)-\mathcal{P}_f(P,A+B)\\ &-f'_+(0)\left[\mathcal{P}_\ell(P,A)+\mathcal{P}_\ell(P,B)-\mathcal{P}_\ell(P,A+B)\right]\\ &-c\left[\mathcal{P}_{\ell^2}(A,P)+\mathcal{P}_{\ell^2}(B,P)-\mathcal{P}_{\ell^2}(A+B,P)\right].\end{aligned}$$

Observe that

$$\begin{aligned}&\mathcal{P}_\ell(P,A)+\mathcal{P}_\ell(P,B)-\mathcal{P}_\ell(P,A+B)\\ &=A^{1/2}\left(A^{-1/2}PA^{-1/2}\right)A^{1/2}+B^{1/2}\left(B^{-1/2}PB^{-1/2}\right)B^{1/2}\\ &-(A+B)^{1/2}\left((A+B)^{-1/2}P(A+B)^{-1/2}\right)(A+B)^{1/2}\\ &=P.\end{aligned}$$

Since, by (3.2)

$$\mathcal{P}_{g_\mu}(A,P)+\mathcal{P}_{g_\mu}(B,P)-\mathcal{P}_{g_\mu}(A+B,P)\geq 0$$

hence

$$\begin{aligned}&\mathcal{P}_f(P,A)+\mathcal{P}_f(P,B)-\mathcal{P}_f(P,A+B)-f'_+(0)P\\&-cP\left(A^{-1}+B^{-1}-(A+B)^{-1}\right)P\\&\geq 0,\end{aligned}$$

which proves (3.4). □

Let $a>0$. The function $f_a(t)=(t+a)^r$ is operator convex on $[0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$. For $1\leq r\leq 2$, a can be also taken to be zero.

If we apply the inequality (3.4) for the function $(\ell+a)^r$, then we get

$$\mathcal{P}_{(\ell+a)^r}(P,A)+\mathcal{P}_{(\ell+a)^r}(P,B)\geq\mathcal{P}_{(\ell+a)^r}(P,A+B)+ra^{r-1}P \tag{3.5}$$

for all A, B, $P>0$ and $a>0$.

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