OPERATOR SUBADDITIVITY OF THE $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $[0, \infty)$ we consider the following *$\mathcal{D}$-logarithmic integral transform*

$$\mathcal{DL}og\left(w\right)\left(T\right) := \int_0^\infty w\left(\lambda\right) \ln\left(\frac{\lambda + T}{\lambda}\right) d\lambda,$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if A, $B > 0$ with $BA + AB \geq 0$, then

$$\mathcal{DL}og\left(w, \mu\right)\left(A\right) + \mathcal{DL}og\left(w, \mu\right)\left(B\right) \geq \mathcal{DL}og\left(w, \mu\right)\left(A + B\right).$$

In particular we have

$$\frac{1}{6}\pi^2 + \operatorname{dilog}\left(A + B\right) \geq \operatorname{dilog}\left(A\right) + \operatorname{dilog}\left(B\right),$$

where the *dilogarithmic function* $\operatorname{dilog} : [0, \infty) \to \mathbb{R}$ is defined by

$$\operatorname{dilog}\left(t\right) := \int_1^t \frac{\ln s}{1 - s} ds, \ t \geq 0.$$

Some examples for integral transform $\mathcal{DL}og\left(\cdot, \cdot\right)$ related to the operator monotone functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f\left(t\right) = f\left(0\right) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu\left(\lambda\right), \tag{1.1}$$

where $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu\left(\lambda\right) < \infty. \tag{1.2}$$

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For some examples of operator monotone functions see [3]-[5], [8]-[9] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$s^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+s}d\lambda. \tag{1.3}$$

Observe that for $s>0$, $s\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+s)(\lambda+1)}=\frac{\ln s}{s-1}+\frac{1}{1-s}\ln\left(\frac{u+s}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln s}{s-1}=\int_0^\infty \frac{d\lambda}{(\lambda+s)(\lambda+1)}, \tag{1.4}$$

which gives the representation for the logarithm

$$\ln s=(s-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+s)} \tag{1.5}$$

If we integrate (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\frac{t^r}{r}&=\frac{\sin(r\pi)}{\pi}\int_0^\infty\left(\int_0^t\left(\frac{1}{\lambda+s}\right)ds\right)\lambda^{r-1}d\lambda\\&=\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda\end{aligned}$$

giving the identity of interest

$$t^r=\frac{r\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda,\ t>0 \text{ and } r\in(0,1].$$

Recall the *dilogarithmic function* $\operatorname{dilog}:[0,\infty)\to\mathbb{R}$ defined by

$$\operatorname{dilog}(t):=\int_1^t\frac{\ln s}{1-s}ds,\ t\geq 0.$$

Some particular values of interest are

$$\operatorname{dilog}(1)=0,\ \ \operatorname{dilog}(0)=\int_1^0\frac{\ln s}{1-s}ds=\int_0^1\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2,$$

and

$$\operatorname{dilog}\left(\frac{1}{2}\right)=\frac{1}{12}\pi^2-\frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.4) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t\frac{\ln s}{s-1}ds=\int_0^\infty\left(\int_0^t\frac{1}{\lambda+s}ds\right)\frac{1}{(\lambda+1)}d\lambda=\int_0^\infty\frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda$$

and since

$$\begin{aligned}\int_0^t\frac{\ln s}{s-1}ds&=\int_0^1\frac{\ln s}{s-1}ds+\int_1^t\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2-\int_1^t\frac{\ln s}{1-s}ds\\&=\frac{1}{6}\pi^2-\operatorname{dilog}(t)\end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \operatorname{dilog}(t) = \int_0^\infty \frac{1}{(\lambda+1)} \ln\left(\frac{t+\lambda}{\lambda}\right) d\lambda,\ t > 0.$$

Motivated by the above representations, we define the $\mathcal{D}$-*logarithmic transform* for a continuous and positive function $w(\lambda),\ \lambda > 0$ by

$$\mathcal{D}\mathcal{L}og\,(w,\mu)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda), \tag{1.6}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.10) exists for all $t > 0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}\mathcal{L}og\,(w)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\lambda. \tag{1.7}$$

Obviously,

$$\begin{aligned} \mathcal{D}\mathcal{L}og\,(w,\mu)(t) &= \int_0^\infty w(\lambda) \ln\left(1+\frac{t}{\lambda}\right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[\ln(\lambda+t) - \ln(\lambda)\right] d\mu(\lambda) \end{aligned}$$

and one can use either of these representations when is needed.

By utilising the continuous functional calculus for selfadjoint operators, we can define the operator $\mathcal{D}$-*logarithmic transform* by

$$\mathcal{D}\mathcal{L}og\,(w,\mu)(T) = \int_0^\infty w(\lambda) \ln\left(1+\frac{1}{\lambda}T\right) d\mu(\lambda)$$

for $T > 0$.

If we use the D-logarithmic transform for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$ we have

$$\mathcal{D}\mathcal{L}og\,(w_{\ell^{r-1}})(T) = T^r,\ T \geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{D}\mathcal{L}og\left(w_{(\ell+1)^{-1}}\right)(T) = \frac{1}{6}\pi^2 - \operatorname{dilog}(T),\ T \geq 0. \tag{1.8}$$

In the recent paper [2] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\mu(\lambda),\ s > 0, \tag{1.9}$$

for a continuous and positive function $w(\lambda),\ \lambda > 0$,where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $s > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\lambda,\ s > 0. \tag{1.10}$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [2].

If we integrate the identity (1.4) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t \mathcal{D}(w,\mu)(s)\,ds := \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s}ds\right) w(\lambda)\,d\mu(\lambda) = \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda) \tag{1.11}$$

for $t>0$, which provides the equality of interest

$$\mathcal{D}\mathcal{L}og(w,\mu)(t) = \int_0^t \mathcal{D}(w,\mu)(s)\,ds,\ t>0, \tag{1.12}$$

provided that the integral on the right side exists for all $t>0$.

2. Main Results

We have the following identity of interest:

Lemma 1. *For all A, $B>0$ and $\lambda>0$ we have*

$$\begin{aligned} &\ln\left(\frac{A+\lambda}{\lambda}\right)+\ln\left(\frac{B+\lambda}{\lambda}\right)-\ln\left(\frac{A+B+\lambda}{\lambda}\right) \\ &= \ln(A+\lambda)+\ln(B+\lambda)-\ln(A+B+\lambda)-\ln\lambda \\ &= \int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1} \\ &\times\left[A(B+s+\lambda)^{-1}BA+B(A+s+\lambda)^{-1}AB\right](A+B+s+\lambda)^{-1}ds \\ &+\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1}ds. \end{aligned} \tag{2.1}$$

Proof. We have, by (1.5), that

$$\begin{aligned} \ln(T+\lambda) &= \int_0^\infty \frac{1}{(s+1)}(T+\lambda-1)(s+T+\lambda)^{-1}ds \\ &= \int_0^\infty \frac{1}{(s+1)}(T+\lambda+s-1-s)(s+T+\lambda)^{-1}ds \\ &= \int_0^\infty \frac{1}{(s+1)}\left[1-(1+s)(s+T+\lambda)^{-1}\right]ds \\ &= \int_0^\infty \left[\frac{1}{s+1}-(s+T+\lambda)^{-1}\right]ds. \end{aligned}$$

For A, $B>0$ and $u\geq 0$, define

$$K_u := (A+u)^{-1}+(B+u)^{-1}-(A+B+u)^{-1} \tag{2.2}$$

and $W_u := 1-uK_u$.

Therefore

$$\begin{aligned}(2.3)\qquad &\ln(A+\lambda)+\ln(B+\lambda)-\ln(A+B+\lambda)-\ln\lambda\\ &=\int_0^\infty\left[\frac{1}{(s+1)}-(s+A+\lambda)^{-1}\right]ds\\ &+\int_0^\infty\left[\frac{1}{(s+1)}-(s+B+\lambda)^{-1}\right]ds\\ &-\int_0^\infty\left[\frac{1}{(s+1)}-(s+A+B+\lambda)^{-1}\right]ds\\ &-\int_0^\infty\left[\frac{1}{(s+1)}-(s+\lambda)^{-1}\right]ds\end{aligned}$$

$$\begin{aligned}&=\int_0^\infty\Big[(s+A+B+\lambda)^{-1}+(s+\lambda)^{-1}\\ &-(s+A+\lambda)^{-1}-(s+B+\lambda)^{-1}\Big]ds\\ &=\int_0^\infty\left[(s+\lambda)^{-1}-K_{s+\lambda}\right]ds=\int_0^\infty\left(\frac{1}{s+\lambda}-K_{s+\lambda}\right)ds\\ &=\int_0^\infty\frac{1}{s+\lambda}\left[1-(s+\lambda)K_{s+\lambda}\right]ds=\int_0^\infty\frac{1}{s+\lambda}W_{s+\lambda}ds.\end{aligned}$$

We have successively

$$\begin{aligned}&(A+B+\lambda)W_\lambda(A+B+\lambda)\\ &=(A+B+\lambda)(1-\lambda K_\lambda)(A+B+\lambda)\\ &=(A+B+\lambda)^2-\lambda(A+B+\lambda)K_\lambda(A+B+\lambda)\\ &=(A+B+\lambda)(A+B+\lambda)\\ &-\lambda\left[B(A+\lambda)^{-1}B+A(B+\lambda)^{-1}A+2(A+B)+\lambda\right]\\ &=A^2+BA+\lambda A+AB+B^2+\lambda B+\lambda A+\lambda B+\lambda^2\\ &-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A-2\lambda(A+B)-\lambda^2\\ &=A^2+B^2+BA+AB-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A\\ &=A(B+\lambda)^{-1}(B+\lambda)A-\lambda A(B+\lambda)^{-1}A\\ &+B(A+\lambda)^{-1}(A+\lambda)B-\lambda B(A+\lambda)^{-1}B+BA+AB\\ &=A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB,\end{aligned}$$

therefore

$$\begin{aligned}(2.4)\qquad W_\lambda=&(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right]\\ &\times(A+B+\lambda)^{-1}.\end{aligned}$$

From (2.4) we obtain

$$
\begin{aligned}
W_{s+\lambda} &= (A+B+s+\lambda)^{-1} \\
&\times \left[A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB + BA + AB\right] \\
&\times (A+B+s+\lambda)^{-1} \\
&= (A+B+s+\lambda)^{-1} \left[A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB\right] \\
&\times (A+B+s+\lambda)^{-1} \\
&+ (A+B+s+\lambda)^{-1} (BA+AB)(A+B+s+\lambda)^{-1}.
\end{aligned} \tag{2.5}
$$

On making use of (2.3) and (2.5) we obtain the desired result (2.1). □

Theorem 2. *For all A, $B>0$ we have*

$$
\begin{aligned}
&\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A+B) \\
&= \int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right. \\
&\times \left[A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB\right] \\
&\times \left.(A+B+s+\lambda)^{-1} ds\right) d\mu(\lambda) \\
&+ \int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right. \\
&\times \left.(BA+AB)(A+B+s+\lambda)^{-1} ds\right) d\mu(\lambda) \\
&\geq \int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right. \\
&\times \left.(BA+AB)(A+B+s+\lambda)^{-1} ds\right) d\mu(\lambda).
\end{aligned} \tag{2.6}
$$

If $BA+AB\geq 0$, then

$$
\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B) \geq \mathcal{DL}og(w,\mu)(A+B). \tag{2.7}
$$

Proof. The identity (2.6) follows by multiplying the equality (2.1) with $w(\lambda)$ and integrating on $[0,\infty)$ over the measure $d\mu(\lambda)$.

Let s, $\lambda \geq 0$. Since $(B+s+\lambda)^{-1} B > 0$ and $(A+s+\lambda)^{-1} A > 0$ hence $A(B+s+\lambda)^{-1} BA > 0$ and $B(A+s+\lambda)^{-1} AB > 0$. Therefore

$$
A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB > 0
$$

and by multiplying both sides by $(A+B+s+\lambda)^{-1}$ we get

$$
\begin{aligned}
&(A+B+s+\lambda)^{-1}\left[A(B+s+\lambda)^{-1} BA + B(A+s+\lambda)^{-1} AB\right] \\
&\times (A+B+s+\lambda)^{-1} > 0.
\end{aligned}
$$

By multiplying with $\frac{1}{s+\lambda}$ and $w(\lambda)$ and integrating twice, we obtain

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right.\\&\times\left[A(B+s+\lambda)^{-1}BA+B(A+s+\lambda)^{-1}AB\right]\\&\left.\times(A+B+s+\lambda)^{-1}ds\right)d\mu(\lambda)\\&\geq 0,\end{aligned}$$

which proves the inequality in (2.6).

If $BA+AB\geq 0$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$ we get

$$(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1}\geq 0$$

for s, $\lambda\geq 0$ and by integration twice, we derive

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right.\\&\left.\times(BA+AB)(A+B+s+\lambda)^{-1}ds\right)d\mu(\lambda)\\&\geq 0\end{aligned}$$

and the subadditivity property (2.7) is proved. □

Remark 1. *If we write the inequality (2.7) for the transform* $\mathcal{DL}og\left(w_{\ell^{r-1}}\right)$ *we get*

$$A^r+B^r\geq(A+B)^r,\ r\in(0,1] \tag{2.8}$$

provided A, $B>0$ *with* $BA+AB\geq 0$.

If we write the inequality (2.7) for the transform $\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)$ *we get*

$$\frac{1}{6}\pi^2+\operatorname{dilog}(A+B)\geq\operatorname{dilog}(A)+\operatorname{dilog}(B) \tag{2.9}$$

provided A, $B>0$ *with* $BA+AB\geq 0$.

We define the function

$$G_{w,\mu}(t):=\int_0^\infty w(\lambda)\left(\int_0^\infty\frac{ds}{(s+\lambda)(s+t+\lambda)^2}\right)d\mu(\lambda),\ t>0. \tag{2.10}$$

Observe that for a, $b>0$ we have

$$\int_0^\infty\frac{ds}{(s+a)(s+b)^2}=\frac{\ln b-\ln a}{(b-a)^2}-\frac{1}{b(b-a)}.$$

This gives that

$$\int_0^\infty\frac{ds}{(s+\lambda)(s+t+\lambda)^2}=\frac{\ln(t+\lambda)-\ln\lambda}{t^2}-\frac{1}{t(t+\lambda)},\ t>0.$$

Therefore

$$G_{w,\mu}(t)=\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{(s+\lambda)(s+t+\lambda)^2}ds\right)d\mu(\lambda) \tag{2.11}$$
$$=\int_0^\infty w(\lambda)\left(\frac{\ln(t+\lambda)-\ln\lambda}{t^2}-\frac{1}{t(t+\lambda)}\right)d\mu(\lambda)$$
$$=\frac{1}{t^2}\int_0^\infty w(\lambda)\ln\left(\frac{t+\lambda}{\lambda}\right)d\mu(\lambda)-\frac{1}{t}\int_0^\infty \frac{w(\lambda)}{t+\lambda}d\mu(\lambda)$$
$$=\frac{1}{t^2}\mathcal{DL}og(w,\mu)(t)-\frac{1}{t}\mathcal{D}(w)(t)\geq 0,$$

for all $t>0$.

Corollary 1. *If A, $B>0$ with $BA+AB\geq k$, where k is a real number, then*

$$\mathcal{DL}og(w,\mu)(A)+\mathcal{DL}og(w,\mu)(B)-\mathcal{DL}og(w,\mu)(A+B) \tag{2.12}$$
$$\geq k\left(\mathcal{DL}og(w,\mu)(A+B)-(A+B)\mathcal{D}(w)(t)\right)(A+B)^{-2}.$$

Proof. If $BA+AB\geq k$, then by multiplying both sides by $(A+B+s+\lambda)^{-1}$, we get

$$(A+B+s+\lambda)^{-1}(BA+AB)(A+B+s+\lambda)^{-1}$$
$$\geq k(A+B+s+\lambda)^{-2},$$

for s, $\lambda\geq 0$, which by integration gives that

$$\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-1}\right.$$
$$\left.\times(BA+AB)(A+B+s+\lambda)^{-1}ds\right)d\mu(\lambda)$$
$$\geq k\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-2}\right)d\mu(\lambda).$$

Observe that, by continuous functional calculus and by (2.10) and (2.11), we get

$$\int_0^\infty w(\lambda)\left(\int_0^\infty \frac{1}{s+\lambda}(A+B+s+\lambda)^{-2}\right)d\mu(\lambda)$$
$$=G_{w,\mu}(A+B)$$
$$=\left(\mathcal{DL}og(w,\mu)(A+B)-(A+B)\mathcal{D}(w)(t)\right)(A+B)^{-2}$$

and the inequality (2.12) is proved. □

Remark 2. *If A, $B>0$ with $BA+AB\geq k\geq 0$, then we have the following refinement of (2.7)*

$$\mathcal{DL}og(w,\mu)(A)+\mathcal{DL}og(w,\mu)(B)-\mathcal{DL}og(w,\mu)(A+B) \tag{2.13}$$
$$\geq k\left(\mathcal{DL}og(w,\mu)(A+B)-(A+B)\mathcal{D}(w)(t)\right)(A+B)^{-2}\geq 0.$$

If we write the inequality (2.12) for the transform $\mathcal{DL}og(w_{\ell^{r-1}})$ we get for $r\in(0,1]$ that

$$A^r+B^r-(A+B)^r\geq(1-r)k(A+B)^{r-2}, \tag{2.14}$$

provided $A,\ B>0$ with $BA+AB\geq k$. If $k\geq 0$, then we obtain the following refinement of (2.8)

$$A^{r}+B^{r}-(A+B)^{r}\geq(1-r)\,k\,(A+B)^{r-2}\geq 0. \tag{2.15}$$

3. Some Examples Via Operator Monotone Functions

We have the following class of examples that are of interest:

Lemma 2. *Assume that function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1), where $b\geq 0$ and μ is a positive measure on $[0,\infty)$. Then*

$$\mathcal{D}\mathcal{L}og\,(\ell,\mu)\,(t)=F_{f}\,(t)-bt \tag{3.1}$$

provided the function

$$F_{f}\,(t):=\int_{0}^{t}\frac{f\,(s)-f\,(0)}{s}ds \tag{3.2}$$

is defined for all $t\in(0,\infty)$.

Proof. From (1.1) we have

$$\frac{f\,(s)-f\,(0)}{s}-b=\int_{0}^{\infty}\frac{\lambda}{s+\lambda}d\mu\,(\lambda)=\mathcal{D}\,(\ell,\mu)\,(s)\,, \tag{3.3}$$

where $\ell\,(\lambda)=\lambda,\ \lambda\geq 0$.

By taking the integral over s on $(0,t)$, we have

$$F_{f}\,(t)=\int_{0}^{t}\frac{f\,(s)-f\,(0)}{s}ds-bt=\int_{0}^{t}\mathcal{D}\,(\ell,\mu)\,(s)\,ds=\mathcal{D}\mathcal{L}og\,(\ell,\mu)\,(t)$$

for $t>0$, and the proposition is proved. □

Remark 3. *If we take $f\,(t)=\ln\,(t+a)\,,$ for $a,\ t>0,$ then we have*

$$F_{\ln(t+a)}\,(t):=\int_{0}^{t}\frac{\ln\,(s+a)-\ln\,(a)}{s}ds=\int_{0}^{t}\frac{1}{s}\ln\left(\frac{s}{a}+1\right)ds.$$

If we change the variable $u=\frac{s}{a}$, then we get

$$\int_{0}^{t}\frac{1}{s}\ln\left(\frac{s}{a}+1\right)ds=\int_{0}^{t/a}\frac{1}{ua}\ln\,(u+1)\,adu=\int_{0}^{t/a}\frac{1}{u}\ln\,(u+1)\,du$$
$$=-\operatorname{dilog}\left(\frac{t}{a}+1\right),$$

which gives

$$F_{\ln(t+a)}\,(t)=-\operatorname{dilog}\left(\frac{t}{a}+1\right),\ t>0.$$

If $f\,(t)=t^{r}$, $r\in(0,1]$, then $F_{f}\,(t):=\frac{t^{r}}{r}$, $t>0$.

Proposition 1. *Assume that function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If $A,B>0$ with $BA+AB\geq 0$, then*

$$F_{f}\,(A)+F_{f}\,(B)\geq F_{f}\,(A+B)\,. \tag{3.4}$$

Proof. By Lemma 2 we have for all A, $B>0$ that

$$\begin{aligned}&\mathcal{DL}og\left(\ell,\mu\right)\left(A\right)+\mathcal{DL}og\left(\ell,\mu\right)\left(B\right)-\mathcal{DL}og\left(\ell,\mu\right)\left(A+B\right)\\&=F_{f}\left(A\right)-bA+F_{f}\left(B\right)-bB-F_{f}\left(A+B\right)+b\left(A+B\right)\\&=F_{f}\left(A\right)+F_{f}\left(B\right)-F_{f}\left(A+B\right).\end{aligned}$$

By making use of (2.7) we derive the desired result (3.4). □

Proposition 2. *If A, $B>0$ with $BA+AB\geq k$, where k is a real number, then*

$$\begin{aligned}&F_{f}\left(A\right)+F_{f}\left(B\right)-F_{f}\left(A+B\right)\\&\geq k\left[F_{f}\left(A+B\right)-f\left(A+B\right)+f\left(0\right)\right]\left(A+B\right)^{-2}.\end{aligned}\tag{3.5}$$

If $k\geq0$, then we have the refinement of (3.4)

$$\begin{aligned}&F_{f}\left(A\right)+F_{f}\left(B\right)-F_{f}\left(A+B\right)\\&\geq k\left[F_{f}\left(A+B\right)-f\left(A+B\right)+f\left(0\right)\right]\left(A+B\right)^{-2}\geq0.\end{aligned}\tag{3.6}$$

Remark 4. *If we take $f\left(t\right)=\ln\left(t+a\right)$, for a, $t>0$, then we have*

$$\begin{aligned}&\operatorname{dilog}\left(\frac{1}{a}\left(A+B\right)+1\right)-\operatorname{dilog}\left(\frac{1}{a}A+1\right)-\operatorname{dilog}\left(\frac{1}{a}B+1\right)\\&\geq k\left[\ln a-\operatorname{dilog}\left(\frac{1}{a}\left(A+B\right)+1\right)-\ln\left(A+B\right)\right]\left(A+B\right)^{-2}\geq0\end{aligned}\tag{3.7}$$

provided A, $B>0$ with $BA+AB\geq k\geq0$.

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