LIPSCHITZ TYPE INEQUALITIES FOR $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following $\mathcal{D}$-*logarithmic integral transform*

$$\mathcal{DL}og(w)(T) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+T}{\lambda}\right)d\lambda,$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\begin{aligned}&\|\mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\|\\ &\leq \|B-A\| \times \begin{cases} \frac{\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ \mathcal{D}(w,\mu)(m) \text{ if } m_1 = m_2 = m,\end{cases}\end{aligned}$$

where

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\mu(\lambda),\ s>0.$$

Some examples for integral transforms $\mathcal{DL}og(\cdot,\cdot)$ related to power function, dilogarithmic function and exponential integral are also provided.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A|-|B|\| \leq L\|A-B\|$$

for any A, $B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds

$$\||A|-|B|\| \leq \frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right)$$

for any A, $B \in \mathcal{B}(H)$ with $A \neq B$.

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If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr} C^{*}C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\,\|A-B\|_{HS}$$

for any $A,\ B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\|\leq a_1\|A-B\|+a_2\|A-B\|^2+O\left(\|A-B\|^3\right)$$

where

$$a_1=\left\|A^{-1}\right\|\|A\|\text{ and }a_2=\left\|A^{-1}\right\|+\left\|A^{-1}\right\|^3\|A\|^2.$$

An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$. In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\|\leq f'(a)\|A-B\|$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,\ B\geq a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following representation of operator monotone functions [15], see for instance [5, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

For some recent results related to operator monotone functions we refer to [10], [11] [7] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [5, p. 145]

$$s^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+s}d\lambda. \tag{1.2}$$

Observe that for $s>0$, $s\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+s)(\lambda+1)}=\frac{\ln s}{s-1}+\frac{1}{1-s}\ln\left(\frac{u+s}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln s}{s-1} = \int_0^\infty \frac{d\lambda}{(\lambda + s)(\lambda + 1)}, \tag{1.3}$$

which gives the representation for the logarithm

$$\ln s = (s-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + s)} \tag{1.4}$$

If we integrate (1.2) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\begin{aligned} \frac{t^r}{r} &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \left(\int_0^t \left(\frac{1}{\lambda + s} \right) ds \right) \lambda^{r-1} d\lambda \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \ln\left(\frac{t+\lambda}{\lambda} \right) d\lambda \end{aligned}$$

giving the identity of interest

$$t^r = \frac{r \sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \ln\left(\frac{t+\lambda}{\lambda} \right) d\lambda, \ t > 0 \text{ and } r \in (0,1].$$

Recall the *dilogarithmic function* $\text{dilog} : [0,\infty) \to \mathbb{R}$ defined by

$$\text{dilog}(t) := \int_1^t \frac{\ln s}{1-s} ds, \ t \geq 0.$$

Some particular values of interest are

$$\text{dilog}(1) = 0, \ \text{dilog}(0) = \int_1^0 \frac{\ln s}{1-s} ds = \int_0^1 \frac{\ln s}{s-1} ds = \frac{1}{6}\pi^2,$$

and

$$\text{dilog}\left(\frac{1}{2}\right) = \frac{1}{12}\pi^2 - \frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.3) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\int_0^t \frac{\ln s}{s-1} ds = \int_0^\infty \left(\int_0^t \frac{1}{\lambda + s} ds \right) \frac{1}{(\lambda + 1)} d\lambda = \int_0^\infty \frac{1}{(\lambda + 1)} \ln\left(\frac{t+\lambda}{\lambda} \right) d\lambda$$

and since

$$\begin{aligned} \int_0^t \frac{\ln s}{s-1} ds &= \int_0^1 \frac{\ln s}{s-1} ds + \int_1^t \frac{\ln s}{s-1} ds = \frac{1}{6}\pi^2 - \int_1^t \frac{\ln s}{1-s} ds \\ &= \frac{1}{6}\pi^2 - \text{dilog}(t) \end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \text{dilog}(t) = \int_0^\infty \frac{1}{(\lambda + 1)} \ln\left(\frac{t+\lambda}{\lambda} \right) d\lambda, \ t > 0.$$

Motivated by the above representations, we define the *$\mathcal{D}$-logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by

$$\mathcal{DL}og(w,\mu)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + t}{\lambda} \right) d\mu(\lambda), \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t>0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{DL}og\,(w)\,(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\lambda. \tag{1.6}$$

Obviously,

$$\begin{aligned}\mathcal{DL}og\,(w,\mu)\,(t) &= \int_0^\infty w(\lambda)\ln\left(1+\frac{t}{\lambda}\right)d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\left[\ln(\lambda+t)-\ln(\lambda)\right]d\mu(\lambda)\end{aligned}$$

and one can use either of these representations when is needed.

If we use the *$\mathcal{D}$-logarithmic transform* for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$ we have

$$\mathcal{DL}og\,(w_{\ell^{r-1}})\,(t) = t^r,\ t\geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \operatorname{dilog}(t),\ t\geq 0. \tag{1.7}$$

In the recent paper [6] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\mu(\lambda),\ s>0, \tag{1.8}$$

for a continuous and positive function $w(\lambda)$, $\lambda>0$,where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $s>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\lambda,\ s>0. \tag{1.9}$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [6].

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\int_0^t \mathcal{D}(w,\mu)(s)\,ds &:= \int_0^\infty\left(\int_0^t \frac{1}{\lambda+s}ds\right)w(\lambda)\,d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda)\end{aligned} \tag{1.10}$$

for $t>0$, which provides the equality of interest

$$\mathcal{DL}og\,(w,\mu)\,(t) = \int_0^t \mathcal{D}(w,\mu)(s)\,ds,\ t>0, \tag{1.11}$$

provided that the integral on the right side exists for all $t>0$.

2. Preliminary Facts

Start to the following identity for the logarithmic function:

Lemma 1. *For all A, $B>0$ we have the identity:*

$$\begin{aligned}&\ln B - \ln A \\ &= \int_0^\infty\left(\int_0^1 (s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}dt\right)ds.\end{aligned} \tag{2.1}$$

Proof. We have from (1.4) for $A,\ B>0$ that

$$\ln B-\ln A=\int_0^\infty \frac{1}{s+1}\left[(B-1)(s+B)^{-1}-(A-1)(s+A)^{-1}\right]ds. \tag{2.2}$$

Since

$$\begin{aligned}&(B-1)(s+B)^{-1}-(A-1)(s+A)^{-1}\\&=B(s+B)^{-1}-A(s+A)^{-1}-\left((s+B)^{-1}-(s+A)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&B(s+B)^{-1}-A(s+A)^{-1}\\&=(B+s-s)(s+B)^{-1}-(A+s-s)(s+A)^{-1}\\&=1-s(s+B)^{-1}-1+s(s+A)^{-1}=s(s+A)^{-1}-s(s+B)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(B-1)(s+B)^{-1}-(A-1)(s+A)^{-1}\\&=s(s+A)^{-1}-s(s+B)^{-1}-\left((s+B)^{-1}-(s+A)^{-1}\right)\\&=(s+1)\left[(s+A)^{-1}-(s+B)^{-1}\right]\end{aligned}$$

and by (2.2) we get

$$\ln B-\ln A=\int_0^\infty \left[(s+A)^{-1}-(s+B)^{-1}\right]ds. \tag{2.3}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Since, by (2.6) we have

$$(s+A)^{-1}-(s+B)^{-1} = \int_0^1 (s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}dt, \quad (2.7)$$

for all $s \geq 0$, hence by (2.3) and (2.7) we get (2.1). □

Lemma 2. *For all A, $B>0$ we have the identity:*

$$\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A) = \int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A) \times (s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda). \quad (2.8)$$

Proof. For all A, $B>0$ we have

$$\begin{aligned}&\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln\lambda]d\mu(\lambda)-\int_0^\infty w(\lambda)[\ln(\lambda+A)-\ln\lambda]d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]d\mu(\lambda).\end{aligned} \quad (2.9)$$

Since, by (2.1) we get

$$\begin{aligned}&\ln(\lambda+B)-\ln(\lambda+A)\\ &=\int_0^\infty\left(\int_0^1(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\right.\\ &\quad\left.\times(\lambda+B-(\lambda+A))(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}dt\right)ds\end{aligned}$$

for all $\lambda \geq 0$, then by multiplying with $w(\lambda)$ and integrating over $\mu(\lambda)$ we obtain

$$\begin{aligned}&\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]d\mu(\lambda)\\ &=\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\ &\quad\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda).\end{aligned} \quad (2.10)$$

Finally, by (2.9) and (2.10) we get (2.8). □

Corollary 1. *If $B \geq A > 0$, then $\mathcal{D}\mathcal{L}og(w,\mu)(B) \geq \mathcal{D}\mathcal{L}og(w,\mu)(A)$, namely $\mathcal{D}\mathcal{L}og(w,\mu)(\cdot)$ is operator monotone on $(0,\infty)$.*

Proof. If $B-A \geq 0$, then by multiplying both sides with $(s+\lambda+(1-t)A+tB)^{-1}$ we get

$$(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1} \geq 0$$

for all $t \in [0,1]$ and s, $\lambda \geq 0$.

If we integrate of over $t \in [0,1]$ and $s \in [0,\infty)$ we obtain

$$\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds \geq 0.$$

Further, if we multiply this inequality by $w(\lambda) \geq 0$, integrate over the positive measure $\mu(\lambda)$ and use the identity (2.8) we derive the desired inequality. □

Remark 1. *Since, by (1.7),*

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \operatorname{dilog}(t), \ t \geq 0$$

and $\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)$ *is operator monotone, then the function* $-\operatorname{dilog}$ *is operator monotone on* $(0, \infty)$.

3. Main Results

We have the following Lipschitz type inequality:

Theorem 2. *Assume that* $A \geq m_1 > 0$, $B \geq m_2 > 0$, *then*

$$\begin{aligned} &\|\mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\| \\ &\leq \|B-A\| \times \begin{cases} \frac{\mathcal{DL}og(w,\mu)(m_2) - \mathcal{DL}og(w,\mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ \mathcal{D}(w,\mu)(m) & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.1}$$

Proof. By taking the norm in (2.8) we get

$$\begin{aligned} &\|\mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\| \\ &\leq \int_0^\infty w(\lambda)\left(\int_0^\infty \left\|\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right. \\ &\left.\left.\left.\times (s+\lambda+(1-t)A+tB)^{-1}dt\right)\right\| ds\right) d\mu(\lambda) \\ &\leq \int_0^\infty w(\lambda)\left(\int_0^\infty \left(\int_0^1 \left\|(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right. \\ &\left.\left.\left.\times (s+\lambda+(1-t)A+tB)^{-1}\right\| dt\right) ds\right) d\mu(\lambda) \\ &\leq \|B-A\| \int_0^\infty w(\lambda) \\ &\times \left(\int_0^\infty \left(\int_0^1 \left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right) ds\right) d\mu(\lambda), \end{aligned} \tag{3.2}$$

for all A, $B > 0$.

Assume that $m_2 > m_1$. Then

$$s+\lambda+(1-t)A+tB \geq (1-t)m_1 + tm_2 + s + \lambda,$$

for $t \in [0,1]$ and s, $\lambda \geq 0$.

This implies that

$$(s+\lambda+(1-t)A+tB)^{-1} \leq ((1-t)m_1 + tm_2 + s + \lambda)^{-1}$$

and

$$\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2 \leq ((1-t)m_1 + tm_2 + s + \lambda)^{-2}$$

for $t \in [0,1]$ and s, $\lambda \geq 0$.

Therefore

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)ds\right)d\mu(\lambda) \tag{3.3}$$
$$\le\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+s+\lambda)^{-2}dt\right)ds\right)d\mu(\lambda)$$
$$=\frac{1}{m_2-m_1}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+s+\lambda)^{-1}\right.\right.$$
$$\left.\left.\times\ (m_2-m_1)((1-t)m_1+tm_2+s+\lambda)^{-1}dt\right)ds\right)d\mu(\lambda).$$

If in the identity (2.8) we take $A=m_1$, $B=m_2$, then we get

$$\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)$$
$$=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)m_1+tm_2)^{-1}(m_2-m_1)\right.\right.$$
$$\left.\left.\times(s+\lambda+(1-t)m_1+tm_2)^{-1}dt\right)ds\right)d\mu(\lambda).$$

and by (3.3) we get

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)ds\right)d\mu(\lambda) \tag{3.4}$$
$$\le\frac{1}{m_2-m_1}\left[\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)\right].$$

The case $m_2<m_1$ goes in a similar way and we also obtain (3.4).

Assume that $m_2=m_1=m$. Let $\epsilon>0$, then $B+\epsilon\ge m+\epsilon>m$. From (3.4) we get

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)ds\right)d\mu(\lambda)$$
$$\le\frac{1}{m+\epsilon-m}\left[\mathcal{DL}og(w,\mu)(m+\epsilon)-\mathcal{DL}og(w,\mu)(m)\right]$$

and by taking the limit over $\epsilon\to0+$, using the continuity and differentiability of $\mathcal{DL}og$ we deduce

$$\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)ds\right)d\mu(\lambda)$$
$$\le(\mathcal{DL}og(w,\mu))'(m)=\mathcal{D}(w,\mu)(m),$$

which proves the second part of (3.1). □

We have:

Lemma 3. *Assume that function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1), where $b\ge0$ and μ is a positive measure on $[0,\infty)$. Then*

$$\mathcal{DL}og(\ell,\mu)(t)=F_f(t)-bt \tag{3.5}$$

provided the function

$$F_f(t):=\int_0^t\frac{f(s)-f(0)}{s}ds \tag{3.6}$$

is defined for all $t\in(0,\infty)$.

Proof. From (1.1) we have

$$\frac{f(s)-f(0)}{s}-b=\int_0^\infty\frac{\lambda}{s+\lambda}d\mu(\lambda)=\mathcal{D}(\ell,\mu)(s) \tag{3.7}$$

where $\ell(\lambda)=\lambda$, $\lambda\geq 0$.

By taking the integral over s on $(0,t)$, we have

$$\int_0^t\frac{f(s)-f(0)}{s}ds-bt=\int_0^t\mathcal{D}(\ell,\mu)(s)\,ds=\mathcal{DL}og(\ell,\mu)(t)$$

for $t>0$, and the proposition is proved. □

Corollary 2. *With the assumptions of Lemma 3 and if* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\begin{aligned}&\|F_f(B)-F_f(A)-b(B-A)\|\\&\leq\|B-A\|\times\begin{cases}\left(\frac{F_f(m_2)-F_f(m_1)}{m_2-m_1}-b\right) & \text{if } m_1\neq m_2,\\[2ex] \left(\frac{f(m)-f(0)}{s}-b\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.8}$$

Moreover,

$$\begin{aligned}&\|F_f(B)-F_f(A)\|\\&\leq\|B-A\|\times\begin{cases}\frac{F_f(m_2)-F_f(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\[2ex] \frac{f(m)-f(0)}{s} & \text{if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.9}$$

Proof. The inequality (3.8) follows by (3.1) for $\mathcal{DL}og(\ell,\mu)(t)=F_f(t)-bt$, $t>0$. By the triangle inequality we have

$$\|F_f(B)-F_f(A)\|-b\|B-A\|\leq\|F_f(B)-F_f(A)-b(B-A)\|$$

and by (3.8) we derive (3.9). □

Remark 2. *Assume that* $A\geq m_1>0$, $B\geq m_2>0$. *Consider the kernel* $w_{\ell^{r-1}}(\lambda):=\frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$. *Then we have*

$$\mathcal{DL}og(w_{\ell^{r-1}})(t)=t^r,\ t\geq 0$$

and by (3.1),

$$\|B^r-A^r\|\leq\|B-A\|\times\begin{cases}\frac{m_2^r-m_1^r}{m_2-m_1} & \text{if } m_1\neq m_2,\\[2ex] rm^r & \text{if } m_1=m_2=m.\end{cases} \tag{3.10}$$

For the kernel $w_{(\ell+1)^{-1}}(\lambda):=\frac{1}{\lambda+1}$ *we have*

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t)=\frac{1}{6}\pi^2-\operatorname{dilog}(t),\ t\geq 0$$

and by (3.1),

$$\|\operatorname{dilog}(B)-\operatorname{dilog}(A)\|\leq\|B-A\|\times\begin{cases}\frac{\operatorname{dilog}(m_1)-\operatorname{dilog}(m_2)}{m_2-m_1} & \text{if } m_1\neq m_2,\\[2ex] u(m) & \text{if } m_1=m_2=m,\end{cases} \tag{3.11}$$

where

$$u(t) := \begin{cases} \frac{\ln t}{t-1}, \ t \neq 1, \ t > 0, \\ \\ 1, \ t = 1. \end{cases}$$

If we take $f(t) = \ln(t+a)$, *for* a, $t > 0$, *then we have*

$$F_{\ln(t+a)}(t) := \int_0^t \frac{\ln(s+a) - \ln(a)}{s} ds = \int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds.$$

If we change the variable $u = \frac{s}{a}$, *then we get*

$$\int_0^t \frac{1}{s} \ln\left(\frac{s}{a} + 1\right) ds = \int_0^{t/a} \frac{1}{ua} \ln(u+1)\, adu = \int_0^{t/a} \frac{1}{u} \ln(u+1)\, du$$
$$= -\operatorname{dilog}\left(\frac{t}{a} + 1\right),$$

which gives

$$F_{\ln(t+a)}(t) = -\operatorname{dilog}\left(\frac{t}{a} + 1\right), \ t > 0.$$

By (3.9) we then get

$$(3.12) \qquad \left\| \operatorname{dilog}\left(\frac{1}{a}B + 1\right) - \operatorname{dilog}\left(\frac{1}{a}A + 1\right) \right\|$$
$$\leq \|B - A\| \times \begin{cases} \frac{\operatorname{dilog}\left(\frac{m_1}{a}+1\right) - \operatorname{dilog}\left(\frac{m_2}{a}+1\right)}{m_2 - m_1} \ \textit{if } m_1 \neq m_2, \\ \\ \frac{\ln(m+a) - \ln a}{m} \ \textit{if } m_1 = m_2 = m. \end{cases}$$

4. More Examples

If we consider the positive kernel $w_{\exp(-a\cdot)}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$, then, after some calculations

$$\int_0^\infty \exp(-a\lambda) \ln(\lambda + t)\, d\lambda = \frac{1}{a}\left[\ln t + E_1(at)\exp(at)\right],$$

for $t > 0$,where the *exponential integral* is given by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

For $a = 1$ we have

$$\int_0^\infty \exp(-\lambda) \ln(\lambda + t)\, d\lambda = \ln t + E_1(t)\exp(t),$$

For $t = 0$, we derive

$$\int_0^\infty \exp(-\lambda) \ln(\lambda)\, d\lambda = -\gamma,$$

where γ is Euler–Mascheroni constant.

For $a>0$, by changing the variable $a\lambda=\nu$, then

$$\begin{aligned}\int_0^\infty \exp(-a\lambda)\ln(\lambda)\,d\lambda &= \int_0^\infty \exp(-\nu)\ln\left(\frac{\nu}{a}\right)\frac{1}{a}d\nu\\ &= \frac{1}{a}\int_0^\infty \left[\exp(-\nu)\ln\nu-\exp(-\nu)\ln a\right]d\nu\\ &= \frac{1}{a}(-\gamma-\ln a) = -\frac{\ln a+\gamma}{a}.\end{aligned}$$

We then have

$$\begin{aligned}\mathcal{D}\mathcal{L}og\left(w_{\exp(-a\cdot)}\right)(t) &= \int_0^\infty \exp(-a\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda)\\ &= \frac{1}{a}\left[\ln(at)+E_1(at)\exp(at)+\gamma\right]\end{aligned}$$

and, for $a=1$,

$$\begin{aligned}\mathcal{D}\mathcal{L}og\left(w_{\exp(-\cdot)}\right)(t) &= \int_0^\infty \exp(-\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda)\\ &= \ln(t)+E_1(t)\exp(t)+\gamma.\end{aligned}$$

Using Corollary 1 we conclude that the function $\ln(t)+E_1(t)\exp(t)$ is *operator monotone on* $(0,\infty)$.

Observe that

$$\begin{aligned}\left(\mathcal{D}\mathcal{L}og\left(w_{\exp(-\cdot)}\right)\right)'(t) &= \frac{1}{t}+E_1'(t)\exp(t)+E_1(t)\exp(t)\\ &= \frac{1}{t}+\exp(t)\left[E_1(t)-\frac{e^{-t}}{t}\right],\ t>0.\end{aligned}$$

Assume that $A\geq m_1>0$, $B\geq m_2>0$, then by (3.1) we obtain:

$$\begin{aligned}&\|\ln(B)+E_1(B)\exp(B)-\ln(A)-E_1(A)\exp(A)\|\\ &\leq\|B-A\|\\ &\times\begin{cases}\frac{\ln(m_2)+E_1(m_2)\exp(m_2)-\ln(m_1)-E_1(m_1)\exp(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{1}{m}+\exp(m)\left[E_1(m)-\frac{e^{-m}}{m}\right] & \text{if } m_1=m_2=m.\end{cases}\end{aligned}\tag{4.1}$$

If we consider the positive kernel $w_{(\cdot+a)^{-2}}(\lambda):=\frac{1}{(\lambda+a)^2}$, $\lambda\geq 0$, $a>0$, then, after some calculations

$$\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2}d\lambda = \begin{cases}\frac{t\ln t-a\ln a}{a(t-a)}, & \text{if } t\neq a,\\ \frac{\ln a+1}{a}, & \text{if } t=a\end{cases}$$

for $t>0$.

If $a=1$, then

$$\int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+1)^2}d\lambda = \begin{cases}\frac{t\ln t}{t-1}, & \text{if } t\neq 1,\\ 1, & \text{if } t=1\end{cases}$$

for $t>0$.

For $t=0$, we derive

$$\int_0^\infty \frac{\ln(\lambda)}{(\lambda+a)^2}d\lambda = \frac{\ln a}{a}$$

for $a>0$.

Therefore

$$\mathcal{DL}og\left(w_{(\cdot+a)^{-2}}\right)(t) = \begin{cases} \frac{t(\ln t-\ln a)}{a(t-a)}, \text{ if } t\neq a, \\ \\ \frac{1}{a}, \text{ if } t=a. \end{cases}$$

For $a=1$,

$$\mathcal{DL}og\left(w_{(\cdot+1)^{-2}}\right)(t) = \begin{cases} \frac{t\ln t}{t-1}, \text{ if } t\neq 1, \\ \\ 1, \text{ if } t=1. \end{cases}$$

Using Corollary 1 we conclude that the function $\mathcal{DL}og\left(w_{(\cdot+1)^{-2}}\right)$ is *operator monotone on* $(0,\infty)$.

Assume that $1>A\geq m_1>0$, $1>B\geq m_2>0$, then by (3.1) we obtain:

$$\begin{aligned} &\left\|(1-B)^{-1}B\ln B-(1-A)^{-1}A\ln A\right\| \\ &\leq \|B-A\| \\ &\times\begin{cases} \frac{1}{m_2-m_1}\left(\frac{m_2\ln m_2}{m_2-1}-\frac{m_2\ln m_2}{m_2-1}\right) \text{ if } m_1\neq m_2, \\ \\ \left(\frac{1}{m-1}-\frac{\ln m}{(m-1)^2}\right) \text{ if } m_1=m_2=m. \end{cases} \end{aligned} \tag{4.2}$$

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