LOWER AND UPPER BOUNDS FOR THE PERTURBED JENSEN'S GAP OF CONVEX FUNCTIONS

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Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some lower and upper bounds for the perturbed Jensen's gap

$$\int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right)$$

for some classes of twice differentiable convex functions Φ defined on an interval I and $v \in I$. Applications for exponential and logarithm are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in [8] and [10] the following result:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to [m, M]$ is so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:*

$$0 \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega (\Phi' \circ f) f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \tag{1.1}$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu$$
$$\le \frac{1}{2}\left[\Phi'_-(M) - \Phi'_+(m)\right] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\left[\Phi'_-(M) - \Phi'_+(m)\right](M - m).$$

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Remark 1. *We notice that the inequality between the first and the second term in (1.1) in the discrete case was proved in 1994 by Dragomir & Ionescu, see* [13].

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points α, $\beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

Upper and lower bounds for the Jensen's gap were also obtained in [11]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $f : \Omega \to [m, M]$, is μ-measurable and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then by assuming that $\int_\Omega f d\mu \neq m$, M, we have*

$$\begin{aligned}
(1.2)\qquad & \left| \int_\Omega \left| \Phi(f) - \Phi\left(\int_\Omega f d\mu\right) \right| \operatorname{sgn}\left(f - \int_\Omega f d\mu\right) d\mu \right| \\
& \leq \int_\Omega (\Phi \circ f) d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
& \leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu \\
& \leq \frac{1}{2}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \\
& \leq \frac{1}{4}\left(\left[\int_\Omega f d\mu, M; \Phi\right] - \left[m, \int_\Omega f d\mu; \Phi\right]\right) (M - m).
\end{aligned}$$

The constant $\frac{1}{2}$ in the second inequality from (1.2) is best possible.

For other recent reverses of Jensen inequality and applications to divergence measures see [9], [10], [11] and the survey paper [12]. More related results may be found in [1]-[4], [12] and [12]-[14].

In the recent paper [ ] we obtained the following result for the perturbation of Jensen's gap

$$\int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right),$$

where $v \in \mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a differentiable convex function on the interior of I.

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior of I denoted by $\mathring{I}$. Assume that $f : \Omega \to I$ is μ-measurable and such that f, $\Phi \circ f$,*

$f \cdot \Phi' \circ f \in L(\Omega, \mu)$. *Then for all* $v \in \mathring{I}$,

$$
\begin{aligned}
0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v) \left(\int_\Omega f d\mu - v \right) \\
&\leq \int_\Omega \left[\Phi' \circ f - \Phi'(v) \right] (f - v) \, d\mu \\
&\leq \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(v)| \int_\Omega |f - v| \, d\mu, \\[2ex] \left(\int_\Omega |\Phi' \circ f - \Phi'(v)|^p \, d\mu \right)^{1/p} \left(\int_\Omega |f - v|^q \, d\mu \right)^{1/q} \\ p, \ q > 1 \ \text{with } \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \operatorname{essup}_\Omega |f - v| \int_\Omega |\Phi' \circ f - \Phi'(v)| \, d\mu, \end{cases}
\end{aligned} \tag{1.3}
$$

provided the integrals in the last term are finite.

In particular, we have the reverse of Jensen's inequality

$$
\begin{aligned}
0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi \left(\int_\Omega f d\mu \right) \\
&\leq \int_\Omega \left[\Phi' \circ f - \Phi' \left(\int_\Omega f d\mu \right) \right] \left(f - \int_\Omega f d\mu \right) d\mu \\
&= \int_\Omega f \Phi' \circ f d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\
&\leq \begin{cases} \operatorname{essup}_\Omega \left| \Phi' \circ f - \Phi' \left(\int_\Omega f d\mu \right) \right| \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu, \\[2ex] \left(\int_\Omega \left| \Phi' \circ f - \Phi' \left(\int_\Omega f d\mu \right) \right|^p d\mu \right)^{1/p} \left(\int_\Omega \left| f - \int_\Omega f d\mu \right|^q d\mu \right)^{1/q} \\ p, \ q > 1 \ \text{with } \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \operatorname{essup}_\Omega \left| f - \int_\Omega f d\mu \right| \int_\Omega \left| \Phi' \circ f - \Phi' \left(\int_\Omega f d\mu \right) \right| d\mu. \end{cases}
\end{aligned} \tag{1.4}
$$

Motivated by the above results, in this paper we establish some new upper and lower bounds for the perturbed Jensen's gap for some classes of differentiable convex functions Φ defined on an interval I and $v \in I$. Applications for exponential, logarithm and power functions are also given.

2. General Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let* $I \subset \mathbb{R}$ *be a closed interval,* $a \in I$ *and let* n *be a positive integer. If* $g : I \longrightarrow \mathbb{C}$ *is such that the* n*-derivative* $g^{(n)}$ *is absolutely continuous on* I*, then for each* $x \in I$

$$g(x) = T_n(g; a, x) + R_n(g; a, x), \tag{2.1}$$

where $T_n(g; c, y)$ *is Taylor's polynomial, i.e.,*

$$T_n(g; a, x) := \sum_{k=0}^{n} \frac{(x - a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(g; a, x) := \frac{1}{n!} \int_a^x (x - t)^n g^{(n+1)}(t) \, dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1 - s) c + sd$, $s \in [0, 1]$ that

$$\int_c^d h(t) \, dt = (d - c) \int_0^1 h((1 - s) c + sd) \, ds.$$

Therefore,

$$\begin{aligned} &\int_a^x g^{(n+1)}(t) (x - t)^n \, dt \\ &= (x - a) \int_0^1 g^{(n+1)}((1 - s) a + sx) (x - (1 - s) a - sx)^n \, ds \\ &= (x - a)^{n+1} \int_0^1 g^{(n+1)}((1 - s) a + sx) (1 - s)^n \, ds. \end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned} g(x) &= \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(a) (x - a)^k \\ &\quad + \frac{1}{n!} (x - a)^{n+1} \int_0^1 g^{(n+1)}((1 - s) a + sx) (1 - s)^n \, ds \end{aligned} \tag{2.4}$$

for all x, $a \in I$.

We have the following result concerning lower and upper bounds for the Jensen's gap:

Theorem 4. *Let* $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable convex function on* $\mathring{I}$ *and* $f : \Omega \to I$ *so that* f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. *Then for all* $v \in \mathring{I}$,

$$\begin{aligned} 0 &\leq \operatorname*{essinf}_{t \in \Omega} \left(\int_0^1 \Phi''((1 - s) v + sf(t)) (1 - s) \, ds \right) \left(\int_\Omega f d\mu - v \right)^2 \\ &\leq \operatorname*{essinf}_{t \in \Omega} \left(\int_0^1 \Phi''((1 - s) v + sf(t)) (1 - s) \, ds \right) \int_\Omega (f - v)^2 \, d\mu \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v) \left(\int_\Omega f d\mu - v \right) \\ &\leq \operatorname*{essup}_{t \in \Omega} \left(\int_0^1 \Phi''((1 - s) v + sf(t)) (1 - s) \, ds \right) \int_\Omega (f - v)^2 \, d\mu \end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
0 &\le \operatorname*{essinf}_{t\in\Omega} \left(f\left(t\right)-v\right)^{2} \int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu \\
&\le \int_{\Omega}\Phi\circ f d\mu-\Phi\left(v\right)-\Phi'\left(v\right)\left(\int_{\Omega}fd\mu-v\right) \\
&\le \operatorname*{essup}_{t\in\Omega} \left(f\left(t\right)-v\right)^{2} \int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu .
\end{aligned} \tag{2.6}
$$

Proof. We have from (2.4) for $n=2$ that

$$
\Phi\left(x\right)=\Phi\left(c\right)+\Phi'\left(c\right)\left(x-c\right)+\left(x-c\right)^{2}\int_{0}^{1}\Phi''\left(\left(1-s\right)c+sx\right)\left(1-s\right)ds
$$

for all x, $c\in\mathring{I}$, where Φ is twice differentiable on $\mathring{I}$.

This implies that

$$
\begin{aligned}
\Phi\left(f\left(t\right)\right)&=\Phi\left(v\right)+\Phi'\left(v\right)\left(f\left(t\right)-v\right) \\
&\quad+\left(f\left(t\right)-v\right)^{2}\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds
\end{aligned} \tag{2.7}
$$

for all $v\in\mathring{I}$ and μ-a.e. $t\in\Omega$.

By taking the integral in (2.7) we get

$$
\begin{aligned}
&\int_{\Omega}\Phi\circ f d\mu-\Phi\left(v\right)-\Phi'\left(v\right)\left(\int_{\Omega}fd\mu-v\right) \\
&=\int_{\Omega}\left(f-v\right)^{2}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu
\end{aligned} \tag{2.8}
$$

for all $v\in\mathring{I}$, which is an equality of interest in itself.

We observe that for μ-a.e. $t\in\Omega$ we have

$$
\begin{aligned}
&\operatorname*{essinf}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds\right) \\
&\le\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds \\
&\le\operatorname*{essup}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\operatorname*{essinf}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds\right)\int_{\Omega}\left(f-v\right)^{2}d\mu \\
&\le\int_{\Omega}\left(f-v\right)^{2}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu \\
&\le\operatorname*{essup}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds\right)\int_{\Omega}\left(f-v\right)^{2}d\mu
\end{aligned}
$$

and by the identity (2.8) we get the third and the fourth inequalities in (2.5).

The second inequality is obvious by Jensen's inequality

$$
\left(\int_{\Omega}fd\mu-v\right)^{2}\le\int_{\Omega}\left(f-v\right)^{2}d\mu,
$$

while the first one follows from the nonnegativity of the second derivative since Φ is convex.

We also have

$$\begin{aligned}
&\operatorname*{essinf}_{t\in\Omega}\left(f\left(t\right)-v\right)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu\\
&\leq\int_{\Omega}\left(f-v\right)^{2}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu\\
&\leq\operatorname*{essup}_{t\in\Omega}\left(f\left(t\right)-v\right)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\right)\left(1-s\right)ds\right)d\mu
\end{aligned}$$

for all $v\in\mathring{I}$, which, by identity (2.8), produces the desired inequality (2.6). $\square$

Corollary 1. *With the assumptions of Theorem 4 we have the following lower and upper bounds for the Jensen's gap*

$$\begin{aligned}
0&\leq\operatorname*{essinf}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)\int_{\Omega}fd\mu+sf\left(t\right)\right)\left(1-s\right)ds\right)\\
&\quad\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]\\
&\leq\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\
&\leq\operatorname*{essup}_{t\in\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)v+sf\left(t\right)\right)\left(1-s\right)ds\right)\\
&\quad\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]
\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}
0&\leq\operatorname*{essinf}_{t\in\Omega}\left(f\left(t\right)-\int_{\Omega}fd\mu\right)^{2}\\
&\quad\times\int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)\int_{\Omega}fd\mu+sf\right)\left(1-s\right)ds\right)d\mu\\
&\leq\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\int_{\Omega}fd\mu\right)\\
&\leq\operatorname*{essup}_{t\in\Omega}\left(f\left(t\right)-\int_{\Omega}fd\mu\right)^{2}\\
&\quad\times\int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)\int_{\Omega}fd\mu+sf\right)\left(1-s\right)ds\right)d\mu.
\end{aligned}\tag{2.10}$$

The proof follows by Theorem 4 by taking $v=\int_{\Omega}fd\mu$ and observing that

$$\int_{\Omega}\left(f-\int_{\Omega}fd\mu\right)^{2}d\mu=\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2},$$

which is the variance of f.

Corollary 2. *With the assumptions of Theorem 4 and if* $\int_\Omega \Phi' \circ f d\mu \neq 0$ *and the Slater point*

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in I, \tag{2.11}$$

then

$$\begin{aligned} 0 &\leq \operatorname*{essinf}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s)\, \sigma + sf\,(t) \right) (1-s)\, ds \right) \left(\int_\Omega f d\mu - \sigma \right)^2 \\ &\leq \operatorname*{essinf}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s)\, \sigma + sf\,(t) \right) (1-s)\, ds \right) \int_\Omega (f-\sigma)^2\, d\mu \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi\,(\sigma) - \Phi'\,(\sigma) \left(\int_\Omega f d\mu - \sigma \right) \\ &\leq \operatorname*{essup}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s)\, \sigma + sf\,(t) \right) (1-s)\, ds \right) \int_\Omega (f-\sigma)^2\, d\mu \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} 0 &\leq \operatorname*{essinf}_{t\in\Omega} (f\,(t) - \sigma)^2 \int_\Omega \left(\int_0^1 \Phi'' \left((1-s)\, \sigma + sf \right) (1-s)\, ds \right) d\mu \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi\,(\sigma) - \Phi'\,(\sigma) \left(\int_\Omega f d\mu - \sigma \right) \\ &\leq \operatorname*{essup}_{t\in\Omega} (f\,(t) - \sigma)^2 \int_\Omega \left(\int_0^1 \Phi'' \left((1-s)\, \sigma + sf \right) (1-s)\, ds \right) d\mu. \end{aligned} \tag{2.13}$$

Corollary 3. *With the assumptions of Theorem 4 and if there exists an interval* $[m, M] \subset I$ *such that* $m \leq f \leq M$ *μ-a.e. on* Ω*, then*

$$\begin{aligned} 0 &\leq \operatorname*{essinf}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s) \frac{m+M}{2} + sf\,(t) \right) (1-s)\, ds \right) \\ &\times \left(\int_\Omega f d\mu - \frac{m+M}{2} \right)^2 \\ &\leq \operatorname*{essinf}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s) \frac{m+M}{2} + sf\,(t) \right) (1-s)\, ds \right) \\ &\times \int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi \left(\frac{m+M}{2} \right) - \Phi' \left(\frac{m+M}{2} \right) \left(\int_\Omega f d\mu - \frac{m+M}{2} \right) \\ &\leq \operatorname*{essup}_{t\in\Omega} \left(\int_0^1 \Phi'' \left((1-s) \frac{m+M}{2} + sf\,(t) \right) (1-s)\, ds \right) \\ &\times \int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu \end{aligned} \tag{2.14}$$

and

$$0\leq\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\frac{m+M}{2}\right)^{2}\times\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\frac{m+M}{2}+sf\right)(1-s)\,ds\right)d\mu$$
$$\leq\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\frac{m+M}{2}\right)-\Phi'\left(\frac{m+M}{2}\right)\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right)$$
$$\leq\operatorname*{essup}_{t\in\Omega}\left(f(t)-\frac{m+M}{2}\right)^{2}\times\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\frac{m+M}{2}+sf\right)(1-s)\,ds\right)d\mu. \tag{2.15}$$

Remark 2. *Since $m\leq f\leq M$ μ-a.e. on Ω is equivalent to*

$$\left|f-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)\ \ \mu\text{-a.e.on }\Omega,$$

then from the last inequality in (2.15) we obtain

$$(0\leq)\int_{\Omega}\Phi\circ fd\mu-\Phi\left(\frac{m+M}{2}\right)-\Phi'\left(\frac{m+M}{2}\right)\left(\int_{\Omega}fd\mu-\frac{m+M}{2}\right)$$
$$\leq\frac{1}{4}(M-m)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''\left((1-s)\frac{m+M}{2}+sf\right)(1-s)\,ds\right)d\mu. \tag{2.16}$$

Corollary 4. *With the assumptions of Corollary 3 and if there is a $\tau\in I$, a trapezoid point, such that*

$$\Phi(\tau)=\frac{\Phi(m)+\Phi(M)}{2}, \tag{2.17}$$

then we have

$$0\leq\operatorname*{essinf}_{t\in\Omega}\left(\int_{0}^{1}\Phi''((1-s)\tau+sf(t))(1-s)\,ds\right)\left(\int_{\Omega}fd\mu-\tau\right)^{2}$$
$$\leq\operatorname*{essinf}_{t\in\Omega}\left(\int_{0}^{1}\Phi''((1-s)\tau+sf(t))(1-s)\,ds\right)\int_{\Omega}(f-\tau)^{2}d\mu$$
$$\leq\int_{\Omega}\Phi\circ fd\mu-\frac{\Phi(m)+\Phi(M)}{2}-\Phi'(\tau)\left(\int_{\Omega}fd\mu-\tau\right)$$
$$\leq\operatorname*{essup}_{t\in\Omega}\left(\int_{0}^{1}\Phi''((1-s)\tau+sf(t))(1-s)\,ds\right)\int_{\Omega}(f-\tau)^{2}d\mu \tag{2.18}$$

and

$$0\leq\operatorname*{essinf}_{t\in\Omega}(f(t)-\tau)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''((1-s)\tau+sf)(1-s)\,ds\right)d\mu$$
$$\leq\int_{\Omega}\Phi\circ fd\mu-\frac{\Phi(m)+\Phi(M)}{2}-\Phi'(\tau)\left(\int_{\Omega}fd\mu-\tau\right)$$
$$\leq\operatorname*{essup}_{t\in\Omega}(f(t)-\tau)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''((1-s)\tau+sf)(1-s)\,ds\right)d\mu. \tag{2.19}$$

Remark 3. *If Φ is strictly monotonic and continuous on I, then the trapezoid point τ can be calculated by*

$$\tau = \Phi^{-1}\left(\frac{\Phi(m)+\Phi(M)}{2}\right).$$

We also have:

Theorem 5. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f : \Omega \to I$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. Then for all $v \in \mathring{I}$,*

$$\begin{aligned} (2.20) \qquad 0 &\le \frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right) \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \\ &\le \frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right). \end{aligned}$$

In particular, we have

$$\begin{aligned} (2.21) \qquad 0 &\le \frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_\Omega \left(f-\int_\Omega f d\mu\right)^2 \Phi''\left((1-s)\int_\Omega f d\mu + sf\right) d\mu\right) \\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_\Omega \left(f-\int_\Omega f d\mu\right)^2 \Phi''\left((1-s)\int_\Omega f d\mu + sf\right) d\mu\right). \end{aligned}$$

Proof. From the equality (2.8) and Fubini's theorem, we have

$$\begin{aligned} (2.22) \qquad &\int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \\ &= \int_0^1 \left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right)(1-s)\,ds, \end{aligned}$$

for all $v \in \mathring{I}$.

Observe that

$$\begin{aligned} &\operatorname*{essinf}_{s\in[0,1]}\left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right)\int_0^1 (1-s)\,ds \\ &\le \int_0^1 \left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right)(1-s)\,ds \\ &\le \operatorname*{essup}_{s\in[0,1]}\left(\int_\Omega (f-v)^2 \Phi''((1-s)v+sf)\,d\mu\right)\int_0^1 (1-s)\,ds, \end{aligned}$$

namely

$$\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}(f-v)^{2}\,\Phi''\left((1-s)\,v+sf\right)d\mu\right)$$
$$\leq\int_{0}^{1}\left(\int_{\Omega}(f-v)^{2}\,\Phi''\left((1-s)\,v+sf\right)d\mu\right)(1-s)\,ds$$
$$\leq\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}(f-v)^{2}\,\Phi''\left((1-s)\,v+sf\right)d\mu\right),$$

which by (2.22) gives the desired result (2.20). □

We also have:

Theorem 6. *Let $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and if there exists an interval $[m,M]\subset I$ such that $m\leq f\leq M$ μ-a.e. on Ω, then*

$$\begin{aligned}(2.23)\quad &\frac{1}{4\,(M-m)}\\ &\times\operatorname*{essinf}_{s\in[0,1]}\left[\int_{m}^{M}\left(\int_{\Omega}(f-v)^{2}\left(\Phi''\left((1-s)\,v+sf\right)\right)d\mu\right)dv\right]\\ &\leq\frac{1}{2}\left\{\int_{\Omega}\Phi\circ f d\mu\right.\\ &\left.+\frac{1}{M-m}\left[\Phi(m)\left(\int_{\Omega}fd\mu-m\right)+\Phi(M)\left(M-\int_{\Omega}fd\mu\right)\right]\right\}\\ &-\frac{1}{M-m}\int_{m}^{M}\Phi(v)\,dv\\ &\leq\frac{1}{4\,(M-m)}\operatorname*{essup}_{s\in[0,1]}\left[\int_{m}^{M}\left(\int_{\Omega}(f-v)^{2}\,\Phi''\left((1-s)\,v+sf\right)d\mu\right)dv\right].\end{aligned}$$

Proof. If we take the integral mean $\frac{1}{M-m}\int_{m}^{M}$ over the variable $v\in[m,M]$ in the equality (2.8), then we get

$$\begin{aligned}(2.24)\quad &\int_{\Omega}\Phi\circ f d\mu-\frac{1}{M-m}\int_{m}^{M}\Phi(v)\,dv-\frac{1}{M-m}\int_{m}^{M}\Phi'(v)\left(\int_{\Omega}fd\mu-v\right)dv\\ &=\frac{1}{M-m}\int_{m}^{M}\left(\int_{\Omega}(f-v)^{2}\left(\int_{0}^{1}\Phi''\left((1-s)\,v+sf\right)(1-s)\,ds\right)d\mu\right)dv.\end{aligned}$$

Integrating by parts in the integral $\int_{m}^{M}$ we have

$$\begin{aligned}&\int_{m}^{M}\Phi'(v)\left(\int_{\Omega}fd\mu-v\right)dv\\ &=\Phi(v)\left(\int_{\Omega}fd\mu-v\right)\Big|_{m}^{M}+\int_{m}^{M}\Phi(v)\,dv\\ &=\Phi(M)\left(\int_{\Omega}fd\mu-M\right)-\Phi(m)\left(\int_{\Omega}fd\mu-m\right)+\int_{m}^{M}\Phi(v)\,dv\\ &=\int_{m}^{M}\Phi(v)\,dv-\Phi(m)\left(\int_{\Omega}fd\mu-m\right)-\Phi(M)\left(M-\int_{\Omega}fd\mu\right),\end{aligned}$$

which gives that

$$\int_\Omega \Phi\circ f d\mu - \frac{1}{M-m}\int_m^M \Phi(v)\,dv - \frac{1}{M-m}\int_m^M \Phi'(v)\left(\int_\Omega f d\mu - v\right)dv$$
$$= \int_\Omega \Phi\circ f d\mu - \frac{1}{M-m}\int_m^M \Phi(v)\,dv$$
$$- \frac{1}{M-m}\left[\int_m^M \Phi(v)\,dv - \Phi(m)\left(\int_\Omega f d\mu - m\right) - \Phi(M)\left(M - \int_\Omega f d\mu\right)\right]$$
$$= \int_\Omega \Phi\circ f d\mu + \frac{1}{M-m}\left[\Phi(m)\left(\int_\Omega f d\mu - m\right) + \Phi(M)\left(M - \int_\Omega f d\mu\right)\right]$$
$$- \frac{2}{M-m}\int_m^M \Phi(v)\,dv.$$

Therefore, by (2.24) we obtain the following identity of interest

$$\begin{aligned}(2.25)\quad & \frac{1}{2}\left\{\int_\Omega \Phi\circ f d\mu \right.\\ &\left. + \frac{1}{M-m}\left[\Phi(m)\left(\int_\Omega f d\mu - m\right) + \Phi(M)\left(M - \int_\Omega f d\mu\right)\right]\right\}\\ &\qquad - \frac{1}{M-m}\int_m^M \Phi(v)\,dv\\ &= \frac{1}{2(M-m)}\int_m^M \left(\int_\Omega (f-v)^2\left(\int_0^1 \Phi''((1-s)v+sf)(1-s)\,ds\right)d\mu\right)dv\\ &= \frac{1}{2(M-m)}\int_0^1 \left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right](1-s)\,ds,\end{aligned}$$

where for the last equality we used Fubini's theorem.

Now, observe that

$$\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right]\int_0^1 (1-s)\,ds$$
$$\le \int_0^1 \left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right](1-s)\,ds$$
$$\le \operatorname*{essup}_{s\in[0,1]}\left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right]\int_0^1 (1-s)\,ds,$$

namely

$$\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right]$$
$$\le \int_0^1 \left[\int_m^M \left(\int_\Omega (f-v)^2\Phi''((1-s)v+sf)\,d\mu\right)dv\right](1-s)\,ds$$
$$\le \frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left[\int_m^M \left(\int_\Omega (f-v)^2(\Phi''((1-s)v+sf))\,d\mu\right)dv\right],$$

which by the equality (2.25) gives the desired result (2.25). □

3. Related Results

When the second derivative is bounded, we have the following lower and upper bounds of interest:

Theorem 7. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f : \Omega \to I$ so that $f,\ f^2,\ \Phi \circ f,\ \Phi'' \circ f \in L(\Omega, \mu)$. If there exists the constants $0 < \gamma < \Gamma < \infty$ such that $\gamma \le \Phi''(x) \le \Gamma$ for a.e. $x \in I$, then for all $v \in \mathring{I}$,*

$$
\begin{aligned}
(3.1)\qquad 0 &\le \frac{1}{2}\gamma\left(\int_\Omega f d\mu - v\right)^2 \le \frac{1}{2}\gamma\int_\Omega (f-v)^2\, d\mu \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \le \frac{1}{2}\Gamma\int_\Omega (f-v)^2\, d\mu
\end{aligned}
$$

and, in particular

$$
\begin{aligned}
(3.2)\qquad 0 &\le \frac{1}{2}\gamma\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \frac{1}{2}\Gamma\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
(3.3)\qquad 0 &\le \frac{1}{2}\gamma\left(\int_\Omega f d\mu - \sigma\right)^2 \le \frac{1}{2}\gamma\int_\Omega (f-\sigma)^2\, d\mu \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi(\sigma) - \Phi'(\sigma)\left(\int_\Omega f d\mu - \sigma\right) \le \frac{1}{2}\Gamma\int_\Omega (f-\sigma)^2\, d\mu,
\end{aligned}
$$

where σ is the Slater's point defined by (2.11).

If there exists an interval $[m, M] \subset I$ such that $m \le f \le M$ μ-a.e. on Ω, then

$$
\begin{aligned}
(3.4)\qquad 0 &\le \frac{1}{2}\gamma\left(\int_\Omega f d\mu - \frac{m+M}{2}\right)^2 \le \frac{1}{2}\gamma\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\frac{m+M}{2}\right) - \Phi'\left(\frac{m+M}{2}\right)\left(\int_\Omega f d\mu - \frac{m+M}{2}\right) \\
&\le \frac{1}{2}\Gamma\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu \le \frac{1}{8}\Gamma(M-m)^2.
\end{aligned}
$$

If $\tau \in I$ is a trapezoid point as defined by (2.17), then

$$
\begin{aligned}
(3.5)\qquad 0 &\le \frac{1}{2}\gamma\left(\int_\Omega f d\mu - \tau\right)^2 \le \frac{1}{2}\gamma\int_\Omega (f-\tau)^2\, d\mu \\
&\le \int_\Omega \Phi\circ f d\mu - \frac{\Phi(m)+\Phi(M)}{2} - \Phi'(\tau)\left(\int_\Omega f d\mu - \tau\right) \\
&\le \frac{1}{2}\Gamma\int_\Omega (f-\tau)^2\, d\mu.
\end{aligned}
$$

We have the inequalities

$$\frac{1}{12}\gamma\int_\Omega\left[(M-f)^2-(M-f)(f-m)+(f-m)^2\right]d\mu \tag{3.6}$$
$$\leq\frac{1}{2}\left\{\int_\Omega\Phi\circ f d\mu\right.$$
$$\left.+\frac{1}{M-m}\left[\Phi(m)\left(\int_\Omega fd\mu-m\right)+\Phi(M)\left(M-\int_\Omega fd\mu\right)\right]\right\}$$
$$-\frac{1}{M-m}\int_m^M\Phi(v)\,dv$$
$$\leq\frac{1}{12}\Gamma\int_\Omega\left[(M-f)^2-(M-f)(f-m)+(f-m)^2\right]d\mu.$$

Proof. Observe that

$$\frac{1}{2}\gamma=\gamma\int_0^1(1-s)\,ds\leq\int_0^1\Phi''((1-s)v+sf(t))(1-s)\,ds$$
$$\leq\Gamma\int_0^1(1-s)\,ds=\frac{1}{2}\Gamma$$

and by the inequality (2.5) we get (3.1).

The inequality (3.3) follows by (3.1) for $v=\sigma$. The inequality (3.4) follows by (3.1) for $v=\frac{m+M}{2}$. The inequality (3.5) follows by (3.1) for $v=\tau$.

Observe that

$$\int_m^M\left(\int_\Omega(f-v)^2(\Phi''((1-s)v+sf))\,d\mu\right)dv$$
$$\geq\gamma\int_m^M\left(\int_\Omega(f-v)^2\,d\mu\right)dv=\gamma\int_\Omega\left(\int_m^M(f-v)^2\,dv\right)d\mu$$
$$=\gamma\int_\Omega\left(\frac{(M-f)^3+(f-m)^3}{3}\right)d\mu$$
$$=\frac{1}{3}\gamma(M-m)\int_\Omega\left[(M-f)^2-(M-f)(f-m)+(f-m)^2\right]d\mu$$

and, similarly

$$\int_m^M\left(\int_\Omega(f-v)^2(\Phi''((1-s)v+sf))\,d\mu\right)dv$$
$$\leq\frac{1}{3}\gamma(M-m)\int_\Omega\left[(M-f)^2-(M-f)(f-m)+(f-m)^2\right]d\mu.$$

By employing the inequality (2.23) we deduce the desired result (3.6). □

Remark 4. *We observe that*

$$(M-f)^2-(M-f)(f-m)+(f-m)^2$$
$$=3\left[\left(f-\frac{m+M}{2}\right)^2+\frac{1}{12}(M-m)^2\right].$$

If $m \leq f \leq M$ μ-a.e. on Ω, then

$$\frac{1}{4}(M-m)^2 \leq (M-f)^2 - (M-f)(f-m) + (f-m)^2$$
$$\leq 3\left[\frac{1}{4}(M-m)^2 + \frac{1}{12}(M-m)^2\right] = (M-m)^2$$

and by (3.6) we get the following double inequality of interest

$$\frac{1}{48}\gamma (M-m)^2 \leq \frac{1}{2}\left\{\int_\Omega \Phi \circ f d\mu + \frac{1}{M-m}\left[\Phi(m)\left(\int_\Omega f d\mu - m\right) + \Phi(M)\left(M - \int_\Omega f d\mu\right)\right]\right\} - \frac{1}{M-m}\int_m^M \Phi(v)\,dv \leq \frac{1}{12}\Gamma (M-m)^2. \tag{3.7}$$

If some monotonicity properties for the second derivative are assumed, then we have the following results as well.

Theorem 8. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f : \Omega \to I$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. If the second derivative Φ'' is monotonic nondecreasing on an interval $[m, M] \subset \mathring{I}$, then we have*

$$\begin{aligned} 0 &\leq \frac{1}{v-m}\left\{\Phi'(v) - \frac{\Phi(v)-\Phi(m)}{v-m}\right\}\left(\int_\Omega f d\mu - v\right)^2 \\ &\leq \frac{1}{v-m}\left\{\Phi'(v) - \frac{\Phi(v)-\Phi(m)}{v-m}\right\}\int_\Omega (f-v)^2 d\mu \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \\ &\leq \frac{1}{M-v}\left\{\frac{\Phi(M)-\Phi(v)}{M-v} - \Phi'(v)\right\}\int_\Omega (f-v)^2 d\mu \end{aligned} \tag{3.8}$$

for all $v \in (m, M)$.

In particular, we have

$$\begin{aligned} 0 &\leq \frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right) - \frac{\Phi\left(\int_\Omega f d\mu\right) - \Phi(m)}{\int_\Omega f d\mu - m}\right\} \\ &\quad \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \\ &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \frac{1}{M - \int_\Omega f d\mu}\left\{\frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \Phi'\left(\int_\Omega f d\mu\right)\right\} \\ &\quad \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]. \end{aligned} \tag{3.9}$$

If the second derivative Φ'' is monotonic nonincreasing on the interval $[m, M] \subset \mathring{I}$, then we have

$$0 \leq \frac{1}{M - v}\left\{\frac{\Phi(M) - \Phi(v)}{M - v} - \Phi'(v)\right\}\left(\int_\Omega f d\mu - v\right)^2 \tag{3.10}$$
$$\leq \frac{1}{M - v}\left\{\frac{\Phi(M) - \Phi(v)}{M - v} - \Phi'(v)\right\}\int_\Omega (f - v)^2 d\mu$$
$$\leq \int_\Omega \Phi \circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right)$$
$$\leq \frac{1}{v - m}\left\{\Phi'(v) - \frac{\Phi(v) - \Phi(m)}{v - m}\right\}\int_\Omega (f - v)^2 d\mu$$

for all $v \in (m, M)$.

In particular, we have

$$0 \leq \frac{1}{M - \int_\Omega f d\mu}\left\{\frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} - \Phi'\left(\int_\Omega f d\mu\right)\right\} \tag{3.11}$$
$$\times\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]$$
$$\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)$$
$$\leq \frac{1}{\int_\Omega f d\mu - m}\left\{\Phi'\left(\int_\Omega f d\mu\right) - \frac{\Phi\left(\int_\Omega f d\mu\right) - \Phi(m)}{\int_\Omega f d\mu - m}\right\}$$
$$\times\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].$$

Proof. First, observe that for u, $v \in [m, M]$ with $u \neq v$ we have

$$\int_0^1 \Phi''((1 - s)v + su)(1 - s) ds \tag{3.12}$$
$$= \frac{1}{u - v}\int_0^1 (1 - s) d\left(\Phi'((1 - s)v + su)\right)$$
$$= \frac{1}{u - v}\left[(1 - s)\Phi'((1 - s)v + su)\Big|_0^1 + \int_0^1 \Phi'((1 - s)v + su) ds\right]$$
$$= \frac{1}{u - v}\left\{-\Phi'(v) + \int_0^1 \Phi'((1 - s)v + su) ds\right\}$$
$$= \frac{1}{v - u}\left\{\Phi'(v) - \int_0^1 \Phi'((1 - s)v + su) ds\right\}$$
$$= \frac{1}{v - u}\left\{\Phi'(v) - \frac{\Phi(v) - \Phi(u)}{v - u}\right\}.$$

If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M]\subset \mathring{I}$, then for μ-a.e. $t\in\Omega$

$$\int_0^1 \Phi''\left((1-s)v+sm\right)(1-s)\,ds \le \int_0^1 \Phi''\left((1-s)v+sf(t)\right)(1-s)\,ds \le \int_0^1 \Phi''\left((1-s)v+sM\right)(1-s)\,ds.$$

By (3.12) we get for $v\in(m,M)$ that

$$\int_0^1 \Phi''\left((1-s)v+sm\right)(1-s)\,ds = \frac{1}{v-m}\left\{\Phi'(v)-\frac{\Phi(v)-\Phi(m)}{v-m}\right\}$$

and

$$\int_0^1 \Phi''\left((1-s)v+sM\right)(1-s)\,ds = \frac{1}{v-M}\left\{\Phi'(v)-\frac{\Phi(v)-\Phi(M)}{v-M}\right\} = \frac{1}{M-v}\left\{\frac{\Phi(M)-\Phi(v)}{M-v}-\Phi'(v)\right\}.$$

Then

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\tau+sf(t)\right)(1-s)\,ds\right)\\ &\ge \int_0^1 \Phi''\left((1-s)v+sm\right)(1-s)\,ds\\ &= \frac{1}{v-m}\left\{\Phi'(v)-\frac{\Phi(v)-\Phi(m)}{v-m}\right\}\end{aligned}$$

and

$$\begin{aligned}&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\sigma+sf(t)\right)(1-s)\,ds\right)\\ &\le \int_0^1 \Phi''\left((1-s)v+sM\right)(1-s)\,ds\\ &= \frac{1}{M-v}\left\{\frac{\Phi(M)-\Phi(v)}{M-v}-\Phi'(v)\right\}\end{aligned}$$

and by (2.5) we get (3.8).

The inequality (3.8) can be proved in a similar way, we omit the details. □

Theorem 9. *Let $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\ f^2,\ \Phi\circ f,\ \Phi''\circ f\in L(\Omega,\mu)$. If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M]\subset\mathring{I}$, then we have*

$$\begin{aligned}(3.13)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-v\right)^2\int_\Omega \frac{1}{f-m}\left\{\frac{\Phi\circ f-\Phi(m)}{f-m}-\Phi'(m)\right\}d\mu\\ &\le \int_\Omega \Phi\circ f d\mu-\Phi(v)-\Phi'(v)\left(\int_\Omega f d\mu - v\right)\\ &\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-v\right)^2\int_\Omega \frac{1}{M-f}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi\circ f}{M-f}\right\}.\end{aligned}$$

In particular,

$$
\begin{aligned}
(3.14)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{f-m}\left\{\frac{\Phi\circ f-\Phi(m)}{f-m}-\Phi'(m)\right\}d\mu \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{M-f}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi\circ f}{M-f}\right\}.
\end{aligned}
$$

If the second derivative Φ'' is monotonic nonincreasing on an interval $[m,M]\subset \mathring{I}$, then we have

$$
\begin{aligned}
(3.15)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}(f(t)-v)^2 \int_\Omega \frac{1}{M-f}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi\circ f}{M-f}\right\} \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi(v) - \Phi'(v)\left(\int_\Omega f d\mu - v\right) \\
&\le \operatorname*{essup}_{t\in\Omega}(f(t)-v)^2 \int_\Omega \frac{1}{f-m}\left\{\frac{\Phi\circ f-\Phi(m)}{f-m}-\Phi'(m)\right\}d\mu.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
(3.16)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{M-f}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi\circ f}{M-f}\right\} \\
&\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{f-m}\left\{\frac{\Phi\circ f-\Phi(m)}{f-m}-\Phi'(m)\right\}d\mu.
\end{aligned}
$$

Proof. If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M]\subset \mathring{I}$, then for μ-a.e. $t\in\Omega$ we have

$$
\begin{aligned}
(3.17)\qquad &\int_0^1 \Phi''((1-s)m+sf(t))(1-s)ds \\
&\le \int_0^1 \Phi''((1-s)v+sf(t))(1-s)ds \\
&\le \int_0^1 \Phi''((1-s)M+sf(t))(1-s)ds.
\end{aligned}
$$

By (3.12) we get

$$
\begin{aligned}
(3.18)\qquad &\int_0^1 \Phi''((1-s)m+sf(t))(1-s)ds \\
&= \frac{1}{m-f(t)}\left\{\Phi'(m)-\frac{\Phi(m)-\Phi(f(t))}{m-f(t)}\right\} \\
&= \frac{1}{f(t)-m}\left\{\frac{\Phi(f(t))-\Phi(m)}{f(t)-m}-\Phi'(m)\right\}
\end{aligned}
$$

for μ-a.e. $t\in\Omega$ and

$$\int_0^1 \Phi''((1-s)M+sf(t))(1-s)\,ds = \frac{1}{M-f(t)}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi(f(t))}{M-f(t)}\right\} \tag{3.19}$$

for μ-a.e. $t\in\Omega$.

Therefore

$$\int_\Omega\left(\int_0^1 \Phi''((1-s)m+sf)(1-s)\,ds\right)d\mu = \int_\Omega \frac{1}{f-m}\left\{\frac{\Phi\circ f-\Phi(m)}{f-m}-\Phi'(m)\right\}d\mu \tag{3.20}$$

and

$$\int_\Omega\left(\int_0^1 \Phi''((1-s)M+sf)(1-s)\,ds\right)d\mu = \int_\Omega \frac{1}{M-f}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi\circ f}{M-f}\right\}. \tag{3.21}$$

Using the inequality (2.6) and (3.17) we get

$$\begin{aligned}0 &\le \operatorname*{essinf}_{t\in\Omega}(f(t)-v)^2\int_\Omega\left(\int_0^1 \Phi''((1-s)m+sf)(1-s)\,ds\right)d\mu\\ &\le \int_\Omega \Phi\circ f d\mu-\Phi(v)-\Phi'(v)\left(\int_\Omega f d\mu - v\right)\\ &\le \operatorname*{essup}_{t\in\Omega}(f(t)-v)^2\int_\Omega\left(\int_0^1 \Phi''((1-s)M+sf)(1-s)\,ds\right)d\mu\end{aligned}$$

and by (3.20) and (3.21) we obtain (3.13). □

We also have:

Theorem 10. *Let $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\ f^2,\ \Phi\circ f,\ \Phi''\circ f\in L(\Omega,\mu)$. If the second derivative Φ'' is convex, then*

$$\begin{aligned}0 &\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\Phi''\left(\int_\Omega f d\mu\right)\\ &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)\\ &\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\int_\Omega \Phi''\circ f d\mu.\end{aligned} \tag{3.22}$$

If the second derivative Φ'' is concave, then

$$
\begin{aligned}
(3.23)\qquad 0 &\le \frac{1}{2}\operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\int_\Omega \Phi''\circ f d\mu \\
&\le \int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right) \\
&\le \frac{1}{2}\operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2\Phi''\left(\int_\Omega f d\mu\right).
\end{aligned}
$$

Proof. We have by Fubini's theorem and inequality (2.10) that

$$
\begin{aligned}
(3.24)\qquad 0 &\le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \\
&\times\int_0^1\left(\int_\Omega \Phi''\left((1-s)\int_\Omega f d\mu+sf\right)d\mu\right)(1-s)\,ds \\
&\le \int_\Omega \Phi\circ f d\mu-\Phi\left(\int_\Omega f d\mu\right) \\
&\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega f d\mu\right)^2 \\
&\times\int_0^1\left(\int_\Omega \Phi''\left((1-s)\int_\Omega f d\mu+sf\right)d\mu\right)(1-s)\,ds.
\end{aligned}
$$

By Jensen's inequality for the convex function Φ'' we have

$$
\begin{aligned}
&\int_\Omega \Phi''\left((1-s)\int_\Omega f d\mu+sf\right)d\mu \\
&\ge \Phi''\left(\int_\Omega\left[(1-s)\int_\Omega f d\mu+sf\right]d\mu\right) \\
&= \Phi''\left((1-s)\int_\Omega f d\mu+s\int_\Omega f d\mu\right)=\Phi''\left(\int_\Omega f d\mu\right).
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int_\Omega \Phi''\left((1-s)\int_\Omega f d\mu+sf\right)d\mu \\
&\le \int_\Omega\left[(1-s)\Phi''\left(\int_\Omega f d\mu\right)+s\Phi''\circ f\right]d\mu \\
&= (1-s)\Phi''\left(\int_\Omega f d\mu\right)+s\int_\Omega \Phi''\circ f d\mu \\
&\le (1-s)\Phi''\int_\Omega \Phi''\circ f d\mu+s\int_\Omega \Phi''\circ f d\mu=\int_\Omega \Phi''\circ f d\mu,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_0^1\left(\int_\Omega \Phi''\left((1-s)\int_\Omega f d\mu+sf\right)d\mu\right)(1-s)\,ds \\
&\ge \Phi''\left(\int_\Omega f d\mu\right)\int_0^1(1-s)\,ds=\frac{1}{2}\Phi''\left(\int_\Omega f d\mu\right)
\end{aligned}
$$

and

$$\int_0^1 \left(\int_\Omega \Phi'' \left((1-s) \int_\Omega f d\mu + sf \right) d\mu \right) (1-s)\, ds$$
$$\leq \int_\Omega \Phi'' \circ f d\mu \int_0^1 (1-s)\, ds = \frac{1}{2} \int_\Omega \Phi'' \circ f d\mu.$$

On utilising (3.24) we get (3.22). □

4. Some Examples for the Discrete Case

Let $\Phi : [m, M] \to \mathbb{R}$ be a twice differentiable convex function on (m, M) and $x_k \in [m, M]$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$. If there exists the constants $0 < \gamma < \Gamma < \infty$ such that $\gamma \leq \Phi''(x) \leq \Gamma$ for a.e. $x \in [m, M]$, then for all $v \in I$, we have by (3.1) and (3.2) for the discrete measure

$$\begin{aligned} 0 &\leq \frac{1}{2}\gamma \left(\sum_{k=1}^n p_k x_k - v \right)^2 \leq \frac{1}{2}\gamma \sum_{k=1}^n p_k (x_k - v)^2 \\ &\leq \sum_{k=1}^n p_k \Phi(x_k) - \Phi(v) - \Phi'(v) \left(\sum_{k=1}^n p_k x_k - v \right) \\ &\leq \frac{1}{2}\Gamma \sum_{k=1}^n p_k (x_k - v)^2 \end{aligned} \tag{4.1}$$

and, in particular (see also [6])

$$\begin{aligned} 0 &\leq \frac{1}{2}\gamma \left[\sum_{k=1}^n p_k x_k^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right] \leq \sum_{k=1}^n p_k \Phi(x_k) - \Phi\left(\sum_{i=1}^n p_i x_i \right) \\ &\leq \frac{1}{2}\Gamma \left[\sum_{k=1}^n p_k x_k^2 - \left(\sum_{i=1}^n p_i x_i \right)^2 \right]. \end{aligned} \tag{4.2}$$

By the inequality (3.4) we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\gamma \left(\sum_{k=1}^n p_k x_k - \frac{m+M}{2} \right)^2 \leq \frac{1}{2}\gamma \sum_{k=1}^n p_k \left(x_k - \frac{m+M}{2} \right)^2 \\ &\leq \sum_{k=1}^n p_k \Phi(x_k) - \Phi\left(\frac{m+M}{2} \right) - \Phi'\left(\frac{m+M}{2} \right) \left(\sum_{k=1}^n p_k x_k - \frac{m+M}{2} \right) \\ &\leq \frac{1}{2}\Gamma \sum_{k=1}^n p_k \left(x_k - \frac{m+M}{2} \right)^2 \leq \frac{1}{8}\Gamma (M-m)^2. \end{aligned} \tag{4.3}$$

By the inequality (3.7) we have

$$
\begin{aligned}
(4.4)\quad & \frac{1}{48}\gamma\left(M-m\right)^{2}\leq\frac{1}{2}\left\{\sum_{k=1}^{n}p_{k}\Phi\left(x_{k}\right)\right. \\
&\left.+\frac{1}{M-m}\left[\Phi\left(m\right)\left(\sum_{k=1}^{n}p_{k}x_{k}-m\right)+\Phi\left(M\right)\left(M-\sum_{k=1}^{n}p_{k}x_{k}\right)\right]\right\} \\
&-\frac{1}{M-m}\int_{m}^{M}\Phi\left(v\right)dv\leq\frac{1}{12}\Gamma\left(M-m\right)^{2}.
\end{aligned}
$$

If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M]\subset\mathring{I}$, then we have

$$
\begin{aligned}
(4.5)\quad 0 &\leq\frac{1}{v-m}\left\{\Phi'\left(v\right)-\frac{\Phi\left(v\right)-\Phi\left(m\right)}{v-m}\right\}\left(\sum_{k=1}^{n}p_{k}x_{k}-v\right)^{2} \\
&\leq\frac{1}{v-m}\left\{\Phi'\left(v\right)-\frac{\Phi\left(v\right)-\Phi\left(m\right)}{v-m}\right\}\sum_{k=1}^{n}p_{k}\left(x_{k}-v\right)^{2} \\
&\leq\sum_{k=1}^{n}p_{k}\Phi\left(x_{k}\right)-\Phi\left(v\right)-\Phi'\left(v\right)\left(\sum_{k=1}^{n}p_{k}x_{k}-v\right) \\
&\leq\frac{1}{M-v}\left\{\frac{\Phi\left(M\right)-\Phi\left(v\right)}{M-v}-\Phi'\left(v\right)\right\}\sum_{k=1}^{n}p_{k}\left(x_{k}-v\right)^{2}
\end{aligned}
$$

for all $v\in\left(m,M\right)$.

In particular, we have

$$
\begin{aligned}
(4.6)\quad 0 &\leq\frac{1}{\sum_{k=1}^{n}p_{k}x_{k}-m}\left\{\Phi'\left(\sum_{k=1}^{n}p_{k}x_{k}\right)-\frac{\Phi\left(\sum_{k=1}^{n}p_{k}x_{k}\right)-\Phi\left(m\right)}{\sum_{k=1}^{n}p_{k}x_{k}-m}\right\} \\
&\times\left[\sum_{k=1}^{n}p_{k}x_{k}^{2}-\left(\sum_{i=1}^{n}p_{i}x_{i}\right)^{2}\right] \\
&\leq\sum_{k=1}^{n}p_{k}\Phi\left(x_{k}\right)-\Phi\left(\sum_{i=1}^{n}p_{i}x_{i}\right) \\
&\leq\frac{1}{M-\sum_{k=1}^{n}p_{k}x_{k}}\left\{\frac{\Phi\left(M\right)-\Phi\left(\sum_{k=1}^{n}p_{k}x_{k}\right)}{M-\sum_{k=1}^{n}p_{k}x_{k}}-\Phi'\left(\sum_{k=1}^{n}p_{k}x_{k}\right)\right\} \\
&\times\left[\sum_{k=1}^{n}p_{k}x_{k}^{2}-\left(\sum_{i=1}^{n}p_{i}x_{i}\right)^{2}\right].
\end{aligned}
$$

If the second derivative Φ'' is monotonic nonincreasing on the interval $[m,M]\subset \mathring{I}$, then we have

$$\begin{aligned}
0 &\leq \frac{1}{M-v}\left\{\frac{\Phi(M)-\Phi(v)}{M-v}-\Phi'(v)\right\}\left(\sum_{k=1}^{n}p_kx_k-v\right)^2 \\
&\leq \frac{1}{M-v}\left\{\frac{\Phi(M)-\Phi(v)}{M-v}-\Phi'(v)\right\}\sum_{k=1}^{n}p_k(x_k-v)^2 \\
&\leq \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi(v)-\Phi'(v)\left(\sum_{k=1}^{n}p_kx_k-v\right) \\
&\leq \frac{1}{v-m}\left\{\Phi'(v)-\frac{\Phi(v)-\Phi(m)}{v-m}\right\}\sum_{k=1}^{n}p_k(x_k-v)^2
\end{aligned}\tag{4.7}$$

for all $v\in(m,M)$.

In particular, we have

$$\begin{aligned}
0 &\leq \frac{1}{M-\sum_{k=1}^{n}p_kx_k}\left\{\frac{\Phi(M)-\Phi\left(\sum_{k=1}^{n}p_kx_k\right)}{M-\sum_{k=1}^{n}p_kx_k}-\Phi'\left(\sum_{k=1}^{n}p_kx_k\right)\right\} \\
&\quad\times\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right] \\
&\leq \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right) \\
&\leq \frac{1}{\sum_{k=1}^{n}p_kx_k-m}\left\{\Phi'\left(\sum_{k=1}^{n}p_kx_k\right)-\frac{\Phi\left(\sum_{k=1}^{n}p_kx_k\right)-\Phi(m)}{\sum_{k=1}^{n}p_kx_k-m}\right\} \\
&\quad\times\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right].
\end{aligned}\tag{4.8}$$

If the second derivative Φ'' is convex, then

$$\begin{aligned}
0 &\leq \frac{1}{2}\min_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\Phi''\left(\sum_{i=1}^{n}p_ix_i\right) \\
&\leq \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right) \\
&\leq \frac{1}{2}\max_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\sum_{k=1}^{n}p_k\Phi''(x_k).
\end{aligned}\tag{4.9}$$

If the second derivative Φ'' is concave, then

$$
\begin{aligned}
0 &\le \frac{1}{2}\min_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\sum_{k=1}^{n}p_k\Phi''(x_k) \\
&\le \sum_{k=1}^{n}p_k\Phi(x_k)-\Phi\left(\sum_{i=1}^{n}p_ix_i\right) \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\Phi''\left(\sum_{i=1}^{n}p_ix_i\right).
\end{aligned}
\tag{4.10}
$$

If we consider the exponential function $\Phi(x)=\exp x$ on the interval $[m,M]$ of real numbers, then for all $x_k\in[m,M]$, $p_k\ge 0$ for $k\in\{1,\dots,n\}$ with $\sum_{k=1}^{n}p_k=1$, we have

$$
\begin{aligned}
0 &\le \frac{1}{2}\left(\sum_{k=1}^{n}p_kx_k-v\right)^2\exp m\le\frac{1}{2}\exp m\sum_{k=1}^{n}p_k(x_k-v)^2 \\
&\le \sum_{k=1}^{n}p_k\exp(x_k)-\left(1+\sum_{k=1}^{n}p_kx_k-v\right)\exp v \\
&\le \frac{1}{2}\exp M\sum_{k=1}^{n}p_k(x_k-v)^2
\end{aligned}
\tag{4.11}
$$

and, in particular

$$
\begin{aligned}
0 &\le \frac{1}{2}\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\exp m\le\sum_{k=1}^{n}p_k\exp(x_k)-\exp\left(\sum_{i=1}^{n}p_ix_i\right) \\
&\le \frac{1}{2}\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\exp M.
\end{aligned}
\tag{4.12}
$$

We have

$$
\begin{aligned}
0 &\le \frac{1}{2}\left(\sum_{k=1}^{n}p_kx_k-\frac{m+M}{2}\right)^2\exp m\le\frac{1}{2}\exp m\sum_{k=1}^{n}p_k\left(x_k-\frac{m+M}{2}\right)^2 \\
&\le \sum_{k=1}^{n}p_k\exp(x_k)-\left(1+\sum_{k=1}^{n}p_kx_k-\frac{m+M}{2}\right)\exp\left(\frac{m+M}{2}\right) \\
&\le \frac{1}{2}\exp M\sum_{k=1}^{n}p_k\left(x_k-\frac{m+M}{2}\right)^2\le\frac{1}{8}(M-m)^2\exp M.
\end{aligned}
\tag{4.13}
$$

Also,

$$\frac{1}{48}(M-m)^2\exp m \leq \frac{1}{2}\left\{\sum_{k=1}^{n} p_k \exp(x_k) + \frac{1}{M-m}\left[\exp(m)\left(\sum_{k=1}^{n} p_k x_k - m\right) + \exp(M)\left(M - \sum_{k=1}^{n} p_k x_k\right)\right]\right\} - \frac{\exp M - \exp m}{M-m} \leq \frac{1}{12}(M-m)^2 \exp M. \tag{4.14}$$

Moreover, since the second derivative of the exponential function is increasing and convex, we have

$$\begin{aligned} 0 &\leq \frac{1}{v-m}\left\{\exp v - \frac{\exp v - \exp m}{v-m}\right\}\left(\sum_{k=1}^{n} p_k x_k - v\right)^2 \\ &\leq \frac{1}{v-m}\left\{\exp v - \frac{\exp v - \exp m}{v-m}\right\}\sum_{k=1}^{n} p_k (x_k - v)^2 \\ &\leq \sum_{k=1}^{n} p_k \exp(x_k) - \left(1 + \sum_{k=1}^{n} p_k x_k - v\right)\exp v \\ &\leq \frac{1}{M-v}\left\{\frac{\exp M - \exp v}{M-v} - \exp v\right\}\sum_{k=1}^{n} p_k (x_k - v)^2 \end{aligned} \tag{4.15}$$

for all $v \in [m, M]$, in particular

$$\begin{aligned} 0 &\leq \frac{1}{\sum_{k=1}^{n} p_k x_k - m}\left\{\exp \sum_{k=1}^{n} p_k x_k - \frac{\exp \sum_{k=1}^{n} p_k x_k - \exp m}{v-m}\right\} \\ &\quad \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \\ &\leq \sum_{k=1}^{n} p_k \exp(x_k) - \exp \sum_{k=1}^{n} p_k x_k \\ &\leq \frac{1}{M - \sum_{k=1}^{n} p_k x_k}\left\{\frac{\exp M - \exp \sum_{k=1}^{n} p_k x_k}{M - \sum_{k=1}^{n} p_k x_k} - \exp v\right\} \\ &\quad \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \end{aligned} \tag{4.16}$$

and

$$
\begin{aligned}
(4.17) \qquad 0 &\le \frac{1}{2}\min_{k\in\{1,...,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2 \\
&\le \frac{\sum_{k=1}^{n}p_k\exp(x_k)}{\exp\left(\sum_{i=1}^{n}p_ix_i\right)}-1 \\
&\le \frac{1}{2}\max_{k\in\{1,...,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\frac{\sum_{k=1}^{n}p_k\exp(x_k)}{\exp\left(\sum_{i=1}^{n}p_ix_i\right)}.
\end{aligned}
$$

If we take in (4.17) $x_i=\ln y_i$ with $y_i>0$ and $i\in\{1,...,n\}$, then we get

$$
\begin{aligned}
(4.18) \qquad 0 &\le \frac{1}{2}\min_{i\in\{1,...,n\}}\left[\ln\left(\frac{y_i}{\prod_{k=1}^{n}y_k^{p_k}}\right)\right]^2 \\
&\le \frac{\sum_{k=1}^{n}p_ky_k}{\prod_{k=1}^{n}y_k^{p_k}}-1 \\
&\le \frac{1}{2}\max_{k\in\{1,...,n\}}\left[\ln\left(\frac{y_i}{\prod_{k=1}^{n}y_k^{p_k}}\right)\right]^2\frac{\sum_{k=1}^{n}p_ky_k}{\prod_{k=1}^{n}y_k^{p_k}}.
\end{aligned}
$$

From this inequality we can derive a refinement and a reverse of the celebrated *Young's inequality*

$$a^{1-t}b^t\le(1-t)\,a+tb,\ a,\ b\ge 0 \text{ and } t\in[0,1].$$

Indeed, if we take in (4.18) $n=2$, $y_1=a$, $y_2=b$, $p_1=1-t$ and $p_2=t$, then we get

$$
\begin{aligned}
(4.19) \qquad 0 &\le \frac{1}{2}\min\left\{\ln\left(\frac{a}{a^{1-t}b^t}\right),\ln\left(\frac{b}{a^{1-t}b^t}\right)\right\}^2 \\
&\le \frac{(1-t)\,a+tb}{a^{1-t}b^t}-1 \\
&\le \frac{1}{2}\max\left\{\ln\left(\frac{a}{a^{1-t}b^t}\right),\ln\left(\frac{b}{a^{1-t}b^t}\right)\right\}^2\frac{(1-t)\,a+tb}{a^{1-t}b^t},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(4.20) \qquad 0 &\le \frac{1}{2}\min\left\{t\ln\left(\frac{a}{b}\right),(1-t)\ln\left(\frac{b}{a}\right)\right\}^2 \\
&\le \frac{(1-t)\,a+tb}{a^{1-t}b^t}-1 \\
&\le \frac{1}{2}\max\left\{t\ln\left(\frac{a}{b}\right),(1-t)\ln\left(\frac{b}{a}\right)\right\}^2\frac{(1-t)\,a+tb}{a^{1-t}b^t},
\end{aligned}
$$

namely

$$
\begin{aligned}
(4.21) \qquad 0 &\le \frac{1}{2}\left(\min\{t,1-t\}\right)^2\left[\ln\left(\frac{b}{a}\right)\right]^2 \\
&\le \frac{(1-t)\,a+tb}{a^{1-t}b^t}-1 \\
&\le \frac{1}{2}\left(\max\{t,1-t\}\right)^2\left[\ln\left(\frac{b}{a}\right)\right]^2\frac{(1-t)\,a+tb}{a^{1-t}b^t},
\end{aligned}
$$

for $a,b>0$ and $t\in[0,1]$.

The second inequality is also equivalent to

$$1-\frac{a^{1-t}b^{t}}{(1-t)\,a+tb}\leq\frac{1}{2}\left(\max\{t,1-t\}\right)^{2}\left[\ln\left(\frac{b}{a}\right)\right]^{2}, \tag{4.22}$$

for $a,b>0$ and $t\in[0,1]$.

Consider also the convex function $\Phi:[m,M]\subset(0,\infty)\to\mathbb{R}$, $\Phi(x)=-\ln x$. We have $\frac{1}{M^{2}}\leq\Phi''(x)\leq\frac{1}{m^{2}}$ for $x\in[m,M]$. Then we obtain the following inequalities

$$\begin{aligned}0&\leq\frac{1}{2M^{2}}\left(\sum_{k=1}^{n}p_{k}x_{k}-v\right)^{2}\leq\frac{1}{2M^{2}}\sum_{k=1}^{n}p_{k}\left(x_{k}-v\right)^{2}\\&\leq\ln(v)+\frac{1}{v}\left(\sum_{k=1}^{n}p_{k}x_{k}-v\right)-\sum_{k=1}^{n}p_{k}\ln x_{k}\\&\leq\frac{1}{2m^{2}}\sum_{k=1}^{n}p_{k}\left(x_{k}-v\right)^{2}\end{aligned} \tag{4.23}$$

and, in particular

$$\begin{aligned}0&\leq\frac{1}{2M^{2}}\left[\sum_{k=1}^{n}p_{k}x_{k}^{2}-\left(\sum_{i=1}^{n}p_{i}x_{i}\right)^{2}\right]\leq\ln\left(\sum_{k=1}^{n}p_{k}x_{k}\right)-\sum_{k=1}^{n}p_{k}\ln x_{k}\\&\leq\frac{1}{2m^{2}}\left[\sum_{k=1}^{n}p_{k}x_{k}^{2}-\left(\sum_{i=1}^{n}p_{i}x_{i}\right)^{2}\right].\end{aligned} \tag{4.24}$$

We also have

$$\begin{aligned}0&\leq\frac{1}{2M^{2}}\left(\sum_{k=1}^{n}p_{k}x_{k}-\frac{m+M}{2}\right)^{2}\leq\frac{1}{2M^{2}}\sum_{k=1}^{n}p_{k}\left(x_{k}-\frac{m+M}{2}\right)^{2}\\&\leq\ln\left(\frac{m+M}{2}\right)+\left(\frac{m+M}{2}\right)^{-1}-\left(\sum_{k=1}^{n}p_{k}x_{k}-\frac{m+M}{2}\right)\sum_{k=1}^{n}p_{k}\ln x_{k}\\&\leq\frac{1}{2m^{2}}\sum_{k=1}^{n}p_{k}\left(x_{k}-\frac{m+M}{2}\right)^{2}\leq\frac{1}{8}\left(\frac{M}{m}-1\right)^{2}\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\frac{1}{48}\left(1-\frac{m}{M}\right)^{2}&\leq\frac{1}{2}\left\{\frac{M\ln M-M-m\ln m+m}{M-m}\right.\\&\left.-\sum_{k=1}^{n}p_{k}\ln(x_{k})-\frac{1}{M-m}\left[\ln(m)\left(\sum_{k=1}^{n}p_{k}x_{k}-m\right)+\ln(M)\left(M-\sum_{k=1}^{n}p_{k}x_{k}\right)\right]\right\}\\&\leq\frac{1}{12}\left(\frac{M}{m}-1\right)^{2}.\end{aligned} \tag{4.26}$$

Since the second derivative of $-\ln$ is decreasing and convex, then we also have

$$
\begin{aligned}
0 &\leq \frac{1}{M-v}\left\{\frac{1}{v}-\frac{\ln(M)-\ln(v)}{M-v}\right\}\left(\sum_{k=1}^{n}p_kx_k-v\right)^2 \\
&\leq \frac{1}{M-v}\left\{\ln(v)-\frac{\ln(M)-\ln(v)}{M-v}\right\}\sum_{k=1}^{n}p_k\left(x_k-v\right)^2 \\
&\leq \ln(v)+\frac{1}{v}\left(\sum_{k=1}^{n}p_kx_k-v\right)-\sum_{k=1}^{n}p_k\ln x_k \\
&\leq \frac{1}{v-m}\left\{\frac{\ln(v)-\ln(m)}{v-m}-\frac{1}{v}\right\}\sum_{k=1}^{n}p_k\left(x_k-v\right)^2
\end{aligned} \tag{4.27}
$$

for all $v\in(m,M)$.

In particular, we have

$$
\begin{aligned}
0 &\leq \frac{1}{M-\sum_{k=1}^{n}p_kx_k}\left\{\left(\sum_{k=1}^{n}p_kx_k\right)^{-1}-\frac{\ln(M)-\ln\left(\sum_{k=1}^{n}p_kx_k\right)}{M-\sum_{k=1}^{n}p_kx_k}\right\} \\
&\quad\times\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right] \\
&\leq \ln\left(\sum_{k=1}^{n}p_kx_k\right)-\sum_{k=1}^{n}p_k\ln x_k \\
&\leq \frac{1}{\sum_{k=1}^{n}p_kx_k-m}\left\{\frac{\ln\left(\sum_{k=1}^{n}p_kx_k\right)-\ln(m)}{\sum_{k=1}^{n}p_kx_k-m}-\left(\sum_{k=1}^{n}p_kx_k\right)^{-1}\right\} \\
&\quad\times\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right].
\end{aligned} \tag{4.28}
$$

Finally, we have

$$
\begin{aligned}
0 &\leq \frac{1}{2}\min_{k\in\{1,\dots,n\}}\left(\frac{x_k}{\sum_{i=1}^{n}p_ix_i}-1\right)^2 \\
&\leq \ln\left(\sum_{k=1}^{n}p_kx_k\right)-\sum_{k=1}^{n}p_k\ln x_k \\
&\leq \frac{1}{2}\max_{k\in\{1,\dots,n\}}\left(\frac{x_k}{\sum_{i=1}^{n}p_ix_i}-1\right)^2\frac{\sum_{k=1}^{n}p_kx_k^{-2}}{\left(\sum_{i=1}^{n}p_ix_i\right)^{-2}},
\end{aligned} \tag{4.29}
$$

which is equivalent to

$$0\le\exp\left[\frac{1}{2}\min_{k\in\{1,\dots,n\}}\left(\frac{x_k}{\sum_{i=1}^n p_i x_i}-1\right)^2\right] \le \frac{\sum_{k=1}^n p_k x_k}{\prod_{k=1}^n x_k^{p_k}} \le \exp\left[\frac{1}{2}\max_{k\in\{1,\dots,n\}}\left(\frac{x_k}{\sum_{i=1}^n p_i x_i}-1\right)^2\frac{\sum_{k=1}^n p_k x_k^{-2}}{\left(\sum_{i=1}^n p_i x_i\right)^{-2}}\right]. \tag{4.30}$$

If we take in (4.18) $n=2$, $x_1=a$, $x_2=b$, $p_1=1-t$ and $p_2=t$, then we get the following refinement and reverse of Young's inequality

$$\begin{aligned}0&\le\exp\left[\frac{1}{2}\min\left\{\left(\frac{a}{(1-t)\,a+tb}-1\right)^2,\left(\frac{b}{(1-t)\,a+tb}-1\right)^2\right\}\right]\\&\le\frac{(1-t)\,a+tb}{a^{1-t}b^t}\\&\le\exp\left\{\frac{1}{2}\max\left\{\left(\frac{a}{(1-t)\,a+tb}-1\right)^2,\left(\frac{b}{(1-t)\,a+tb}-1\right)^2\right\}\right.\\&\left.\times\frac{(1-t)\,a^{-2}+tb^{-2}}{((1-t)\,a+tb)^{-2}}\right\},\end{aligned}$$

namely

$$\begin{aligned}0&\le\exp\left[\frac{1}{2}\frac{(b-a)^2}{((1-t)\,a+tb)^2}\min\left\{t^2,(1-t)^2\right\}\right]\\&\le\frac{(1-t)\,a+tb}{a^{1-t}b^t}\\&\le\exp\left\{\frac{1}{2}\frac{(b-a)^2}{((1-t)\,a+tb)^2}\max\left\{t^2,(1-t)^2\right\}\right.\\&\left.\times\frac{(1-t)\,a^{-2}+tb^{-2}}{((1-t)\,a+tb)^{-2}}\right\},\end{aligned}$$

which is equivalent to

$$0\le\exp\left[\frac{1}{2}\left(\min\{t,1-t\}\right)^2\frac{(b-a)^2}{((1-t)\,a+tb)^2}\right] \le\frac{(1-t)\,a+tb}{a^{1-t}b^t} \le\exp\left\{\frac{1}{2}\left(\max\{t,1-t\}\right)^2(b-a)^2\left[(1-t)\,a^{-2}+tb^{-2}\right]\right\}. \tag{4.31}$$

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