QUADRATIC OPERATOR MONOTONICITY OF THE $\mathcal{AT}$-INTEGRAL TRANSFORM FOR SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. For μ a positive measure on $[0,1]$ and $p:[0,1]\to[0,\infty)$ a continuous function, we consider the following $\mathcal{AT}$-*integral transform*

$$\mathcal{AT}(p,\mu)(T):=\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\mu(\lambda),$$

where the integral is assumed to exist for T a selfadjoint operator on a complex Hilbert space H.

In this paper, we show among others that, if $B^2\geq A^2$, then

$$\mathcal{AT}(p,\mu)(B)\geq \mathcal{AT}(p,\mu)(A),$$

namely $\mathcal{AT}(p,\mu)$ is quadratic operator monotone. Also, if

$$\alpha^2\leq A^2,\ B^2\leq \beta^2 \text{ and } m^2\leq B^2-A^2\leq M^2$$

for some constants $\alpha,\ \beta,\ m,\ M>0$, then

$$0\leq \frac{m^2}{2\beta}\mathcal{AT}'(p,\mu)(\beta)\leq \mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\leq \frac{M^2}{2\alpha}\mathcal{AT}'(p,\mu)(\alpha).$$

Some examples for integral transforms $\mathcal{AT}(p,\mu)$ related to the arctan function are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduced in [2], for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, then we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator [2]

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

We also defined the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda>0$ by [3]

$$\mathcal{L}og(w,\mu)(t):=\int_0^\infty w(\lambda)\ln(\lambda+t)d\mu(\lambda), \tag{1.9}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t):=\int_0^\infty w(\lambda)\ln(\lambda+t)d\lambda. \tag{1.10}$$

Now, for a continuous and positive function $p(\lambda)$, $\lambda\in[0,1]$ and a positive measure μ on $[0,1]$, we define the integral transform

$$\mathcal{AT}(p,\mu)(t):=\begin{cases}\int_0^1\frac{p(\lambda)}{\lambda^2+t^{-2}}d\mu(\lambda),\ t\neq0,\\[2ex] 0,\ t=0.\end{cases} \tag{1.11}$$

If μ is the usual Lebesgue measure, then we put

$$\mathcal{AT}(p)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} \tag{1.12}$$

For $p(\lambda) = 1$, $\lambda \in [0,1]$ and μ is the usual Lebesgue measure, we have

$$\mathcal{AT}(t) := \begin{cases} \int_0^1 \frac{1}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = t \arctan t, \ t \in \mathbb{R}. \tag{1.13}$$

For $p(\lambda) = \ell(\lambda) := \lambda$, we get

$$\mathcal{AT}(\ell)(t) := \begin{cases} \int_0^1 \frac{\lambda}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = \frac{1}{2} \ln\left(t^2+1\right), \ t \in \mathbb{R}.$$

In the case when $p(\lambda) = \ell^2(\lambda) = \lambda^2$, we derive

$$\mathcal{AT}\left(\ell^2\right)(t) = \begin{cases} \int_0^1 \frac{\lambda^2}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = \begin{cases} 1 - \frac{\arctan t}{t}, \ t \neq 0, \\ 0, \ t = 0. \end{cases}$$

Also, for $p(\lambda) = \ell^3(\lambda) = \lambda^3$, we obtain

$$\mathcal{AT}\left(\ell^3\right)(t) = \begin{cases} \int_0^1 \frac{\lambda^3}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = \begin{cases} \frac{1}{2}\left[1 - \frac{\ln\left(t^2+1\right)}{t^2}\right], \ t \neq 0, \\ 0, \ t = 0. \end{cases}$$

We observe that (1.11) can be written in equivalent form as

$$\begin{aligned} \mathcal{AT}(p,\mu)(t) &:= \int_0^1 \frac{t^2 p(\lambda)}{t^2\lambda^2+1} d\mu(\lambda) \\ &= \int_0^1 \left(1 - \frac{1}{t^2\lambda^2+1}\right) \frac{p(\lambda)}{\lambda^2} d\mu(\lambda), \ t \in \mathbb{R}. \end{aligned} \tag{1.14}$$

For a selfadjoint operator T we have that $\lambda^2 T^2 + 1$ is invertible for all $\lambda \in [0,1]$ and by the continuous functional calculus for selfadjoint operators we can define the $\mathcal{AT}$-*operator transform* by

$$\mathcal{AT}(p,\mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\mu(\lambda), \tag{1.15}$$

and, in particular, for the Lebesgue measure

$$\mathcal{AT}(p)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda. \tag{1.16}$$

We observe that

$$\mathcal{AT}(T) = \int_0^1 \frac{1}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = T \arctan T,$$

$$\mathcal{AT}(\ell)(T) = \int_0^1 \frac{1}{\lambda} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = \frac{1}{2} \ln\left(T^2+1\right)$$

for any selfadjoint operator T.

If T is invertible, then

$$\mathcal{AT}\left(\ell^{2}\right)(T)=\int_{0}^{1}\left[1-\left(\lambda^{2}T^{2}+1\right)^{-1}\right]d\lambda=1-T^{-1}\arctan T$$

and

$$\mathcal{AT}\left(\ell^{3}\right)(T)=\int_{0}^{1}\lambda\left[1-\left(\lambda^{2}T^{2}+1\right)^{-1}\right]d\lambda=\frac{1}{2}\left[1-T^{-2}\ln\left(T^{2}+1\right)\right].$$

In this paper, we show among others that, if $B^2\geq A^2$, then

$$\mathcal{AT}(p,\mu)(B)\geq\mathcal{AT}(p,\mu)(A),$$

namely $\mathcal{AT}(p,\mu)$ is *quadratic operator monotone*. Also, if

$$\alpha^{2}\leq A^{2},\ B^{2}\leq\beta^{2}\text{ and }m^{2}\leq B^{2}-A^{2}\leq M^{2}$$

for some constants α, β, m, $M>0$, then

$$0\leq\frac{m^{2}}{2\beta}\mathcal{AT}'(p,\mu)(\beta)\leq\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\leq\frac{M^{2}}{2\alpha}\mathcal{AT}'(p,\mu)(\alpha).$$

Some examples for integral transforms $\mathcal{AT}(p,\mu)$ related to the arctan function are also provided.

For some classical results concerning operator monotone functions, see [7] and [8]. For recent results, see [4]-[6], [9]-[10] and the references therein.

2. Main Results

We have the following result concerning the quadratic operator monotonicity of the transform $\mathcal{AT}(p,\mu)$.

Theorem 1. *Let A, B be selfadjoint operators, then*

$$\begin{aligned}(2.1)\quad &\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\\ &=\int_{0}^{1}p(\lambda)\left[\int_{0}^{1}\left(1+\lambda^{2}\left[(1-t)A^{2}+tB^{2}\right]\right)^{-1}\left(B^{2}-A^{2}\right)\right.\\ &\left.\times\left(1+\lambda^{2}\left[(1-t)A^{2}+tB^{2}\right]\right)^{-1}dt\right]d\mu(\lambda).\end{aligned}$$

If $B^2\geq A^2$, then

$$\mathcal{AT}(p,\mu)(B)\geq\mathcal{AT}(p,\mu)(A).\tag{2.2}$$

Proof. Let A, B selfadjoint operators. By employing (1.15) we have

$$\begin{aligned}(2.3)\quad &\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\\ &=\int_{0}^{1}\frac{p(\lambda)}{\lambda^{2}}\left[1-\left(\lambda^{2}B^{2}+1\right)^{-1}\right]d\mu(\lambda)\\ &-\int_{0}^{1}\frac{p(\lambda)}{\lambda^{2}}\left[1-\left(\lambda^{2}A^{2}+1\right)^{-1}\right]d\mu(\lambda)\\ &=\int_{0}^{1}\frac{p(\lambda)}{\lambda^{2}}\left[\left(\lambda^{2}A^{2}+1\right)^{-1}-\left(\lambda^{2}B^{2}+1\right)^{-1}\right]d\mu(\lambda).\end{aligned}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]:\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

If in (2.6) we take $C=\lambda^2A^2+1$ and $D=\lambda^2B^2+1$, then we get

$$\begin{aligned}&\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\\&=\int_0^1\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}\left(\lambda^2B^2-\lambda^2A^2\right)\\&\times\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}dt\\&=\lambda^2\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\\&\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\end{aligned} \tag{2.7}$$

for all $t,\ \lambda\in[0,1]$.

Therefore

$$\begin{aligned}&\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\&=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\&=\int_0^1p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\&\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)\end{aligned}$$

and by (2.3) we get the representation (2.1).

If $B^2-A^2\geq 0$, then by multiplying both sides by $\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}$ we get

$$\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\geq 0$$

for all $t,\ \lambda\in[0,1]$.

If we integrate over $t \in [0,1]$, then multiply by $p(\lambda) \geq 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have

$$\int_0^1 p(\lambda) \left[\int_0^1 \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \left(B^2 - A^2\right) \times \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} dt \right] d\mu(\lambda) \geq 0$$

and the operator quadratic monotonicity of the transform $\mathcal{AT}(p,\mu)$ is proved. □

If more information is available for the operators A, B, then we can provide better upper and lower bounds for the difference $\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)$.

Theorem 2. *Let A, B be selfadjoint operators such that*

$$\alpha^2 \leq A^2 \leq \beta^2 \text{ and } m^2 \leq B^2 - A^2 \leq M^2 \tag{2.8}$$

for some constants α, β, m, $M > 0$. Then

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2} \left[\mathcal{AT}(p,\mu)\left(\sqrt{M^2+\beta^2}\right) - \mathcal{AT}(p,\mu)(\beta) \right] \\ &\leq \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \\ &\leq \frac{M^2}{m^2} \left[\mathcal{AT}(p,\mu)\left(\sqrt{m^2+\alpha^2}\right) - \mathcal{AT}(p,\mu)(\alpha) \right]. \end{aligned} \tag{2.9}$$

Proof. Since $\alpha^2 \leq A^2 \leq \beta^2$ and $m^2 \leq B^2 - A^2 \leq M^2$, hence

$$\alpha^2 + tm^2 \leq (1-t) A^2 + tB^2 = t\left(B^2 - A^2\right) + A^2 \leq \beta^2 + tM^2$$

for all $t \in [0,1]$.

This implies that

$$1+\lambda^2 \left(\alpha^2 + tm^2\right) \leq 1+\lambda^2 \left[(1-t) A^2 + tB^2\right] \leq 1+\lambda^2 \left(\beta^2 + tM^2\right),$$

namely

$$\begin{aligned} \left[1+\lambda^2 \left(\beta^2 + tM^2\right)\right]^{-1} &\leq \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \\ &\leq \left[1+\lambda^2 \left(\alpha^2 + tm^2\right)\right]^{-1}, \end{aligned} \tag{2.10}$$

for all t, $\lambda \in [0,1]$.

Since $m^2 \leq B^2 - A^2 \leq M^2$, hence by multiplying both sides by

$$\left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1},$$

we get

$$\begin{aligned} &m^2 \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-2} \\ &\leq \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \left(B^2 - A^2\right) \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \\ &\leq M^2 \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-2} \end{aligned}$$

for all t, $\lambda \in [0,1]$.

By utilising (2.10) we obtain the following bounds

$$\begin{aligned} &m^2 \left(1+\lambda^2 \left(\beta^2 + tM^2\right)\right)^{-2} \\ &\leq \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \left(B^2 - A^2\right) \left(1+\lambda^2 \left[(1-t) A^2 + tB^2\right]\right)^{-1} \\ &\leq M^2 \left(1+\lambda^2 \left(\alpha^2 + tm^2\right)\right)^{-2} \end{aligned} \tag{2.11}$$

for all $t,\ \lambda\in[0,1]$.

If we integrate (2.11) over $t\in[0,1]$, then multiply by $p(\lambda)\geq 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have

$$\begin{aligned}&m^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\beta^2+tM^2\right)\right)^{-2}dt\right)d\mu(\lambda)\\&\leq\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\&\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)\\&\leq M^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\alpha^2+tm^2\right)\right)^{-2}dt\right)d\mu(\lambda)\end{aligned}$$

and by the identity (2.1),

$$\begin{aligned}&m^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\beta^2+tM^2\right)\right)^{-2}dt\right)d\mu(\lambda)\\&\leq\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\\&\leq M^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\alpha^2+tm^2\right)\right)^{-2}dt\right)d\mu(\lambda).\end{aligned}\tag{2.12}$$

By (2.1) we have for $B=\sqrt{M^2+\beta^2}$, $A^2=\beta$ that

$$\begin{aligned}&\mathcal{AT}(p,\mu)\left(\sqrt{M^2+\beta^2}\right)-\mathcal{AT}(p,\mu)(\beta)\\&=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)\beta^2+t\left(M^2+\beta^2\right)\right]\right)^{-1}M^2\right.\\&\left.\times\left(1+\lambda^2\left[(1-t)\beta^2+t\left(M^2+\beta^2\right)\right]\right)^{-1}dt\right]d\mu(\lambda)\\&=M^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\beta^2+tM^2\right)\right)^{-2}dt\right)d\mu(\lambda),\end{aligned}$$

namely

$$\begin{aligned}&\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\beta^2+tM^2\right)\right)^{-2}dt\right)d\mu(\lambda)\\&=\frac{1}{M^2}\left[\mathcal{AT}(p,\mu)\left(\sqrt{M^2+\beta^2}\right)-\mathcal{AT}(p,\mu)(\beta)\right].\end{aligned}\tag{2.13}$$

By (2.1) we also have for $B=\sqrt{m^2+\alpha^2}$, $A=\alpha$ that

$$\begin{aligned}&\mathcal{AT}(p,\mu)\left(\sqrt{m^2+\alpha^2}\right)-\mathcal{AT}(p,\mu)(\alpha)\\&=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)\alpha^2+t\left(m^2+\alpha^2\right)\right]\right)^{-1}m^2\right.\\&\left.\times\left(1+\lambda^2\left[1+\lambda^2\left[(1-t)\alpha^2+t\left(m^2+\alpha^2\right)\right]\right]\right)^{-1}dt\right]d\mu(\lambda)\\&=m^2\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left(\alpha^2+tm^2\right)\right)^{-2}dt\right)d\mu(\lambda)\end{aligned}$$

namely

$$\int_0^1 p(\lambda)\left(\int_0^1 \left(1+\lambda^2\left(\alpha^2+tm^2\right)\right)^{-2}dt\right)d\mu(\lambda) = \frac{1}{m^2}\left[\mathcal{AT}(p,\mu)\left(\sqrt{m^2+\alpha^2}\right)-\mathcal{AT}(p,\mu)(\alpha)\right]. \tag{2.14}$$

By making use of (2.12)-(2.14) we derive (2.9). □

We also have:

Theorem 3. *Let A, B be selfadjoint operators such that*

$$\alpha^2 \le A^2,\ \ B^2 \le \beta^2 \ \text{ and } m^2 \le B^2 - A^2 \le M^2 \tag{2.15}$$

for some constants α, β, m, $M > 0$. Then

$$0 \le \frac{m^2}{2\beta}\mathcal{AT}'(p,\mu)(\beta) \le \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \le \frac{M^2}{2\alpha}\mathcal{AT}'(p,\mu)(\alpha), \tag{2.16}$$

where $\mathcal{AT}'(p,\mu)$ is the derivative of $\mathcal{AT}$ as a function of $t > 0$.

Proof. Since $\alpha^2 \le A^2$, $B^2 \le \beta^2$, hence

$$\alpha^2 \le (1-t)A^2 + tB^2 \le \beta^2$$

for all $t \in [0,1]$.

This implies that

$$1+\lambda^2\alpha^2 \le 1+\lambda^2\left[(1-t)A^2+tB^2\right] \le 1+\lambda^2\beta^2,$$

namely

$$\left(1+\lambda^2\beta^2\right)^{-1} \le \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1} \le \left(1+\lambda^2\alpha^2\right)^{-1} \tag{2.17}$$

for all t, $\lambda \in [0,1]$.

Since $m^2 \le B^2 - A^2 \le M^2$, hence by multiplying both sides by

$$\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1},$$

we get

$$\begin{aligned} & m^2\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-2} \\ & \le \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1} \\ & \le M^2\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-2} \end{aligned} \tag{2.18}$$

for all t, $\lambda \in [0,1]$.

By utilising (2.17) and (2.18) we get

$$\begin{aligned} & m^2\left(1+\lambda^2\beta^2\right)^{-2} \\ & \le \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1} \\ & \le M^2\left(1+\lambda^2\alpha^2\right)^{-2} \end{aligned} \tag{2.19}$$

for all t, $\lambda \in [0,1]$.

If we integrate (2.19) over $t \in [0,1]$, then multiply by $p(\lambda) \geq 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have by (2.1) that,

$$m^2 \int_0^1 p(\lambda)\left(1+\lambda^2\beta^2\right)^{-2} d\mu(\lambda) \leq \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \leq M^2 \int_0^1 p(\lambda)\left(1+\lambda^2\alpha^2\right)^{-2} d\mu(\lambda). \tag{2.20}$$

If in (1.14) we take the derivative over t, we get

$$\begin{aligned}
\mathcal{AT}'(p,\mu)(t) &= \int_0^1 \left(1-\frac{1}{t^2\lambda^2+1}\right)' \frac{p(\lambda)}{\lambda^2} d\mu(\lambda) \\
&= \int_0^1 2\lambda^2 t\left(1+\lambda^2 t^2\right)^{-2} \frac{p(\lambda)}{\lambda^2} d\mu(\lambda) \\
&= 2t\int_0^1 p(\lambda)\left(1+\lambda^2 t^2\right)^{-2} d\mu(\lambda).
\end{aligned}$$

Therefore

$$\int_0^1 p(\lambda)\left(1+\lambda^2\beta^2\right)^{-2} d\mu(\lambda) = \frac{\mathcal{AT}'(p,\mu)(\beta)}{2\beta}$$

and

$$\int_0^1 p(\lambda)\left(1+\lambda^2\alpha^2\right)^{-2} d\mu(\lambda) = \frac{\mathcal{AT}'(p,\mu)(\alpha)}{2\alpha}$$

By making use of (2.20) we then get the desired result (2.16). □

The case of separate operators is as follows:

Theorem 4. *Let A, B be selfadjoint operators such that*

$$\alpha^2 \leq A^2 \leq \beta^2 < \gamma^2 \leq B^2 \leq \delta \tag{2.21}$$

for some constants α, β, γ, $\delta > 0$. Then

$$\begin{aligned}
0 &\leq \left(\gamma^2-\beta^2\right)\frac{\mathcal{AT}(p,\mu)(\delta) - \mathcal{AT}(p,\mu)(\beta)}{\delta^2-\beta^2} \\
&\leq \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \\
&\leq \left(\delta^2-\alpha^2\right)\frac{\mathcal{AT}(p,\mu)(\gamma) - \mathcal{AT}(p,\mu)(\alpha)}{\gamma^2-\alpha^2}.
\end{aligned} \tag{2.22}$$

Proof. Since $\alpha^2 \leq A^2 \leq \beta^2 < \gamma^2 \leq B^2 \leq \delta^2$, hence

$$(1-t)\alpha^2 + t\gamma^2 \leq (1-t)A^2 + tB^2 \leq (1-t)\beta^2 + t\delta^2$$

for all $t \in [0,1]$.

This implies that

$$1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right] \leq 1+\lambda^2\left[(1-t)A^2+tB^2\right] \leq 1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right],$$

namely

$$\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-2} \leq \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-2} \leq \left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-2} \tag{2.23}$$

for all t, $\lambda \in [0,1]$.

Since

$$0 < \gamma^2-\beta^2 \leq B^2-A^2 \leq \delta^2-\alpha^2,$$

hence by multiplying both sides by

$$\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1},$$

we get

$$\begin{aligned}(2.24)\quad 0 &< \left(\gamma^2-\beta^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-2}\\ &\le \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\\ &\le \left(\delta^2-\alpha^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-2}\end{aligned}$$

for all $t,\ \lambda\in[0,1]$.

By (2.23) and (2.24) we derive

$$\begin{aligned}(2.25)\quad 0 &< \left(\gamma^2-\beta^2\right)\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-2}\\ &\le \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\\ &\le \left(\delta^2-\alpha^2\right)\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-2}\end{aligned}$$

for all $t,\ \lambda\in[0,1]$.

If we integrate (2.19) over $t\in[0,1]$, then multiply by $p(\lambda)\ge 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have by (2.1) that,

$$\begin{aligned}(2.26)\qquad 0 &< \left(\gamma^2-\beta^2\right)\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-2}dt\right)d\mu(\lambda)\\ &\le \mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\\ &\le \left(\delta^2-\alpha^2\right)\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-2}dt\right)d\mu(\lambda).\end{aligned}$$

Using (2.1) for $B=\delta$ and $A=\beta$ we have,

$$\begin{aligned}(2.27)\qquad &\mathcal{AT}(p,\mu)(\delta)-\mathcal{AT}(p,\mu)(\beta)\\ &=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-1}\left(\delta^2-\beta^2\right)\right.\\ &\quad\left.\times\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-1}dt\right]d\mu(\lambda)\\ &=\left(\delta^2-\beta^2\right)\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-2}dt\right)d\mu(\lambda),\end{aligned}$$

which gives that

$$\begin{aligned}(2.28)\qquad &\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\beta^2+t\delta^2\right]\right)^{-2}dt\right)d\mu(\lambda)\\ &=\frac{\mathcal{AT}(p,\mu)(\delta)-\mathcal{AT}(p,\mu)(\beta)}{\delta^2-\beta^2}.\end{aligned}$$

Using (2.1) for $B=\gamma$ and $A=\alpha$ we have,

$$\begin{aligned}(2.29)\qquad &\mathcal{AT}(p,\mu)(\gamma)-\mathcal{AT}(p,\mu)(\alpha)\\ &=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-1}\left(\gamma^2-\alpha^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-1}dt\right]d\mu(\lambda)\\ &=\left(\gamma^2-\alpha^2\right)\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-2}dt\right)d\mu(\lambda),\end{aligned}$$

which gives that

$$\begin{aligned}(2.30)\qquad &\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)\alpha^2+t\gamma^2\right]\right)^{-2}dt\right)d\mu(\lambda)\\ &=\frac{\mathcal{AT}(p,\mu)(\gamma)-\mathcal{AT}(p,\mu)(\alpha)}{\gamma^2-\alpha^2}.\end{aligned}$$

By making use of (2.26), (2.28) and (2.30) we get (2.22). □

3. Some Examples

For any A, B selfadjoint operators we have by Theorem 1,

$$\begin{aligned}(3.1)\qquad &B\arctan B-A\arctan A\\ &=\int_0^1\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda).\end{aligned}$$

If $B^2\geq A^2$, then

$$(3.2)\qquad B\arctan B\geq A\arctan A.$$

For any A, B selfadjoint and invertible operators, we have by Theorem 1, that

$$\begin{aligned}(3.3)\qquad &A^{-1}\arctan A-B^{-1}\arctan B\\ &=\int_0^1\lambda^2\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda).\end{aligned}$$

If $B^2\geq A^2$, then

$$(3.4)\qquad A^{-1}\arctan A\geq B^{-1}\arctan B.$$

If the operators A, B satisfy the condition (2.8) for some constants α, β, m, $M>0$, then by (2.9) we obtain

$$\begin{aligned}(3.5)\qquad 0&\leq\frac{m^2}{M^2}\left[\sqrt{M^2+\beta^2}\arctan\left(\sqrt{M^2+\beta^2}\right)-\beta\arctan\beta\right]\\ &\leq B\arctan B-A\arctan A\\ &\leq\frac{M^2}{m^2}\left[\sqrt{m^2+\alpha^2}\arctan\left(\sqrt{m^2+\alpha^2}\right)-\alpha\arctan\alpha\right]\end{aligned}$$

and

$$0 \le \frac{m^2}{M^2}\left[\frac{\arctan\beta}{\beta} - \frac{\arctan\left(\sqrt{M^2+\beta^2}\right)}{\sqrt{M^2+\beta^2}}\right] \tag{3.6}$$
$$\le A^{-1}\arctan A - B^{-1}\arctan B.$$
$$\le \frac{M^2}{m^2}\left[\frac{\arctan\alpha}{\alpha} - \frac{\arctan\left(\sqrt{m^2+\alpha^2}\right)}{\sqrt{m^2+\alpha^2}}\right].$$

If the operators A, B satisfy the condition (2.15) for some constants α, β, m, $M > 0$, then by (2.16) we get

$$0 \le \frac{m^2}{2\beta}\left(\arctan\beta + \frac{\beta}{\beta^2+1}\right) \le B\arctan B - A\arctan A \tag{3.7}$$
$$\le \frac{M^2}{2\alpha}\left(\arctan\alpha + \frac{\alpha}{\alpha^2+1}\right)$$

and

$$0 \le \frac{m^2}{2\beta^3}\left(\arctan\beta - \frac{\beta}{\beta^2+1}\right) \le A^{-1}\arctan A - B^{-1}\arctan B \tag{3.8}$$
$$\le \frac{mM^2}{2\alpha^3}\left(\arctan\alpha - \frac{\alpha}{\alpha^2+1}\right).$$

Finally, if we assume that A, B satisfy the separation condition (2.21) for some constants α, β, γ, $\delta > 0$, then

$$0 \le \left(\gamma^2 - \beta^2\right)\frac{\delta\arctan\delta - \beta\arctan\beta}{\delta^2-\beta^2} \tag{3.9}$$
$$\le B\arctan B - A\arctan A$$
$$\le \left(\delta^2 - \alpha^2\right)\frac{\gamma\arctan\gamma - \alpha\arctan\alpha}{\gamma^2-\alpha^2}$$

and

$$0 \le \left(\gamma^2 - \beta^2\right)\frac{\beta^{-1}\arctan\beta - \delta^{-1}\arctan\delta}{\delta^2-\beta^2} \tag{3.10}$$
$$\le A^{-1}\arctan A - B^{-1}\arctan B$$
$$\le \left(\delta^2 - \alpha^2\right)\frac{\alpha^{-1}\arctan\alpha - \gamma^{-1}\arctan\gamma}{\gamma^2-\alpha^2}.$$

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