OPERATOR MONOTONICITY OF LERCH TRANSFORM

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a positive measure μ on $[0,\infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\operatorname{Sp}(T)\subset(-1,1)$, by

$$\Phi_\mu(T,s,v):=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(e^t-T\right)^{-1}d\mu(t),$$

for $\operatorname{Re}v>0$ and $\operatorname{Re}s>0$, provided that the integral exists.

In this paper we how among others that, if $v>0$ and $s>0$, then $\Phi(\cdot,s,v)$ is operator monotone on $(-1,1)$. Moreover, if $-1<\alpha\le A\le\beta<1$, $-1<B<1$ and $0<m\le B-A\le M$, then

$$\begin{aligned}0&\le\Phi_\mu(m+\alpha,s,v)-\Phi_\mu(\alpha,s,v)\\&\le\Phi_\mu(B,s,v)-\Phi_\mu(A,s,v)\le\Phi_\mu(M+\beta,s,v)-\Phi_\mu(\beta,s,\beta)\end{aligned}$$

for $v>0$ and $s>0$. Some examples for *polylogarithm* function and *Legendre chi* function are also given.

1. Introduction

The *Lerch transcendent* function is given by the series

$$\Phi(z,s,\alpha):=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s},\ \ |z|<1,\ \ \alpha\neq0,-1,-2,... \tag{1.1}$$

see for instance [6, Section 1.11, p. 27] or [1, Section 25.14]. This function, defined by Mathias Lerch in 1887 in his paper [8], includes as special cases of the parameters; the Hurwitz, Riemann zeta functions and the polylogarithms, among others. Therefore the transcendent has applications ranging from number theory to physics.

The *Hurwitz zeta* function, formally defined for complex arguments s with $\operatorname{Re}(s)>1$ and α with $\operatorname{Re}(\alpha)>0$ by

$$\zeta(s,\alpha):=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s} \tag{1.2}$$

is a special case, given by

$$\zeta(s,\alpha)=\Phi(1,s,\alpha). \tag{1.3}$$

For $\alpha=1$ we have the *Riemann zeta* function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{1.4}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator inequalities, Logarithmic operator inequalities. Power inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 131, 10 pp. Received 05/11/20

The *polylogarithm* function $\mathrm{Li}(s,z)$ is defined by a power series in z, which is also a *Dirichlet series* in s:

$$\mathrm{Li}(z,s) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1). \tag{1.5}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z|<1$; it can be extended to $|z|\geq 1$ by the process of analytic continuation. The special case $s=1$ involves the ordinary natural logarithm, $\mathrm{Li}(z,1) = -\ln(1-z)$, while the special cases $s=2$ and $s=3$ are called the *dilogarithm* (also referred to as *Spence's function*) and *trilogarithm* respectively.

The *Legendre chi* function is a special case, given by

$$\chi_s(z) = 2^{-s} z\Phi\left(z^2, s, 1/2\right). \tag{1.6}$$

The *Legendre chi* function is a special function whose Taylor series is also a Dirichlet series, given by

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}. \tag{1.7}$$

The following integral representations are valid [6, p. 27]

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{1.8}$$

for $\mathrm{Re}\, v > 0$ and either $|z| \leq 1$, $z \neq 1$, $\mathrm{Re}\, s > 0$ or $z = 1$, $\mathrm{Re}\, s > 1$. Here $\Gamma(\cdot)$ is Euler's Gama function

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{z-1} dt, \ \ \mathrm{Re}\, s > 0.$$

For a selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1,1)$ and by utilising the continuous functional calculus for selfadjoint operators, we define the transform

$$\Phi(T,s,v) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} dt \tag{1.9}$$

for $\mathrm{Re}\, v > 0$ and $\mathrm{Re}\, s > 0$.

We can also define the related transforms

$$\mathrm{Li}(T,s) := T\Phi(T,s,1) \ \text{ and } \chi_s(T) := 2^{-s} T\Phi\left(T^2, s, 1/2\right) \tag{1.10}$$

for a selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1,1)$ and $\mathrm{Re}\, s > 0$.

Now, for a positive measure μ on $[0,\infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1,1)$, by

$$\Phi_\mu(T,s,v) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} d\mu(t), \tag{1.11}$$

for $\mathrm{Re}\, v > 0$ and $\mathrm{Re}\, s > 0$, provided that the integral exists. When μ is Lebesgue measure, then we recapture (1.8).

In this paper we how among others that, if $v > 0$ and $s > 0$, then $\Phi_\mu(\cdot, s, v)$ is operator monotone on $(-1,1)$. Moreover, if $-1 < \alpha \leq A \leq \beta < 1$, $-1 < B < 1$ and $0 < m \leq B - A \leq M$, then

$$\begin{aligned} 0 &\leq \Phi_\mu(m+\alpha, s, v) - \Phi_\mu(\alpha, s, v) \\ &\leq \Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \leq \Phi_\mu(M+\beta, s, v) - \Phi_\mu(\beta, s, \beta) \end{aligned}$$

for $v>0$ and $s>0$. Some examples for *polylogarithm* function and *Legendre chi* function are also given.

2. Main Results

We have the following representation and operator monotonicity result:

Theorem 1. *For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, $\operatorname{Re} v>0$ and $\operatorname{Re} s>0$ we have*

$$\Phi_{\mu}(B,s,v)-\Phi_{\mu}(A,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-(v-1)t} \tag{2.1}$$
$$\times\left[\int_0^1 \left(e^t-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^t-[(1-u)B+uA]\right)^{-1}du\right]d\mu(t).$$

If $v>0$ and $s>0$, then $\Phi(\cdot,s,v)$ is operator monotone on $(-1,1)$.

Proof. For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, we have by (1.9) that

$$\begin{aligned}&\Phi_{\mu}(B,s,v)-\Phi_{\mu}(A,s,v)\\ &=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-(v-1)t}\left[\left(e^t-B\right)^{-1}-\left(e^t-A\right)^{-1}\right]d\mu(t)\end{aligned} \tag{2.2}$$

for $\operatorname{Re} v>0$ and $\operatorname{Re} s>0$.

The function $g(u)=-u^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S):=\lim_{u\to 0}\left[\frac{g(T+uS)-g(T)}{u}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(u)=g((1-u)C+uD),\ u\in[0,1].$$

If $g_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-u)C+uD,\ u\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$g(D)-g(C)=\int_0^1 \frac{d}{du}\left(g_{C,D}(u)\right)du=\int_0^1 \nabla g_{(1-u)C+uD}(D-C)\,du. \tag{2.4}$$

If we write this equality for the function $g(u)=-u^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-u)C+uD)^{-1}(D-C)((1-u)C+uD)^{-1}du. \tag{2.5}$$

By (2.5) for $C=e^{t}-B,\ D=e^{t}-A,\ t\in[0,\infty)$ we have

$$
\begin{aligned}
(2.6)\qquad & \left(e^{t}-B\right)^{-1}-\left(e^{t}-A\right)^{-1}\\
&=\int_{0}^{1}\left((1-u)\left(e^{t}-B\right)+u\left(e^{t}-A\right)\right)^{-1}\left(e^{t}-A-e^{t}+B\right)\\
&\times\left((1-u)\left(e^{t}-B\right)+u\left(e^{t}-A\right)\right)^{-1}du\\
&=\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du.
\end{aligned}
$$

If we multiply this equality by $t^{s-1}e^{-(v-1)t}$ and integrate, then we obtain (2.1).

Assume that $B-A\geq 0$. Then by multiplying both sides by $\left(e^{t}-[(1-u)B+uA]\right)^{-1}$, we derive

$$
\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}\geq 0,
$$

for $t\in[0,\infty)$ and $u\in[0,1]$, which implies that

$$
\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\geq 0
$$

for all $t\in[0,\infty)$.

If $v>0$ and $s>0$, then by multiplying with $t^{s-1}e^{-(v-1)t}\geq 0$ and by integrating over $t\in[0,\infty)$, we get

$$
\begin{aligned}
&\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\\
&\times\left[\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\right]d\mu(t)\\
&\geq 0,
\end{aligned}
$$

which by representation (2.1) gives that $\Phi_{\mu}(B,s,v)-\Phi_{\mu}(A,s,v)\geq 0$. □

Observe that

$$
\begin{aligned}
(2.7)\qquad \operatorname{Li}(T,s)=T\Phi(T,s,1)&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}T\left(e^{t}-T\right)^{-1}dt\\
&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\left(T-e^{t}+e^{t}\right)\left(e^{t}-T\right)^{-1}dt\\
&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\left[e^{t}\left(e^{t}-T\right)^{-1}-1\right]dt
\end{aligned}
$$

for $\operatorname{Re}s>0$ and T with $\operatorname{Sp}(T)\subset(-1,1)$.

Corollary 1. *For A, B with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$ and $\operatorname{Re}s>0$, we have*

$$
\begin{aligned}
(2.8)\quad &\operatorname{Li}(B,s)-\operatorname{Li}(A,s)\\
&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{t}\\
&\times\left[\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\right]dt.
\end{aligned}
$$

If $s>0$, then $\operatorname{Li}(\cdot,s)$ is operator monotone on $(-1,1)$.

Proof. By (2.7) we have

$$\mathrm{Li}\,(B,s)-\mathrm{Li}\,(A,s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left[\left(e^t-B\right)^{-1}-\left(e^t-A\right)^{-1}\right]dt$$

for $\mathrm{Re}\,s>0$ and by employing the identity (2.6) we derive (2.8). □

Corollary 2. *For A, B with $\mathrm{Sp}\,(A)\,,\ \mathrm{Sp}\,(B)\subset(-1,1)\,,$ we have*

$$\begin{aligned}\chi_s\,(B)\,B^{-1}-\chi_s\,(A)\,A^{-1} \\ =\frac{1}{2^s\Gamma(s)}\int_0^\infty t^{s-1}e^{t/2}\left(\int_0^1\left(e^t-\left[(1-u)\,B^2+uA^2\right]\right)^{-1}\right. \\ \left.\times\left(B^2-A^2\right)\left(e^t-\left[(1-u)\,B^2+uA^2\right]\right)^{-1}du\right)dt\end{aligned}\tag{2.9}$$

for $\mathrm{Re}\,s>0$.

If $B^2\geq A^2$, *then* $\chi_s\,(B)\,B^{-1}\geq\chi_s\,(A)\,A^{-1}$ *for* $s>0$.

It follows by Theorem 1 observing that $\chi_s\,(T)\,T^{-1}=2^{-s}\Phi\left(T^2,s,1/2\right).$

When more information is available for the operators A and B, we have the better inequalities below.

Theorem 2. *Assume that* $-1<\alpha\leq A\leq\beta<1$, $-1<B<1$ *and* $0<m\leq B-A\leq M$, *then*

$$\begin{aligned}0&\leq\Phi_\mu\,(m+\alpha,s,v)-\Phi_\mu\,(\alpha,s,v) \\ &\leq\Phi_\mu\,(B,s,v)-\Phi_\mu\,(A,s,v)\leq\Phi_\mu\,(M+\beta,s,v)-\Phi_\mu\,(\beta,s,\beta)\end{aligned}\tag{2.10}$$

for $v>0$ *and* $s>0$.

Proof. Observe, from (2.1), we have by changing u with $1-u$ that

$$\begin{aligned}\Phi_\mu\,(B,s,v)-\Phi_\mu\,(A,s,v)&=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t} \\ &\times\left[\int_0^1\left(e^t-\left[uB+(1-u)\,A\right]\right)^{-1}(B-A)\right. \\ &\left.\times\left(e^t-\left[uB+(1-u)\,A\right]\right)^{-1}du\right]d\mu\,(t)\end{aligned}\tag{2.11}$$

for A, B with $\mathrm{Sp}\,(A)\,,\ \mathrm{Sp}\,(B)\subset(-1,1)\,,$ $\mathrm{Re}\,v>0$ and $\mathrm{Re}\,s>0$.

Notice that

$$0<\alpha+mu\leq uB+(1-u)\,A=A+u\,(B-A)\leq\beta+Mu,$$

which gives that

$$0<e^t-\beta-Mu\leq e^t-(uB+(1-u)\,A)\leq e^t-\alpha-mu$$

namely

$$0<\left(e^t-\alpha-mu\right)^{-1}\leq\left(e^t-\left[uB+(1-u)\,A\right]\right)^{-1}\leq\left(e^t-\beta-Mu\right)^{-1}$$

for $u\in[0,1]$ and $t\geq0$.

By taking the square, we derive

$$0<\left(e^t-\alpha-mu\right)^{-2}\leq\left(e^t-\left[uB+(1-u)\,A\right]\right)^{-2}\leq\left(e^t-\beta-Mu\right)^{-2}\tag{2.12}$$

for $u\in[0,1]$ and $t\geq0$.

Since $0 < m \leq B - A \leq M$, hence by multiplying both sides by $\left(e^t - [uB + (1-u)A]\right)^{-1}$ we derive

$$0 < m\left(e^t - [uB + (1-u)A]\right)^{-2} \leq \left(e^t - [uB + (1-u)A]\right)^{-1}(B - A)\left(e^t - [uB + (1-u)A]\right)^{-1} \leq M\left(e^t - [uB + (1-u)A]\right)^{-2} \tag{2.13}$$

for $u \in [0,1]$ and $t \geq 0$.

By utilising (2.12) and (2.13) we derive

$$0 < m\left(e^t - \alpha - mu\right)^{-2} \leq \left(e^t - [uB + (1-u)A]\right)^{-1}(B - A)\left(e^t - [uB + (1-u)A]\right)^{-1} \leq M\left(e^t - \beta - Mu\right)^{-2} \tag{2.14}$$

for $u \in [0,1]$ and $t \geq 0$.

By integrating over u, then multiplying by $t^{s-1}e^t \geq 0$ and integrating over t in $[0,\infty)$ we get

$$\begin{aligned} 0 &< m\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \alpha - mu\right)^{-2}du\right)d\mu(t) \\ &\leq \Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \\ &\leq M\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \beta - Mu\right)^{-2}du\right)d\mu(t). \end{aligned} \tag{2.15}$$

If we write the equality (2.11) for $B = M + \beta$ and $A = \beta$ we get

$$\begin{aligned} &\Phi_\mu(M+\beta,s,v) - \Phi_\mu(\beta,s,\beta) \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t} \\ &\times\left[\int_0^1 \left(e^t - [u(M+\beta) + (1-u)\beta]\right)^{-1}\right. \\ &\left.\times(M+\beta-\beta)\left(e^t - [u(M+\beta) + (1-u)\beta]\right)^{-1}du\right]d\mu(t) \\ &= \frac{M}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \beta - Mu\right)^{-2}du\right)d\mu(t) \end{aligned} \tag{2.16}$$

and if we write the equality (2.11) for $B = m + \alpha$ and $A = \alpha$ we get

$$\begin{aligned} &\Phi_\mu(m+\alpha,s,v) - \Phi_\mu(\alpha,s,v) \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t} \\ &\times\left[\int_0^1 \left(e^t - [u(m+\alpha) + (1-u)\alpha]\right)^{-1}\right. \\ &\left.\times(m+\alpha-\alpha)\left(e^t - [uB + (1-u)A]\right)^{-1}du\right]d\mu(t) \\ &= \frac{m}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \alpha - um\right)^{-2}du\right)d\mu(t). \end{aligned} \tag{2.17}$$

By utilising (2.15)-(2.17) we derive (2.10). □

Corollary 3. *With the assumptions of Theorem 2 we have*

$$0 \leq \operatorname{Li}(m+\alpha, s) - \operatorname{Li}(\alpha, s) \leq \operatorname{Li}(B, s) - \operatorname{Li}(A, s) \leq \operatorname{Li}(M+\beta, s) - \operatorname{Li}(\beta, s) \tag{2.18}$$

for $s > 0$.

3. Related Results

We also have:

Theorem 3. *Assume that* $-1 < \alpha \leq A$, $B \leq \beta < 1$ *and* $0 < m \leq B - A \leq M$, *then*

$$0 \leq m \frac{\partial \Phi_\mu(\alpha, s, v)}{\partial z} \leq \Phi_\mu(B, s, v) - \Phi_\mu(A, s, v) \leq M \frac{\partial \Phi_\mu(\beta, s, v)}{\partial z} \tag{3.1}$$

for $v > 0$ *and* $s > 0$.

Proof. Since $-1 < \alpha \leq A$, $B \leq \beta < 1$, hence

$$\alpha \leq uB + (1-u)A \leq \beta,$$

which gives that

$$0 < e^t - \beta \leq e^t - (uB + (1-u)A) \leq e^t - \alpha$$

namely

$$0 < \left(e^t - \alpha\right)^{-1} \leq \left(e^t - [uB + (1-u)A]\right)^{-1} \leq \left(e^t - \beta\right)^{-1}$$

for $u \in [0,1]$ and $t \geq 0$.

By taking the square, we derive

$$0 < \left(e^t - \alpha\right)^{-2} \leq \left(e^t - [uB + (1-u)A]\right)^{-2} \leq \left(e^t - \beta\right)^{-2} \tag{3.2}$$

for $u \in [0,1]$ and $t \geq 0$.

Since $0 < m \leq B - A \leq M$, hence by multiplying both sides by $\left(e^t - [uB + (1-u)A]\right)^{-1}$ we derive

$$\begin{aligned} 0 &< m\left(e^t - [uB + (1-u)A]\right)^{-2} \\ &\leq \left(e^t - [uB + (1-u)A]\right)^{-1}(B-A)\left(e^t - [uB + (1-u)A]\right)^{-1} \\ &\leq M\left(e^t - [uB + (1-u)A]\right)^{-2} \end{aligned} \tag{3.3}$$

for $u \in [0,1]$ and $t \geq 0$.

By (3.2) and (3.3) we get

$$\begin{aligned} 0 &< m\left(e^t - \alpha\right)^{-2} \\ &\leq \left(e^t - [uB + (1-u)A]\right)^{-1}(B-A)\left(e^t - [uB + (1-u)A]\right)^{-1} \\ &\leq M\left(e^t - \beta\right)^{-2} \end{aligned} \tag{3.4}$$

for $u \in [0,1]$ and $t \geq 0$.

By integrating over u, then multiplying by $t^{s-1}e^t \geq 0$ and integrating over t in $[0,\infty)$ we get

$$\begin{aligned} 0 &< m\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \alpha\right)^{-2} du\right) d\mu(t) \\ &\leq \Phi_\mu(B, s, v) - \Phi_\mu(A, s, v) \\ &\leq M\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1 \left(e^t - \beta\right)^{-2} du\right) d\mu(t), \end{aligned}$$

namely

$$0 < m \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(e^t - \alpha\right)^{-2} d\mu(t) \le \Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \le M \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(e^t - \beta\right)^{-2} d\mu(t). \tag{3.5}$$

If we take the partial derivative over z in the definition (1.8) of $\Phi_\mu(z,s,v)$, then we get

$$\frac{\partial \Phi_\mu(\alpha,s,v)}{\partial z} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(v-1)t}}{\left(e^t - \alpha\right)^2} d\mu(t)$$

and

$$\frac{\partial \Phi_\mu(\beta,s,v)}{\partial} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(v-1)t}}{\left(e^t - \beta\right)^2} d\mu(t)$$

for $v > 0$ and $s > 0$.

By utilising (3.5) we derive (3.1). □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$0 \le m \frac{\partial \operatorname{Li}(\alpha,s)}{\partial z} \le \operatorname{Li}(B,s) - \operatorname{Li}(A,s) \le M \frac{\partial \operatorname{Li}(\beta,s)}{\partial z} \tag{3.6}$$

for $s > 0$.

The case of separated operators is as follows:

Theorem 4. *Assume that* A *and* B *satisfy the separation condition* $-1 < \alpha \le A \le \beta < \gamma \le B \le \delta < 1$, *then*

$$(\gamma - \beta) \frac{\Phi_\mu(\gamma,s,v) - \Phi_\mu(\alpha,s,v)}{\gamma - \alpha} \le \Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \le (\delta - \alpha) \frac{\Phi_\mu(\delta,s,v) - \Phi_\mu(\beta,s,v)}{\delta - \beta}. \tag{3.7}$$

Proof. Since $-1 < \alpha \le A \le \beta < \gamma \le B \le \delta < 1$, then

$$u\gamma + (1-u)\alpha \le uB + (1-u)A \le u\delta + (1-u)\beta \tag{3.8}$$

for all $u \in [0,1]$ and

$$0 < \gamma - \beta \le B - A \le \delta - \alpha. \tag{3.9}$$

From (3.8) we have

$$e^t - (u\delta + (1-u)\beta) \le e^t - (uB + (1-u)A) \le e^t - (u\gamma + (1-u)\alpha),$$

namely

$$\left[e^t - (u\gamma + (1-u)\alpha)\right]^{-1} \le \left[e^t - (uB + (1-u)A)\right]^{-1} \le \left[e^t - (u\delta + (1-u)\beta)\right]^{-1},$$

for all $u \in [0,1]$ and $t \ge 0$, which gives that

$$\left[e^t - (u\gamma + (1-u)\alpha)\right]^{-2} \le \left[e^t - (uB + (1-u)A)\right]^{-2} \le \left[e^t - (u\delta + (1-u)\beta)\right]^{-2}, \tag{3.10}$$

for all $u \in [0,1]$ and $t \ge 0$.

If we multiply the inequality (3.9) both sides by $\left[e^t-(uB+(1-u)A)\right]^{-1}$, then we get

$$0<(\gamma-\beta)\left[e^t-(uB+(1-u)A)\right]^{-2} \le \left[e^t-(uB+(1-u)A)\right]^{-1}(B-A)\left[e^t-(uB+(1-u)A)\right]^{-1} \le (\delta-\alpha)\left[e^t-(uB+(1-u)A)\right]^{-2} \tag{3.11}$$

for all $u\in[0,1]$ and $t\ge 0$.

By employing (3.10), we deduce from (3.11) that

$$0<(\gamma-\beta)\left[e^t-(u\gamma+(1-u)\alpha)\right]^{-2} \le \left[e^t-(uB+(1-u)A)\right]^{-1}(B-A)\left[e^t-(uB+(1-u)A)\right]^{-1} \le (\delta-\alpha)\left[e^t-(u\delta+(1-u)\beta)\right]^{-2} \tag{3.12}$$

for all $u\in[0,1]$ and $t\ge 0$.

By integrating over u, then multiplying by $t^{s-1}e^t\ge 0$ and integrating over t in $[0,\infty)$ we get

$$\begin{aligned} 0&<\frac{\gamma-\beta}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1\left[e^t-(u\gamma+(1-u)\alpha)\right]^{-2}du\right)d\mu(t)\\ &\le \Phi_\mu(B,s,v)-\Phi_\mu(A,s,v)\\ &\le \frac{\delta-\alpha}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1\left[e^t-(u\delta+(1-u)\beta)\right]^{-2}du\right)d\mu(t). \end{aligned} \tag{3.13}$$

Further, if we use the identity (2.11) for $B=\gamma$ and $A=\alpha$, we get

$$\begin{aligned} &\Phi_\mu(\gamma,s,v)-\Phi_\mu(\alpha,s,v)\\ &=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\\ &\times\left(\int_0^1\left(e^t-[u\gamma+(1-u)\alpha]\right)^{-1}\left(e^t-[u\gamma+(1-u)\alpha]\right)^{-1}du\right)d\mu(t)\\ &=\frac{\gamma-\alpha}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1\left(e^t-[u\gamma+(1-u)\alpha]\right)^{-2}du\right)d\mu(t), \end{aligned}$$

which gives that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1\left(e^t-[u\gamma+(1-u)\alpha]\right)^{-2}du\right)d\mu(t) = \frac{\Phi_\mu(\gamma,s,v)-\Phi_\mu(\alpha,s,v)}{\gamma-\alpha}. \tag{3.14}$$

By the identity (2.11) for $B=\delta$ and $A=\beta$, we also get

$$\begin{aligned}
&\Phi_{\mu}(\delta,s,v)-\Phi_{\mu}(\beta,s,v)\\
&=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(v-1)t}\\
&\times\left(\int_0^1\left(e^t-[u\delta+(1-u)\beta]\right)^{-1}(\delta-\beta)\left(e^t-[u\delta+(1-u)\beta]\right)^{-1}du\right)d\mu(t)\\
&=\frac{\delta-\beta}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(v-1)t}\left(\int_0^1\left(e^t-[u\delta+(1-u)\beta]\right)^{-2}du\right)d\mu(t),
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(v-1)t}\left(\int_0^1\left(e^t-[u\delta+(1-u)\beta]\right)^{-2}du\right)d\mu(t)\\
&\quad=\frac{\Phi_{\mu}(\delta,s,v)-\Phi_{\mu}(\beta,s,v)}{\delta-\beta}.
\end{aligned}\tag{3.15}$$

By utilising (3.13)-(3.15) we derive (3.7). □

Corollary 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
0&\le(\gamma-\beta)\frac{\operatorname{Li}(\gamma,s)-\operatorname{Li}(\alpha,s)}{\gamma-\alpha}\le\operatorname{Li}(B,s)-\operatorname{Li}(A,s)\\
&\le(\delta-\alpha)\frac{\operatorname{Li}(\delta,s)-\operatorname{Li}(\beta,s)}{\delta-\beta}.
\end{aligned}\tag{3.16}$$

References

[1] T. M. Apostol, Zeta and related functions, in: F. W. F. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, National Institute of Standards and Technology, Cambridge, Washington, DC, 2010, pp. 601–616. Available online in http://dlmf.nist.gov/25.

[2] P. Cerone and S. S. Dragomir, Inequalities for Dirichlet series with positive terms, *Acta Math. Vietnam.* **33** (2008), no. 1, 65–84.

[3] S. S. Dragomir, On some inequalities (Romanian), Caiete Metodico-Stiintifice, No. 13, 1984, pp. 20. Faculty of Mathematics, Timisoara University, Romania.

[4] S. S. Dragomir, A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequality, *J. Ineq. Pure & Appl. Math.* **4** (2003), No. 3, Art. 63, pp. 142. [Online http://www.emis.de/journals/JIPAM/article301.html?sid=301].

[5] S. S. Dragomir and J. Sándor, Some generalisations of Cauchy-Buniakowski-Schwartz's inequality (Romanian), *Gaz. Mat. Metod.* (Bucharest) **11** (1990), 104–109.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[7] A. Laforgia and P. Natalini, Turán-type inequalities for some special functions, *J. Inequal. Pure & Appl. Math.*, **7** (2006),Issue 1, Article 22.

[8] M. Lerch, Note sur la fonction $\mathfrak{k}(w,x,s)=\sum_{k=0}^{\infty}e^{2\pi ikx}(w+k)^{-s}$, *Acta Math.* **11** (1887) 19–24.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.