QUADRATIC LIPSCHITZIAN INEQUALITIES FOR THE $\mathcal{AT}$-INTEGRAL TRANSFORM OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. For μ a positive measure on $[0,1]$ and $p:[0,1]\to[0,\infty)$ a continuous function, we consider the following $\mathcal{AT}$*-integral transform*

$$\mathcal{AT}(p,\mu)(T):=\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\mu(\lambda),$$

where the integral is assumed to exist for T a selfadjoint operator on a complex Hilbert space H.

In this paper, we show among others that, if A, B are selfadjoint operators with $A^2\geq m_1^2$, $B^2\geq m_2^2$ and m_1, $m_2>0$, then

$$\begin{aligned}&\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\|\\ &\leq\left\|B^2-A^2\right\|\begin{cases}\frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} \text{ if } m_2^2\neq m_1^2,\\ \\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) \text{ if } m_2^2=m_1^2=m^2,\end{cases}\end{aligned}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t. Some examples for integral transforms $\mathcal{AT}(p,\mu)$ related to the arctan function are also provided.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

for any A, $B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [11], the following inequality holds

$$\||A|-|B|\|\leq\frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.1}$$

for any A, $B\in\mathcal{B}(H)$ with $A\neq B$.

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator inequalities, Lipschitzian inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 132, 11 pp. Received 05/11/20

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \le \sqrt{2}\,\|A - B\|_{HS}$$

for any $A,\ B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \le a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right)$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \le f'(a)\|A - B\|$$

where f is an *operator monotone function* on $(0, \infty)$ and $A,\ B \ge a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.2}$$

Observe that for $t > 0$, $t \ne 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.3}$$

for all $t > 0$.

Motivated by these representations, we introduced in [6], for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda),\ t > 0, \tag{1.4}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$.

For μ the Lebesgue measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda,\ t > 0. \tag{1.5}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t),\ t > 0. \tag{1.6}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, then we have the representation

$$\ln t = (t - 1)\mathcal{D}(w_{\ln})(t),\ t > 0. \tag{1.7}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator [6]

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.8}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.9}$$

for $T > 0$.

We also defined the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by [7]

$$\mathcal{L}og(w, \mu)(t) := \int_0^\infty w(\lambda)\ln(\lambda + t) d\mu(\lambda), \tag{1.10}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t) := \int_0^\infty w(\lambda)\ln(\lambda + t) d\lambda. \tag{1.11}$$

Now, for a continuous and positive function $p(\lambda)$, $\lambda \in [0, 1]$ and a positive measure μ on $[0, 1]$, we define the integral transform

$$\mathcal{AT}(p, \mu)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2 + t^{-2}} d\mu(\lambda),\ t \neq 0, \\ 0,\ t = 0. \end{cases} \tag{1.12}$$

If μ is the usual Lebesgue measure, then we put

$$\mathcal{AT}(p)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2 + t^{-2}} d\lambda,\ t \neq 0, \\ 0,\ t = 0. \end{cases} \tag{1.13}$$

For $p(\lambda) = 1$, $\lambda \in [0, 1]$ and μ is the usual Lebesgue measure, we have

$$\mathcal{AT}(t) := \begin{cases} \int_0^1 \frac{1}{\lambda^2 + t^{-2}} d\lambda,\ t \neq 0, \\ 0,\ t = 0. \end{cases} = t\arctan t,\ t \in \mathbb{R}. \tag{1.14}$$

In the case when $p(\lambda) = \ell^2(\lambda) = \lambda^2$, we derive

$$\mathcal{AT}\left(\ell^2\right)(t) = \begin{cases} \int_0^1 \frac{\lambda^2}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ \\ 0, \ t = 0. \end{cases} = \begin{cases} 1 - \frac{\arctan t}{t}, \ t \neq 0, \\ \\ 0, \ t = 0. \end{cases}$$

We observe that (1.12) can be written in equivalent form as

$$\begin{aligned} \mathcal{AT}(p,\mu)(t) &:= \int_0^1 \frac{t^2 p(\lambda)}{t^2\lambda^2+1} d\mu(\lambda) \\ &= \int_0^1 \left(1 - \frac{1}{t^2\lambda^2+1}\right) \frac{p(\lambda)}{\lambda^2} d\mu(\lambda), \ t \in \mathbb{R}. \end{aligned} \tag{1.15}$$

For a selfadjoint operator T we have that $\lambda^2 T^2 + 1$ is invertible for all $\lambda \in [0,1]$ and by the continuous functional calculus for selfadjoint operators we can define the $\mathcal{AT}$-*operator transform* by

$$\mathcal{AT}(p,\mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\mu(\lambda), \tag{1.16}$$

and, in particular, for the Lebesgue measure

$$\mathcal{AT}(p)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda. \tag{1.17}$$

We observe that

$$\mathcal{AT}(T) = \int_0^1 \frac{1}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = T \arctan T,$$

for any selfadjoint operator T.

If T is invertible, then

$$\mathcal{AT}\left(\ell^2\right)(T) = \int_0^1 \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = 1 - T^{-1} \arctan T.$$

In this paper, we show among others that, if A, B are selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and m_1, $m_2 > 0$, then

$$\begin{aligned} &\left\|\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)\right\| \\ &\leq \left\|B^2 - A^2\right\| \begin{cases} \frac{\mathcal{AT}(p,\mu)(m_2) - \mathcal{AT}(p,\mu)(m_1)}{m_2^2 - m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m} \mathcal{AT}'(p,\mu)(m) \text{ if } m_2^2 = m_1^2 = m^2, \end{cases} \end{aligned}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t. Some examples for integral transforms $\mathcal{AT}(p,\mu)$ related to the arctan function are also provided.

2. Main Results

We have the following identity of interest for the difference $\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)$.

Lemma 1. *Let A, B be selfadjoint operators, then*

$$\begin{aligned} &\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A) \\ &=\int_0^1 p(\lambda)\left[\int_0^1 \left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right. \\ &\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda). \end{aligned} \tag{2.1}$$

If $B^2\geq A^2$, then

$$\mathcal{AT}(p,\mu)(B)\geq\mathcal{AT}(p,\mu)(A). \tag{2.2}$$

Proof. Let A, B selfadjoint operators. By employing (1.16) we have

$$\begin{aligned} &\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A) \\ &=\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda) \\ &-\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2A^2+1\right)^{-1}\right]d\mu(\lambda) \\ &=\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda). \end{aligned} \tag{2.3}$$

The function $g(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S):=\lim_{t\to 0}\left[\frac{g(T+tS)-g(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$g(D)-g(C)=\int_0^1\frac{d}{dt}(g_{C,D}(t))\,dt=\int_0^1\nabla g_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $g(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

If in (2.6) we take $C=\lambda^2A^2+1$ and $D=\lambda^2B^2+1$, then we get

$$\begin{aligned}(2.7)\qquad &\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\\ &=\int_0^1\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}\left(\lambda^2B^2-\lambda^2A^2\right)\\ &\times\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}dt\\ &=\lambda^2\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\\ &\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\end{aligned}$$

for all $t,\ \lambda\in[0,1]$.

Therefore

$$\begin{aligned}&\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\ &=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\ &=\int_0^1p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)\end{aligned}$$

and by (2.3) we get the representation (2.1).

If $B^2-A^2\geq0$, then by multiplying both sides by $\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}$ we get

$$\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\geq0$$

for all $t,\ \lambda\in[0,1]$.

If we integrate over $t\in[0,1]$, then multiply by $p(\lambda)\geq0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have

$$\begin{aligned}&\int_0^1p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)\geq0\end{aligned}$$

and the operator quadratic monotonicity of the transform $\mathcal{AT}(p,\mu)$ is proved. $\square$

We have the following main result:

Theorem 1. *For A, B selfadjoint operators with $A^2\geq m_1^2$, $B^2\geq m_2^2$ and m_1, $m_2>0$, we have*

$$\begin{aligned}(2.8)\qquad &\left\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\right\|\\ &\leq\left\|B^2-A^2\right\|\begin{cases}\frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} & \text{if } m_2^2\neq m_1^2,\\ \\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) & \text{if } m_2^2=m_1^2=m^2,\end{cases}\end{aligned}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t.

Proof. From the identity (2.1), we get, by taking the norm, that

$$
\begin{aligned}
&\left\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\right\| \\
&\le \int_0^1 p(\lambda)\left[\int_0^1 \left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\right. \\
&\left.\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\| dt\right] d\mu(\lambda) \\
&\le \left\|B^2-A^2\right\| \\
&\times \int_0^1 p(\lambda)\left(\int_0^1 \left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2 dt\right) d\mu(\lambda)
\end{aligned}
\tag{2.9}
$$

for A, B selfadjoint operators.

Assume that $m_1^2 < m_2^2$, then

$$
(1-t)m_1^2+tm_2^2 \le (1-t)A^2+tB^2
$$

for $t\in[0,1]$, which implies that

$$
1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right] \le 1+\lambda^2\left[(1-t)A^2+tB^2\right],
$$

namely

$$
\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1} \le \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1}
$$

for $t,\ \lambda\in[0,1]$.

This implies that

$$
\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\| \le \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1},
$$

which implies that

$$
\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2 \le \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2},
\tag{2.10}
$$

for $t,\ \lambda\in[0,1]$.

If we integrate (2.10) over $t\in[0,1]$, then multiply by $p(\lambda)\ge 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have by (2.1) that,

$$
\begin{aligned}
&\int_0^1 p(\lambda)\left(\int_0^1 \left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2 dt\right) d\mu(\lambda) \\
&\le \int_0^1 p(\lambda)\left(\int_0^1 \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2} dt\right) d\mu(\lambda).
\end{aligned}
\tag{2.11}
$$

If we use the identity (2.1) for $A=m_1$ and $B=m_2$, then we get

$$
\begin{aligned}
&\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1) \\
&= \int_0^1 p(\lambda)\left[\int_0^1 \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1}\left(m_2^2-m_1^2\right)\right. \\
&\left.\times\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1} dt\right] d\mu(\lambda) \\
&= \left(m_2^2-m_1^2\right)\int_0^1 p(\lambda)\left(\int_0^1 \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2} dt\right) d\mu(\lambda),
\end{aligned}
$$

which gives that

$$\int_0^1 p(\lambda)\left(\int_0^1 \left(1+\lambda^2\left[(1-t)\,m_1^2+tm_2^2\right]\right)^{-2} dt\right) d\mu(\lambda)$$
$$= \frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2}$$

and by (2.9) and (2.11), we get the first inequality in (2.8).

If $m_1^2 > m_2^2$, then we can prove (2.8) in a similar way.

Assume that $m_1^2 = m_2^2 = m^2$ with $m > 0$. Let $\varepsilon > 0$ such that $m^2 - \varepsilon > 0$. Then $B^2 \geq m^2$ and $A^2 \geq m^2 > m^2 - \varepsilon > 0$. If we take $m_2^2 := m^2$, $m_1^2 = m^2 - \varepsilon > 0$, then $m_2^2 > m_1^2$ and by the first inequality in (2.8,)

$$\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\|$$
$$\leq \left\|B^2-A^2\right\| \frac{\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{\varepsilon}$$

and by taking the limit over $\varepsilon \to 0+$, then we get

$$\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\| \tag{2.12}$$
$$\leq \left\|B^2-A^2\right\| \lim_{\varepsilon\to 0+}\frac{\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{\varepsilon}$$
$$= \left\|B^2-A^2\right\| \lim_{\varepsilon\to 0+}\frac{\mathcal{AT}(p,\mu)\,\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{m-\sqrt{m^2-\varepsilon}}$$
$$\times \lim_{\varepsilon\to 0+}\frac{m-\sqrt{m^2-\varepsilon}}{\varepsilon}.$$

Since

$$\lim_{\varepsilon\to 0+}\frac{\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}-m+m\right)}{m-\sqrt{m^2-\varepsilon}}$$
$$= \lim_{h\to 0+}\frac{\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)(m-h)}{h} = \mathcal{AT}'(p,\mu)(m)$$

and

$$\lim_{\varepsilon\to 0+}\frac{m-\sqrt{m^2-\varepsilon}}{\varepsilon} = \lim_{\varepsilon\to 0+}\frac{1}{m+\sqrt{m^2-\varepsilon}} = \frac{1}{2m},$$

hence by (2.12) we obtain the second part of (2.8). □

Remark 1. *For A, B selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and m_1, $m_2 > 0$, we have*

$$\|B\arctan(B)-A\arctan(A)\| \tag{2.13}$$
$$\leq \left\|B^2-A^2\right\| \begin{cases} \frac{m_2\arctan(m_2)-m_1\arctan(m_1)}{m_2^2-m_1^2} & \text{if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m}\left(\arctan m + \frac{m}{m^2+1}\right) & \text{if } m_2^2 = m_1^2 = m^2. \end{cases}$$

For any A, B selfadjoint and invertible operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and m_1, $m_2 > 0$, we have

$$\left\|A^{-1}\arctan A - B^{-1}\arctan B\right\| \leq \left\|B^2 - A^2\right\| \begin{cases} \frac{m_1^{-1}\arctan(m_1) - m_2^{-1}\arctan(m_2)}{m_2^2 - m_1^2} & \text{if } m_2^2 \neq m_1^2, \\ \frac{1}{2m^3}\left(\arctan m - \frac{m}{m^2+1}\right) & \text{if } m_2^2 = m_1^2 = m^2. \end{cases} \tag{2.14}$$

3. A Hermite-Hadamard Type Inequality

We have:

Proposition 1. *For A, B selfadjoint operators, assume that $((1-s)A + sB)^2 \geq m^2$ for $s \in [0,1]$ with $m > 0$, then*

$$\begin{aligned} &\left\|\mathcal{AT}(p,\mu)((1-s)A + sB) - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1\left\|((1-s)A + sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|ds \\ &\leq \frac{1}{12m}\left(\|A\|^2 + \frac{1}{2}\|AB + BA\| + \|B\|^2\right)\mathcal{AT}'(p,\mu)(m). \end{aligned} \tag{3.1}$$

Proof. Since

$$\left(\frac{A+B}{2}\right)^2 \geq m^2,$$

then by (2.8) we have

$$\begin{aligned} &\left\|\mathcal{AT}(p,\mu)((1-s)A + sB) - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \frac{1}{2m}\left\|((1-s)A + sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|\mathcal{AT}'(p,\mu)(m), \end{aligned}$$

for $s \in [0,1]$.

If we take the integral and use the norm properties, then we get

$$\begin{aligned} &\left\|\int_0^1\mathcal{AT}(p,\mu)((1-s)A + sB)\,ds - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \int_0^1\left\|\mathcal{AT}(p,\mu)((1-s)A + sB) - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|ds \\ &\leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1\left\|((1-s)A + sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|ds. \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned}
&((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\\
&=(1-s)^2A^2+s(1-s)(AB+BA)+s^2B^2\\
&-\frac{1}{4}\left(A^2+AB+BA+B^2\right)\\
&=\left[(1-s)^2-\frac{1}{4}\right]A^2+\left[s(1-s)-\frac{1}{4}\right](AB+BA)+\left(s^2-\frac{1}{4}\right)B^2
\end{aligned}$$

for $s\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned}
&\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|\\
&\leq\left|(1-s)^2-\frac{1}{4}\right|\|A\|^2+\left|s(1-s)-\frac{1}{4}\right|\|AB+BA\|\\
&+\left|s^2-\frac{1}{4}\right|\|B\|^2
\end{aligned}$$

and, by integration, we derive

$$\begin{aligned}
&\int_0^1\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|ds \qquad (3.3)\\
&\leq\int_0^1\left|(1-s)^2-\frac{1}{4}\right|ds\|A\|^2+\int_0^1\left|s(1-s)-\frac{1}{4}\right|ds\|AB+BA\|\\
&+\int_0^1\left|s^2-\frac{1}{4}\right|ds\|B\|^2\\
&=\frac{1}{6}\|A\|^2+\frac{1}{12}\|AB+BA\|+\frac{1}{6}\|B\|^2.
\end{aligned}$$

By making use of the inequalities (3.2) and (3.3), we derive the desired result (3.1). □

Remark 2. *We observe that, if* $A, B\geq m>0$, *then* $(1-s)A+sB\geq m>0$, *which implies that* $((1-s)A+sB)^2\geq m^2$ *for* $s\in[0,1]$.

With the assumptions in Proposition 1, we have

$$\begin{aligned}
&\left\|\int_0^1((1-s)A+sB)\arctan((1-s)A+sB)\,ds\right. \qquad (3.4)\\
&\left.-\left(\frac{A+B}{2}\right)\arctan\left(\frac{A+B}{2}\right)\right\|\\
&\leq\frac{1}{2m}\left(\arctan m+\frac{m}{m^2+1}\right)\int_0^1\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|ds\\
&\leq\frac{1}{12m}\left(\arctan m+\frac{m}{m^2+1}\right)\left(\|A\|^2+\frac{1}{2}\|AB+BA\|+\|B\|^2\right).
\end{aligned}$$

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