Abstract. For a positive measure on $[0, 1]$ and $p : [0, 1] \to [0, \infty)$ a continuous function, we consider the following $AT$-integral transform

$$AT(p, \mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[ 1 - (\lambda^2 T^2 + 1)^{-1} \right] d\mu(\lambda),$$

where the integral is assumed to exist for $T$ a selfadjoint operator on a complex Hilbert space $H$.

In this paper, we show among others that, if $A, B$ are selfadjoint operators with $A^2 \geq m_1^2, B^2 \geq m_2^2$ and $m_1, m_2 > 0$, then

$$\|AT(p, \mu)(B) - AT(p, \mu)(A)\| \leq \|B^2 - A^2\| \left\{ \begin{array}{l} \frac{AT(p, \mu)(m_2) - AT(p, \mu)(m_1)}{m_2^2 - m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \frac{1}{2m_2} AT'(p, \mu)(m) \text{ if } m_2^2 = m_1^2 = m_2, \end{array} \right.$$

where $AT'(p, \mu)(t)$ is the derivative of $AT(p, \mu)$ as a function of $t$. Some examples for integral transforms $AT(p, \mu)$ related to the arctan function are also provided.

1. Introduction

Let $B(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space $H$. The absolute value of an operator $A$ is the positive operator $|A|$ defined as $|A| := (A^* A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not Lipschitz continuous on $B(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\|A - |B|\| \leq L \|A - B\|$$

for any $A, B \in B(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [11], the following inequality holds

$$\|A - |B|\| \leq \frac{2}{\pi} \|A - B\| \left( 2 + \log \left( \frac{\|A\| + \|B\|}{\|A - B\|} \right) \right)$$

for any $A, B \in B(H)$ with $A \neq B$.
If the operator norm is replaced with Hilbert-Schmidt norm $\|C\|_{HS} := (\text{tr} C^* C)^{1/2}$ of an operator $C$, then the following inequality is true [1]

$$\|A - B\|_{HS} \leq \sqrt{2} \|A - B\|_{HS}$$

for any $A, B \in B(H)$.

The coefficient $\sqrt{2}$ is best possible for a general $A$ and $B$. If $A$ and $B$ are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if $A$ is an invertible operator, then for all operators $B$ in a neighborhood of $A$ we have

$$\|A - B\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O(\|A - B\|^3)$$

where

$$a_1 = \|A^{-1}\| \|A\| \quad \text{and} \quad a_2 = \|A^{-1}\| + \|A^{-1}\|^3 \|A\|^2.$$

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $\langle T x, x \rangle \geq 0$ for all $x \in H$ and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

In [2] the author also obtained the following Lipschitz type inequality

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\|$$

where $f$ is an operator monotone function on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions $f$ for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin (r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \quad (1.2)$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left( \frac{u + t}{u + 1} \right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \quad \text{for all } t > 0. \quad (1.3)$$

Motivated by these representations, we introduced in [6], for a continuous and positive function $w(\lambda), \lambda > 0$, the following integral transform

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0, \quad (1.4)$$

where $\mu$ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$. 

For $\mu$ the Lebesgue measure, we put

$$D(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0.$$  \hfill (1.5)

If we take $\mu$ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} D(w_r)(t), \quad t > 0.$$  \hfill (1.6)

For the same measure, if we take the kernel $w_{in}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, then we have the representation

$$\ln t = (t - 1) D(w_{in})(t), \quad t > 0.$$  \hfill (1.7)

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator [6]

$$D(w, \mu)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),$$  \hfill (1.8)

where $w$ and $\mu$ are as above. Also, when $\mu$ is the usual Lebesgue measure, then

$$D(w)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\lambda,$$  \hfill (1.9)

for $T > 0$.

We also defined the logarithmic transform for a continuous and positive function $w(\lambda), \lambda > 0$ by [7]

$$\text{Log}(w, \mu)(t) := \int_0^\infty w(\lambda) \ln(\lambda + t) d\mu(\lambda),$$  \hfill (1.10)

where $\mu$ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$. Also, when $\mu$ is the usual Lebesgue measure, then

$$\text{Log}(w)(t) := \int_0^\infty w(\lambda) \ln(\lambda + t) d\lambda,$$  \hfill (1.11)

Now, for a continuous and positive function $p(\lambda), \lambda \in [0, 1]$ and a positive measure $\mu$ on $[0, 1]$, we define the integral transform

$$\mathcal{AT}(p, \mu)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^{t+2}} d\mu(\lambda), & t \neq 0, \\ 0, & t = 0. \end{cases}$$  \hfill (1.12)

If $\mu$ is the usual Lebesgue measure, then we put

$$\mathcal{AT}(p)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^{t+2}} d\lambda, & t \neq 0, \\ 0, & t = 0. \end{cases}$$  \hfill (1.13)

For $p(\lambda) = 1, \lambda \in [0, 1]$ and $\mu$ is the usual Lebesgue measure, we have

$$\mathcal{AT}(t) := \begin{cases} \int_0^1 \frac{1}{\lambda^{t+2}} d\lambda, & t \neq 0, \\ 0, & t = 0. \end{cases} = t \arctan t, \quad t \in \mathbb{R}.$$  \hfill (1.14)
In the case when \( p(\lambda) = \ell^2(\lambda) = \lambda^2 \), we derive

\[
\mathcal{A}T(\ell^2)(t) = \begin{cases} 
\int_0^t \frac{\lambda^2}{\lambda^2 + t^2} d\lambda, & t \neq 0, \\
1 - \frac{\arctan t}{t}, & t \neq 0, \\
0, & t = 0.
\end{cases}
\]

We observe that (1.12) can be written in equivalent form as

\[
(1.15) \quad \mathcal{A}T(p; \mu)(t) := \int_0^1 \frac{t^2 p(\lambda)}{t^2 \lambda^2 + 1} d\mu(\lambda)
= \int_0^1 \left(1 - \frac{1}{t^2 \lambda^2 + 1}\right) \frac{p(\lambda)}{\lambda^2} d\mu(\lambda), \quad t \in \mathbb{R}.
\]

For a selfadjoint operator \( T \) we have that \( \lambda^2 T^2 + 1 \) is invertible for all \( \lambda \in [0,1] \) and by the continuous functional calculus for selfadjoint operators we can define the \( \mathcal{A}T \)-operator transform by

\[
(1.16) \quad \mathcal{A}T(p; \mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - (\lambda^2 T^2 + 1)^{-1}\right] d\mu(\lambda),
\]

and, in particular, for the Lebesgue measure

\[
(1.17) \quad \mathcal{A}T(p)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - (\lambda^2 T^2 + 1)^{-1}\right] d\lambda.
\]

We observe that

\[
\mathcal{A}T(T) = \int_0^1 \frac{1}{\lambda^2} \left[1 - (\lambda^2 T^2 + 1)^{-1}\right] d\lambda = T \arctan T,
\]

for any selfadjoint operator \( T \).

If \( T \) is invertible, then

\[
\mathcal{A}T(\ell^2)(T) = \int_0^1 \left[1 - (\lambda^2 T^2 + 1)^{-1}\right] d\lambda = 1 - T^{-1} \arctan T.
\]

In this paper, we show among others that, if \( A, B \) are selfadjoint operators with \( A^2 \geq m_1^2, B^2 \geq m_2^2 \) and \( m_1, m_2 > 0 \), then

\[
\| \mathcal{A}T(p; \mu)(B) - \mathcal{A}T(p; \mu)(A) \|
\leq \| B^2 - A^2 \|egin{cases} 
\frac{\mathcal{A}T(p; \mu)(m_2) - \mathcal{A}T(p; \mu)(m_1)}{m_2^2 - m_1^2} \quad \text{if } m_2^2 \neq m_1^2, \\
\frac{1}{2m_1} \mathcal{A}T'(p; \mu)(m) \quad \text{if } m_2^2 = m_1^2 = m^2,
\end{cases}
\]

where \( \mathcal{A}T'(p; \mu)(t) \) is the derivative of \( \mathcal{A}T(p; \mu) \) as a function of \( t \). Some examples for integral transforms \( \mathcal{A}T(p; \mu) \) related to the \( \arctan \) function are also provided.

2. Main Results

We have the following identity of interest for the difference \( \mathcal{A}T(p; \mu)(B) - \mathcal{A}T(p; \mu)(A) \).
Lemma 1. Let $A, B$ be selfadjoint operators, then

\begin{align}
(2.1) \quad & \mathcal{A}T(p, \mu)(B) - \mathcal{A}T(p, \mu)(A) \\
& = \int_0^1 p(\lambda) \left[ \int_0^1 \frac{d\lambda}{\lambda^2} \left( (1 + \lambda^2 [(1 - t) A^2 + tB^2])^{-1} (B^2 - A^2) \\
& \times (1 + \lambda^2 [(1 - t) A^2 + tB^2])^{-1} dt \right] d\mu(\lambda) .
\end{align}

If $B^2 \geq A^2$, then

\begin{align}
(2.2) \quad & \mathcal{A}T(p, \mu)(B) \geq \mathcal{A}T(p, \mu)(A) .
\end{align}

Proof. Let $A, B$ selfadjoint operators. By employing (1.16) we have

\begin{align}
(2.3) \quad & \mathcal{A}T(p, \mu)(B) - \mathcal{A}T(p, \mu)(A) \\
& = \int_0^1 p(\lambda) \left[ \int_0^1 \frac{d\lambda}{\lambda^2} \left( 1 - (\lambda^2 B^2 + 1)^{-1} \right) d\mu(\lambda) \\
& - \int_0^1 \frac{d\lambda}{\lambda^2} \left( 1 - (\lambda^2 A^2 + 1)^{-1} \right) d\mu(\lambda) \\
& = \int_0^1 \frac{d\lambda}{\lambda^2} \left( (\lambda^2 A^2 + 1)^{-1} - (\lambda^2 B^2 + 1)^{-1} \right) d\mu(\lambda) .
\end{align}

The function $g(t) = -t^{-1}$ is operator monotone on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

\begin{align}
(2.4) \quad & \nabla g_T(S) := \lim_{t \to 0} \left[ \frac{g(T + tS) - g(T)}{t} \right] = T^{-1} ST^{-1}
\end{align}

for $T, S > 0$.

Consider the continuous function $g$ defined on an interval $I$ for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D] := \{(1 - t) C + tD, \ t \in [0, 1]\}$ for $C, D$ selfadjoint operators with spectra in $I$. We consider the auxiliary function defined on $[0, 1]$ by

\[ g_{C,D}(t) := f((1-t)C + tD), \ t \in [0, 1] . \]

Then we have, by the properties of the Bochner integral, that

\begin{align}
(2.5) \quad & g(D) - g(C) = \int_0^1 \frac{d}{dt} (g_{C,D}(t)) dt = \int_0^1 \nabla g_{(1-t)C+tD}((1-t)C + tD) dt .
\end{align}

If we write this equality for the function $g(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

\begin{align}
(2.6) \quad & C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt .
\end{align}
If in (2.6) we take \( C = \lambda^2 A^2 + 1 \) and \( D = \lambda^2 B^2 + 1 \), then we get

\[
(\lambda^2 A^2 + 1)^{-1} - (\lambda^2 B^2 + 1)^{-1} = \int_0^1 \frac{p(\lambda)}{\lambda} \left[ (\lambda^2 A^2 + 1)^{-1} - (\lambda^2 B^2 + 1)^{-1} \right] d\mu(\lambda)
\]

\[
= \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[ (\lambda^2 A^2 + 1)^{-1} - (\lambda^2 B^2 + 1)^{-1} \right] d\mu(\lambda)
\]

\[
= \int_0^1 p(\lambda) \left[ \int_0^1 (1 + \lambda^2 [(1 - t) A^2 + t B^2])^{-1} (B^2 - A^2) dt \right] d\mu(\lambda)
\]

for all \( t, \lambda \in [0, 1] \).

Therefore

\[
\int_0^1 \frac{p(\lambda)}{\lambda^2} \left[ (\lambda^2 A^2 + 1)^{-1} - (\lambda^2 B^2 + 1)^{-1} \right] d\mu(\lambda)
\]

and by (2.3) we get the representation (2.1).

If \( B^2 - A^2 \geq 0 \), then by multiplying both sides by \( (1 + \lambda^2 [(1 - t) A^2 + t B^2])^{-1} \) we get

\[
(1 + \lambda^2 [(1 - t) A^2 + t B^2])^{-1} (B^2 - A^2) (1 + \lambda^2 [(1 - t) A^2 + t B^2])^{-1} \geq 0
\]

for all \( t, \lambda \in [0, 1] \).

If we integrate over \( t \in [0, 1] \), then multiply by \( p(\lambda) \geq 0 \) and integrate over the positive measure \( d\mu(\lambda) \) on \([0, 1] \), we have

\[
\int_0^1 p(\lambda) \left[ \int_0^1 (1 + \lambda^2 [(1 - t) A^2 + t B^2])^{-1} (B^2 - A^2) dt \right] d\mu(\lambda) \geq 0
\]

and the operator quadratic monotonicity of the transform \( \mathcal{A}T \) is proved.

We have the following main result:

**Theorem 1.** For \( A, B \) selfadjoint operators with \( A^2 \geq m_1^2, B^2 \geq m_2^2 \) and \( m_1, m_2 > 0 \), we have

\[
\| \mathcal{A}T (p, \mu) (B) - \mathcal{A}T (p, \mu) (A) \|
\]

\[
\leq \| B^2 - A^2 \| \left\{ \begin{array}{ll}
\frac{\mathcal{A}T(p,\mu)(m_2) - \mathcal{A}T(p,\mu)(m_1)}{m_2^2 - m_1^2} & \text{if } m_2^2 \neq m_1^2, \\
\frac{1}{2m} \mathcal{A}T' (p, \mu) (m) & \text{if } m_2^2 = m_1^2 = m^2,
\end{array} \right.
\]

where \( \mathcal{A}T' (p, \mu) (t) \) is the derivative of \( \mathcal{A}T (p, \mu) \) as a function of \( t \).
Proof. From the identity (2.1), we get, by taking the norm, that

\[ \| \mathcal{A}T (p, \mu) (B) - \mathcal{A}T (p, \mu) (A) \| 
\leq \int_0^1 p(\lambda) \left[ \int_0^1 \|(1 + \lambda^2 [(1 - t) A^2 + tB^2])^{-1} (B^2 - A^2) \times (1 + \lambda^2 [(1 - t) A^2 + tB^2])^{-1} \| dt \right] d\mu(\lambda) 
\leq \| B^2 - A^2 \| 
\times \int_0^1 p(\lambda) \left( \int_0^1 \|(1 + \lambda^2 [(1 - t) A^2 + tB^2])^{-1} \|^2 dt \right) d\mu(\lambda) \]

for \( A, B \) selfadjoint operators.

Assume that \( m_1^2 < m_2^2 \), then

\[ (1 - t) m_1^2 + tm_2^2 \leq (1 - t) A^2 + tB^2 \]

for \( t \in [0, 1] \), which implies that

\[ 1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right) \leq 1 + \lambda^2 \left( (1 - t) A^2 + tB^2 \right) \]

namely

\[ (1 + \lambda^2 \left( (1 - t) A^2 + tB^2 \right))^{-1} \leq (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-1} \]

for \( t, \lambda \in [0, 1] \).

This implies that

\[ \left\| (1 + \lambda^2 \left( (1 - t) A^2 + tB^2 \right))^{-1} \right\| \leq (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-1} \],

which implies that

\[ \left(1 + \lambda^2 \left( (1 - t) A^2 + tB^2 \right) \right)^{-1} \| \right|^2 \leq (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-2} \]

for \( t, \lambda \in [0, 1] \).

If we integrate (2.10) over \( t \in [0, 1] \), then multiply by \( p(\lambda) \geq 0 \) and integrate over the positive measure \( d\mu(\lambda) \) on \([0, 1] \), we have by (2.1) that,

\[ \int_0^1 p(\lambda) \left( \int_0^1 \left\| (1 + \lambda^2 \left( (1 - t) A^2 + tB^2 \right))^{-1} \right\|^2 dt \right) d\mu(\lambda) \leq \int_0^1 p(\lambda) \left( \int_0^1 (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-2} dt \right) d\mu(\lambda) \]

If we use the identity (2.1) for \( A = m_1 \) and \( B = m_2 \), then we get

\[ \mathcal{A}T (p, \mu) (m_2) - \mathcal{A}T (p, \mu) (m_1) \]

\[ \int_0^1 p(\lambda) \left[ \int_0^1 (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-1} \left( m_2^2 - m_1^2 \right) \times (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-1} dt \right] d\mu(\lambda) \]

\[ = (m_2^2 - m_1^2) \int_0^1 p(\lambda) \left( \int_0^1 (1 + \lambda^2 \left( (1 - t) m_1^2 + tm_2^2 \right))^{-2} dt \right) d\mu(\lambda) \]
which gives that
\[
\int_0^1 p(\lambda) \left( \int_0^1 \left(1 + \lambda^2 \left[(1-t) m_0^2 + tm_2^2\right]\right)^{-2} dt \right) d\mu(\lambda) = \frac{\mathcal{A}T(p,\mu)(m_2) - \mathcal{A}T(p,\mu)(m_1)}{m_2^2 - m_1^2}
\]
and by (2.9) and (2.11), we get the first inequality in (2.8).

If \( m_2^2 > m_2^2 \), then we can prove (2.8) in a similar way.

Assume that \( m_1^2 = m_2^2 = m_2^2 > 0 \), and by (2.9) and (2.11), we get the first inequality in (2.8).

If we take \( m_2^2 := m_2^2 > m_2^2 > m_2^2 > 0 \), then
\[
\| \mathcal{A}T(p,\mu)(B) - \mathcal{A}T(p,\mu)(A) \| \leq \| B^2 - A^2 \| \left( \frac{\mathcal{A}T(p,\mu)(m) - \mathcal{A}T(p,\mu)(\sqrt{m^2 - \varepsilon})}{\varepsilon} \right)
\]
and by taking the limit over \( \varepsilon \to 0^+ \), then we get
\[
(2.12) \quad \| \mathcal{A}T(p,\mu)(B) - \mathcal{A}T(p,\mu)(A) \| \leq \| B^2 - A^2 \| \lim_{\varepsilon \to 0^+} \frac{\mathcal{A}T(p,\mu)(m) - \mathcal{A}T(p,\mu)(\sqrt{m^2 - \varepsilon})}{\varepsilon}
\]
and by taking the limit over \( \varepsilon \to 0^+ \), then we get
\[
\lim_{\varepsilon \to 0^+} \frac{m - \sqrt{m^2 - \varepsilon}}{\varepsilon}
\]
Since
\[
\lim_{\varepsilon \to 0^+} \frac{\mathcal{A}T(p,\mu)(m) - \mathcal{A}T(p,\mu)(\sqrt{m^2 - \varepsilon})}{m - \sqrt{m^2 - \varepsilon}} = \lim_{h \to 0^+} \frac{\mathcal{A}T(p,\mu)(m) - \mathcal{A}T(p,\mu)(m - h)}{h} = \mathcal{A}T'(p,\mu)(m)
\]
and
\[
\lim_{\varepsilon \to 0^+} \frac{m - \sqrt{m^2 - \varepsilon}}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{1}{m + \sqrt{m^2 - \varepsilon}} = \frac{1}{2m},
\]

hence by (2.12) we obtain the second part of (2.8).

\[ \square \]

**Remark 1.** For \( A, B \) selfadjoint operators with \( A^2 \geq m_1^2, B^2 \geq m_2^2 \) and \( m_1, m_2 > 0 \), we have
\[
(2.13) \quad \| B \arctan(B) - A \arctan(A) \| \leq \| B^2 - A^2 \| \left\{ \begin{array}{ll}
m_2 \arctan(m_2) - m_1 \arctan(m_1) & \text{if } m_2^2 \neq m_1^2, \\
\frac{1}{2m} \left( \arctan m + \frac{m}{m^2 + 1} \right) & \text{if } m_2^2 = m_1^2 = m^2.
\end{array} \right.
\]
For any $A, B$ selfadjoint and invertible operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and $m_1, m_2 > 0$, we have

\begin{equation}
\|A^{-1} \arctan A - B^{-1} \arctan B\|
\leq \|B^2 - A^2\|
\begin{cases}
m_1^{-1} \arctan(m_1) - m_2^{-1} \arctan(m_2) & \text{if } m_2^2 \neq m_1^2, \\
\frac{1}{2m^2} \left( \arctan m - \frac{m}{m^2 + 1} \right) & \text{if } m_2^2 = m_1^2 = m^2.
\end{cases}
\end{equation}

3. A HERMITE-HADAMARD TYPE INEQUALITY

We have:

**Proposition 1.** For $A, B$ selfadjoint operators, assume that $((1 - s) A + sB)^2 \geq m^2$ for $s \in [0, 1]$ with $m > 0$, then

\begin{equation}
\left\| AT (p, \mu) ((1 - s) A + sB) - AT (p, \mu) \left( \frac{A + B}{2} \right) \right\|
\leq \frac{1}{2m} AT' (p, \mu) (m) \int_0^1 \left\| ((1 - s) A + sB)^2 - \left( \frac{A + B}{2} \right)^2 \right\| ds
\leq \frac{1}{12m} \left( \|A\|^2 + \frac{1}{2} \|AB + BA\| + \|B\|^2 \right) AT' (p, \mu) (m).
\end{equation}

**Proof.** Since

\[ \left( \frac{A + B}{2} \right)^2 \geq m^2, \]

then by (2.8) we have

\[ \left\| AT (p, \mu) ((1 - s) A + sB) - AT (p, \mu) \left( \frac{A + B}{2} \right) \right\|
\leq \frac{1}{2m} \left\| ((1 - s) A + sB)^2 - \left( \frac{A + B}{2} \right)^2 \right\| AT' (p, \mu) (m), \]

for $s \in [0, 1]$.

If we take the integral and use the norm properties, then we get

\begin{equation}
\int_0^1 \left\| AT (p, \mu) ((1 - s) A + sB) ds - AT (p, \mu) \left( \frac{A + B}{2} \right) \right\|
\leq \int_0^1 \left\| AT (p, \mu) ((1 - s) A + sB) - AT (p, \mu) \left( \frac{A + B}{2} \right) \right\| ds
\leq \frac{1}{2m} AT' (p, \mu) (m) \int_0^1 \left\| ((1 - s) A + sB)^2 - \left( \frac{A + B}{2} \right)^2 \right\| ds.
\end{equation}
We have
\[(1 - s) A + sB)^2 - \left( \frac{A + B}{2} \right)^2 \]
\[= (1 - s)^2 A^2 + s (1 - s) (AB + BA) + s^2 B^2 \]
\[- \frac{1}{4} (A^2 + AB + BA + B^2)\]
\[= \left[(1 - s)^2 - \frac{1}{4}\right] A^2 + \left[s (1 - s) - \frac{1}{4}\right] (AB + BA) + \left(s^2 - \frac{1}{4}\right) B^2\]
for \(s \in [0, 1]\).

If we take the norm, then we get
\[\left\| \left[(1 - s)^2 - \frac{1}{4}\right] A^2 + \left[s (1 - s) - \frac{1}{4}\right] (AB + BA) + \left(s^2 - \frac{1}{4}\right) B^2 \right\| \]
\[\leq \left\| \left[(1 - s)^2 - \frac{1}{4}\right] A^2 + \left[s (1 - s) - \frac{1}{4}\right] (AB + BA) \right\| + \|B\|^2\]
and, by integration, we derive
\[(3.3) \quad \int_0^1 \left\| \left[(1 - s)^2 - \frac{1}{4}\right] A^2 + \left[s (1 - s) - \frac{1}{4}\right] (AB + BA) \right\| ds\]
\[\leq \int_0^1 \left\| \left[(1 - s)^2 - \frac{1}{4}\right] A^2 + \left[s (1 - s) - \frac{1}{4}\right] (AB + BA) \right\| ds + \int_0^1 \|A\|^2 + \|B\|^2 ds\]
\[= \frac{1}{6} \|A\|^2 + \frac{1}{12} \|AB + BA\| + \frac{1}{6} \|B\|^2.\]

By making use of the inequalities (3.2) and (3.3), we derive the desired result (3.1).

\[\square\]

**Remark 2.** We observe that, if \(A, B \geq m > 0\), then \((1 - s) A + sB \geq m > 0\), which implies that \(((1 - s) A + sB)^2 \geq m^2\) for \(s \in [0, 1]\).

With the assumptions in Proposition 1, we have
\[(3.4) \quad \left\| \int_0^1 \left( (1 - s) A + sB \right) \arctan \left( (1 - s) A + sB \right) ds \right\|\]
\[- \frac{1}{2m} \left( \arctan \left( \frac{m}{m^2 + 1} \right) \right) \int_0^1 \left\| (1 - s) A + sB)^2 - \left( \frac{A + B}{2} \right)^2 \right\| ds \]
\[\leq \frac{1}{2m} \left( \arctan \left( \frac{m}{m^2 + 1} \right) \right) \left( \|A\|^2 + \frac{1}{2} \|AB + BA\| + \|B\|^2 \right).\]
References


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