LIPSCHITZ TYPE INEQUALITIES FOR OPERATOR LERCH TRANSFORM

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Abstract. For a positive measure μ on $[0,\infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\mathrm{Sp}(T)\subset(-1,1)$, by

$$\Phi_\mu(T,s,v):=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(e^t-T\right)^{-1}d\mu(t),$$

for $\mathrm{Re}\, v>0$ and $\mathrm{Re}\, s>0$, provided that the integral exists.

In this paper we show among others that, if $-1<A\le m_1<1$ and $-1<B\le m_2<1$, then for $v>0$ and $s>0$,

$$\|\Phi_\mu(B,s,v)-\Phi_\mu(A,s,v)\| \le \|B-A\|\begin{cases}\frac{\Phi_\mu(m_2,s,v)-\Phi_\mu(m_1,s,v)}{m_2-m_1}, & \text{if } m_1\ne m_2\\[2mm] \frac{\partial\Phi_\mu(m,s,v)}{\partial z}, & \text{if } m_1=m_2=m,\end{cases}$$

where $\frac{\partial\Phi_\mu}{\partial z}$ is the partial derivative of Φ_μ in the first variable. Some examples for *polylogarithm* are also given.

1. Introduction

The *Lerch transcendent* function is given by the series

$$\Phi(z,s,\alpha):=\sum_{n=0}^\infty\frac{z^n}{(n+\alpha)^s},\ \ |z|<1,\ \ \alpha\ne 0,-1,-2,... \tag{1.1}$$

see for instance [10, Section 1.11, p. 27] or [1, Section 25.14]. This function, defined by Mathias Lerch in 1887 in his paper [16], includes as special cases of the parameters; the Hurwitz, Riemann zeta functions and the polylogarithms, among others. Therefore the transcendent has applications ranging from number theory to physics.

The *Hurwitz zeta* function, formally defined for complex arguments s with $\mathrm{Re}(s)>1$ and α with $\mathrm{Re}(\alpha)>0$ by

$$\zeta(s,\alpha):=\sum_{n=0}^\infty\frac{1}{(n+\alpha)^s}$$

is a special case, given by

$$\zeta(s,\alpha)=\Phi(1,s,\alpha).$$

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For $\alpha = 1$ we have the *Riemann zeta* function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The *polylogarithm* function $\mathrm{Li}(s, z)$ is defined by a power series in z, which is also a *Dirichlet series* in s:

$$\mathrm{Li}(z, s) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1). \tag{1.2}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$; it can be extended to $|z| \geq 1$ by the process of analytic continuation. The special case $s = 1$ involves the ordinary natural logarithm, $\mathrm{Li}(z, 1) = -\ln(1 - z)$, while the special cases $s = 2$ and $s = 3$ are called the *dilogarithm* (also referred to as *Spence's function*) and *trilogarithm* respectively.

The *Legendre chi* function is a special case, given by

$$\chi_s(z) = 2^{-s} z\Phi\left(z^2, s, 1/2\right). \tag{1.3}$$

The *Legendre chi* function is a special function whose Taylor series is also a Dirichlet series, given by

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}. \tag{1.4}$$

The following integral representations are valid [10, p. 27]

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt \tag{1.5}$$

for $\mathrm{Re}\, v > 0$ and either $|z| \leq 1$, $z \neq 1$, $\mathrm{Re}\, s > 0$ or $z = 1$, $\mathrm{Re}\, s > 1$. Here $\Gamma(\cdot)$ is Euler's Gama function

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \mathrm{Re}\, s > 0.$$

For a selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1, 1)$ and by utilising the continuous functional calculus for selfadjoint operators, we define the transform

$$\Phi(T, s, v) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} dt \tag{1.6}$$

for $\mathrm{Re}\, v > 0$ and $\mathrm{Re}\, s > 0$.

We can also define the related transforms

$$\mathrm{Li}(T, s) := T\Phi(T, s, 1) \text{ and } \chi_s(T) := 2^{-s} T\Phi\left(T^2, s, 1/2\right) \tag{1.7}$$

for a selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1, 1)$ and $\mathrm{Re}\, s > 0$.

Now, for a positive measure μ on $[0, \infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\mathrm{Sp}(T) \subset (-1, 1)$, by

$$\Phi_\mu(T, s, v) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} d\mu(t), \tag{1.8}$$

for $\mathrm{Re}\, v > 0$ and $\mathrm{Re}\, s > 0$, provided that the integral exists. When μ is Lebesgue measure, then we recapture (1.5).

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^* A)^{1/2}$.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any $A,\ B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [11], [12] and Kato in [14], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.9}$$

for any $A,\ B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS} \tag{1.10}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right), \tag{1.11}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.12}$$

where f is an *operator monotone function* on $[0, \infty)$ and $A,\ B \geq aI_H > 0$. Recall that $A \geq B$ means that $\langle (A - B) x, x\rangle \geq 0$ for all $x \in H$ and f is an *operator monotone function* on $[0, \infty)$ if $f(A) \geq f(B)$ for $A \geq B \geq 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [13] and the references therein.

In this paper we show among others that, if $-1 < A \leq m_1 < 1$ and $-1 < B \leq m_2 < 1$, then for $v > 0$ and $s > 0$,

$$\begin{aligned} &\|\Phi_\mu(B, s, v) - \Phi_\mu(A, s, v)\| \\ &\leq \|B - A\| \begin{cases} \frac{\Phi_\mu(m_2,s,v) - \Phi_\mu(m_1,s,v)}{m_2 - m_1}, & \text{if } m_1 \neq m_2 \\ \\ \frac{\partial \Phi_\mu(m,s,v)}{\partial z}, & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

where $\frac{\partial \Phi_\mu}{\partial z}$ is the partial derivative of Φ_μ in the first variable. Some examples for *polylogarithm* are also given.

2. Main Results

We have the following representation and operator monotonicity result:

Lemma 1. *For* A, B *with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, $\operatorname{Re} v > 0$ *and* $\operatorname{Re} s > 0$ *we have*

$$\Phi_\mu (B,s,v) - \Phi_\mu (A,s,v) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \tag{2.1}$$

$$\times \left[\int_0^1 \left(e^t - [(1-u)B + uA]\right)^{-1} (B-A) \left(e^t - [(1-u)B + uA]\right)^{-1} du \right] d\mu(t).$$

If $v > 0$ *and* $s > 0$, *then* $\Phi(\cdot, s, v)$ *is operator monotone on* $(-1,1)$.

Proof. For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, we have by (1.6) that

$$\Phi_\mu (B,s,v) - \Phi_\mu (A,s,v) \tag{2.2}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left[\left(e^t - B\right)^{-1} - \left(e^t - A\right)^{-1} \right] d\mu(t)$$

for $\operatorname{Re} v > 0$ and $\operatorname{Re} s > 0$.

The function $g(u) = -u^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S) := \lim_{u \to 0} \left[\frac{g(T+uS) - g(T)}{u} \right] = T^{-1} S T^{-1} \tag{2.3}$$

for T, $S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(u) = g((1-u)C + uD), \ u \in [0,1].$$

If $g_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-u)C + uD, \ u \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$g(D) - g(C) = \int_0^1 \frac{d}{du} (g_{C,D}(u))\, du = \int_0^1 \nabla g_{(1-u)C+uD} (D-C)\, du. \tag{2.4}$$

If we write this equality for the function $g(u) = -u^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-u)C + uD)^{-1} (D-C) ((1-u)C + uD)^{-1} du. \tag{2.5}$$

By (2.5) for $C = e^t - B$, $D = e^t - A$, $t \in [0,\infty)$ we have

$$\left(e^t - B\right)^{-1} - \left(e^t - A\right)^{-1} \tag{2.6}$$

$$= \int_0^1 \left((1-u)\left(e^t - B\right) + u\left(e^t - A\right)\right)^{-1} \left(e^t - A - e^t + B\right)$$

$$\times \left((1-u)\left(e^t - B\right) + u\left(e^t - A\right)\right)^{-1} du$$

$$= \int_0^1 \left(e^t - [(1-u)B + uA]\right)^{-1} (B-A) \left(e^t - [(1-u)B + uA]\right)^{-1} du.$$

If we multiply this equality by $t^{s-1}e^{-(v-1)t}$ and integrate, then we obtain (2.1).

Assume that $B-A\geq 0$. Then by multiplying both sides by $\left(e^{t}-[(1-u)B+uA]\right)^{-1}$, we derive

$$\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}\geq 0,$$

for $t\in[0,\infty)$ and $u\in[0,1]$, which implies, by integration, that

$$\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\geq 0$$

for all $t\in[0,\infty)$.

If $v>0$ and $s>0$, then by multiplying with $t^{s-1}e^{-(v-1)t}\geq 0$ and integrating over $t\in[0,\infty)$, we get

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\\&\times\left[\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\right]d\mu(t)\\&\geq 0,\end{aligned}$$

which, by representation (2.1), gives that $\Phi_{\mu}(B,s,v)-\Phi_{\mu}(A,s,v)\geq 0$. □

Observe that

$$\begin{aligned}\operatorname{Li}(T,s)=T\Phi(T,s,1)&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}T\left(e^{t}-T\right)^{-1}dt\\&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\left(T-e^{t}+e^{t}\right)\left(e^{t}-T\right)^{-1}dt\\&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\left[e^{t}\left(e^{t}-T\right)^{-1}-1\right]dt\end{aligned}\tag{2.7}$$

for $\operatorname{Re}s>0$ and T with $\operatorname{Sp}(T)\subset(-1,1)$.

Corollary 1. *For A, B with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset(-1,1)$ and $\operatorname{Re}s>0$, we have*

$$\begin{aligned}&\operatorname{Li}(B,s)-\operatorname{Li}(A,s)\\&=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{t}\\&\times\left[\int_{0}^{1}\left(e^{t}-[(1-u)B+uA]\right)^{-1}(B-A)\left(e^{t}-[(1-u)B+uA]\right)^{-1}du\right]dt.\end{aligned}\tag{2.8}$$

If $s>0$, then $\operatorname{Li}(\cdot,s)$ is operator monotone on $(-1,1)$.

Proof. By (2.7) we have

$$\operatorname{Li}(B,s)-\operatorname{Li}(A,s)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{t}\left[\left(e^{t}-B\right)^{-1}-\left(e^{t}-A\right)^{-1}\right]dt$$

for $\operatorname{Re}s>0$ and by employing the identity (2.6) we derive (2.8). □

We have the following Lipschitz type inequalities:

Theorem 1. *Assume that* $-1 < A \leq m_1 < 1$ *and* $-1 < B \leq m_2 < 1$*, then for* $v > 0$ *and* $s > 0$,

$$\left\| \Phi_\mu (B, s, v) - \Phi_\mu (A, s, v) \right\| \leq \|B - A\| \begin{cases} \frac{\Phi_\mu (m_2,s,v) - \Phi_\mu (m_1,s,v)}{m_2 - m_1}, & \text{if } m_1 \neq m_2 \\ \\ \frac{\partial \Phi_\mu (m,s,v)}{\partial z}, & \text{if } m_1 = m_2 = m, \end{cases} \tag{2.9}$$

where $\frac{\partial \Phi_\mu}{\partial z}$ *is the partial derivative of* Φ_μ *in the first variable.*

Proof. By taking the norm in (2.1) and using the properties of the integral, we get

$$\begin{aligned} & \left\| \Phi_\mu (B, s, v) - \Phi_\mu (A, s, v) \right\| \\ & \leq \|B - A\| \\ & \times \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\int_0^1 \left\| \left(e^t - [(1-u) B + uA] \right)^{-1} \right\|^2 du \right) d\mu (t) \\ & = \|B - A\| \\ & \times \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\int_0^1 \left\| \left(e^t - [uB + (1-u) A] \right)^{-1} \right\|^2 du \right) d\mu (t) . \end{aligned} \tag{2.10}$$

Since $-1 < A \leq m_1 < 1$ and $-1 < B \leq m_2 < 1$, hence

$$(1-u) m_1 + u m_2 \geq uB + (1-u) A$$

for $u \in [0, 1]$, namely

$$0 < e^t - [(1-u) m_1 + u m_2] \leq e^t - [uB + (1-u) A]$$

for $u \in [0, 1]$, $t \geq 0$.

This implies that

$$\left(e^t - [uB + (1-u) A] \right)^{-1} \leq \left(e^t - [(1-u) m_1 + u m_2] \right)^{-1}$$

for $u \in [0, 1]$, $t \geq 0$.

By taking the norm, we get

$$\left\| \left(e^t - [uB + (1-u) A] \right)^{-1} \right\| \leq \left(e^t - [(1-u) m_1 + u m_2] \right)^{-1},$$

namely

$$\left\| \left(e^t - [uB + (1-u) A] \right)^{-1} \right\|^2 \leq \left(e^t - [(1-u) m_1 + u m_2] \right)^{-2},$$

for $u \in [0, 1]$, $t \geq 0$.

By taking the integral over u in $[0, 1]$, multiplying by $t^{s-1} e^{-(v-1)t}$ and taking the integral on $[0, \infty)$ over the positive measure $d\mu (t)$, we get

$$\begin{aligned} & \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\int_0^1 \left\| \left(e^t - [uB + (1-u) A] \right)^{-1} \right\|^2 du \right) d\mu (t) \\ & \leq \frac{1}{\Gamma (s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\int_0^1 \left(e^t - [(1-u) m_1 + u m_2] \right)^{-2} du \right) d\mu (t) . \end{aligned} \tag{2.11}$$

Assume that $m_1 < m_2$. If we use the identity (2.1), change the variable, by replacing u with $1-u$ and choose $A = m_1$, $B = m_2$, then we get

$$\begin{aligned}
&\Phi_\mu(m_2, s, v) - \Phi_\mu(m_1, s, v) \\
&= \frac{m_2 - m_1}{\Gamma(s)} \\
&\times \int_0^\infty t^{s-1} e^{-(v-1)t} \left[\int_0^1 \left(e^t - [u m_2 + (1-u) m_1] \right)^{-2} du \right] d\mu(t),
\end{aligned}$$

which gives that

$$\begin{aligned}
(2.12) \qquad &\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\int_0^1 \left(e^t - [(1-u) m_1 + u m_2] \right)^{-2} du \right) d\mu(t) \\
&= \frac{\Phi_\mu(m_2, s, v) - \Phi_\mu(m_1, s, v)}{m_2 - m_1}.
\end{aligned}$$

By making use of (2.10), (2.11) and (2.12) we get the first inequality in (2.9).

The case $m_1 > m_2$ goes in a similar way and we omit the details.

Let $m_1 = m_2 = m$ and $\varepsilon > 0$. Then $B \le m + \varepsilon = m_2$ and $A \le m = m_1$. By using the first inequality in (2.9) for $m_2 > m_1$ we get

$$(2.13) \qquad \|\Phi_\mu(B, s, v) - \Phi_\mu(A, s, v)\| \le \|B - A\| \frac{\Phi_\mu(m+\varepsilon, s, v) - \Phi_\mu(m, s, v)}{\varepsilon}.$$

By taking the limit over $\varepsilon \to 0+$ in (2.13), we derive the second branch of the inequality (2.9). □

Corollary 2. *Assume that* $-1 < A \le m_1 < 1$, $-1 < B \le m_2 < 1$ *and* $\operatorname{Re} s > 0$, *we have*

$$\begin{aligned}
(2.14) \qquad &\|\operatorname{Li}(B, s) - \operatorname{Li}(A, s)\| \\
&\le \|B - A\| \begin{cases} \frac{\operatorname{Li}(m_2,s) - \operatorname{Li}(m_1,s)}{m_2 - m_1}, & \text{if } m_1 \ne m_2 \\[2ex] \frac{\partial \operatorname{Li}(m,s)}{\partial z}, & \text{if } m_1 = m_2 = m. \end{cases}
\end{aligned}$$

3. Some Discrete Inequalities

Assume that $-1 < A_k \le m < 1$ and $p_k \ge 0$, for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$. Then $-1 < \sum_{j=1}^n p_j A_j \le m < 1$ and by the second inequality in (2.9), we get

$$(3.1) \qquad \left\| \Phi_\mu(A_k, s, v) - \Phi_\mu\left(\sum_{j=1}^n p_j A_j, s, v \right) \right\| \le \frac{\partial \Phi_\mu(m, s, v)}{\partial z} \left\| A_k - \sum_{j=1}^n p_j A_j \right\|$$

for $k \in \{1, ..., n\}$.

If we multiply (3.1) by $p_k \ge 0$ and sum, then we get

$$\begin{aligned}
(3.2) \qquad &\sum_{k=1}^n p_k \left\| \Phi_\mu(A_k, s, v) - \Phi_\mu\left(\sum_{j=1}^n p_j A_j, s, v \right) \right\| \\
&\le \frac{\partial \Phi_\mu(m, s, v)}{\partial z} \sum_{k=1}^n p_k \left\| A_k - \sum_{j=1}^n p_j A_j \right\|.
\end{aligned}$$

By using the triangle inequality, we have

$$\left\| \sum_{k=1}^{n} p_k \Phi_\mu\left(A_k, s, v\right) - \Phi_\mu\left(\sum_{j=1}^{n} p_j A_j, s, v\right) \right\| \leq \sum_{k=1}^{n} p_k \left\| \Phi_\mu\left(A_k, s, v\right) - \Phi_\mu\left(\sum_{j=1}^{n} p_j A_j, s, v\right) \right\|. \tag{3.3}$$

By making use of (3.2) and (3.3), we then derive that

$$\left\| \sum_{k=1}^{n} p_k \Phi_\mu\left(A_k, s, v\right) - \Phi_\mu\left(\sum_{j=1}^{n} p_j A_j, s, v\right) \right\| \leq \frac{\partial \Phi_\mu\left(m, s, v\right)}{\partial z} \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|. \tag{3.4}$$

Denote

$$K(p, A) := \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|.$$

By making use of Hölder's inequality, we have the bounds

$$K(p, A) \leq \begin{cases} \max_{k\in\{1,\dots,n\}} \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\| \\ \left(\sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|^p\right)^{1/p}, \ p > 1 \\ \max_{k\in\{1,\dots,n\}} \{p_k\} \sum_{k=1}^{n} \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\|. \end{cases} \tag{3.5}$$

By the triangle inequality we also have

$$\begin{aligned} \sum_{k=1}^{n} p_k \left\| A_k - \sum_{j=1}^{n} p_j A_j \right\| &= \sum_{k=1}^{n} p_k \left\| \sum_{j=1}^{n} p_j \left(A_k - A_j\right) \right\| \\ &\leq \sum_{k=1}^{n} p_k \sum_{j=1}^{n} p_j \left\| A_k - A_j \right\| \\ &= 2 \sum_{1\leq j<k\leq n} p_k p_j \left\| A_k - A_j \right\| =: 2G\left(p, A\right). \end{aligned}$$

Observe that

$$\begin{aligned} G(p,A) &\leq \max_{1\leq j<k\leq n}\|A_k-A_j\| \sum_{1\leq j<k\leq n}^{n} p_k p_j \\ &= \max_{1\leq j<k\leq n}\|A_k-A_j\| \frac{1}{2}\left(\sum_{k,j=1}^{n} p_k p_j - \sum_{k=1}^{n} p_k^2\right) \\ &= \max_{1\leq j<k\leq n}\|A_k-A_j\| \frac{1}{2}\left(1-\sum_{k=1}^{n} p_k^2\right), \end{aligned}$$

which implies that

$$K(p,A) \leq \max_{1\leq j<k\leq n}\|A_k-A_j\| \left(1-\sum_{k=1}^{n} p_k^2\right). \tag{3.6}$$

Utilising Hölder's inequality, we also have for $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned} G(p,A) &\leq \left(\sum_{1\leq j<k\leq n}^{n} (p_k p_j)^q\right)^{1/q} \left(\sum_{1\leq j<k\leq n}^{n} \|A_k-A_j\|^p\right)^{1/p} \\ &= \frac{1}{2^{1/q}}\left(\sum_{k,j=1}^{n} p_k^q p_j^q - \sum_{k=1}^{n} p_k^{2q}\right)^{1/q} \left(\sum_{1\leq j<k\leq n}^{n} \|A_k-A_j\|^p\right)^{1/p} \\ &= \frac{1}{2^{1/q}}\left(\left(\sum_{k=1}^{n} p_k^q\right)^2 - \sum_{k=1}^{n} p_k^{2q}\right)^{1/q} \left(\sum_{1\leq j<k\leq n}^{n} \|A_k-A_j\|^p\right)^{1/p}, \end{aligned}$$

which implies that

$$K(p,A) \leq 2^{1/p}\left[\left(\sum_{k=1}^{n} p_k^q\right)^2 - \sum_{k=1}^{n} p_k^{2q}\right]^{1/q} \left(\sum_{1\leq j<k\leq n}^{n} \|A_k-A_j\|^p\right)^{1/p}. \tag{3.7}$$

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