OPERATOR CONVEXITY OF THE $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following $\mathcal{D}$-*logarithmic integral transform*

$$\mathcal{DL}og(w)(T) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda + T}{\lambda}\right) d\lambda,$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

In this paper, we show among others that the function $\mathcal{DL}og(w,\mu)(\cdot)$ is operator concave on $(0,\infty)$. Upper and lower bounds for the Jensen's difference

$$\mathcal{DL}og(w,\mu)\left(\frac{A+B}{2}\right) - \frac{1}{2}\left[\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B)\right]$$

in the case when $0 < \rho \le A \le \sigma$, $0 < \varsigma \le B \le \tau$ and $0 < n \le B - A \le N$ are given. Some examples for integral transforms $\mathcal{DL}og(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$.

We have the following representation of operator monotone functions [10], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.1}$$

where $b \ge 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.$$

For some recent results related to operator monotone functions we refer to [5], [6] [7], [11], [12] and the references therein.

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if, [8]

$$f\left(\left(1-\lambda \right) A+\lambda B\right) \leq \left(\geq \right) \left(1-\lambda \right) f\left(A\right) +\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following integral representation for the power function when $t>0$, $r\in (0,1]$, see for instance [1, p. 145]

$$s^{r-1}=\frac{\sin \left(r\pi \right) }{\pi }\int_{0}^{\infty }\frac{\lambda ^{r-1}}{\lambda +s}d\lambda . \tag{1.2}$$

Observe that for $s>0$, $s\neq 1$, we have

$$\int_{0}^{u}\frac{d\lambda }{\left(\lambda +s\right) \left(\lambda +1\right) }=\frac{\ln s}{s-1}+\frac{1}{1-s}\ln \left(\frac{u+s}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\rightarrow \infty$ in this equality, we derive

$$\frac{\ln s}{s-1}=\int_{0}^{\infty }\frac{d\lambda }{\left(\lambda +s\right) \left(\lambda +1\right) }, \tag{1.3}$$

which gives the representation for the logarithm

$$\ln s=\left(s-1\right) \int_{0}^{\infty }\frac{d\lambda }{\left(\lambda +1\right) \left(\lambda +s\right) } \tag{1.4}$$

If we integrate (1.2) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned} \frac{t^{r}}{r} &=\frac{\sin \left(r\pi \right) }{\pi }\int_{0}^{\infty }\left(\int_{0}^{t}\left(\frac{1}{\lambda +s}\right) ds\right) \lambda ^{r-1}d\lambda \\ &=\frac{\sin \left(r\pi \right) }{\pi }\int_{0}^{\infty }\lambda ^{r-1}\ln \left(\frac{t+\lambda }{\lambda }\right) d\lambda \end{aligned}$$

giving the identity of interest

$$t^{r}=\frac{r\sin \left(r\pi \right) }{\pi }\int_{0}^{\infty }\lambda ^{r-1}\ln \left(\frac{t+\lambda }{\lambda }\right) d\lambda ,\ t>0\text{ and }r\in (0,1].$$

Recall the *dilogarithmic function* $\operatorname{dilog}:[0,\infty)\rightarrow \mathbb{R}$ defined by

$$\operatorname{dilog}\left(t\right) :=\int_{1}^{t}\frac{\ln s}{1-s}ds,\ t\geq 0.$$

Some particular values of interest are

$$\operatorname{dilog}\left(1\right) =0,\ \operatorname{dilog}\left(0\right) =\int_{1}^{0}\frac{\ln s}{1-s}ds=\int_{0}^{1}\frac{\ln s}{s-1}ds=\frac{1}{6}\pi ^{2},$$

and

$$\operatorname{dilog}\left(\frac{1}{2}\right) =\frac{1}{12}\pi ^{2}-\frac{1}{2}\left(\ln 2\right) ^{2}.$$

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_{0}^{t}\frac{\ln s}{s-1}ds=\int_{0}^{\infty }\left(\int_{0}^{t}\frac{1}{\lambda +s}ds\right) \frac{1}{\left(\lambda +1\right) }d\lambda =\int_{0}^{\infty }\frac{1}{\left(\lambda +1\right) }\ln \left(\frac{t+\lambda }{\lambda }\right) d\lambda$$

and since

$$\int_0^t \frac{\ln s}{s-1} ds = \int_0^1 \frac{\ln s}{s-1} ds + \int_1^t \frac{\ln s}{s-1} ds = \frac{1}{6}\pi^2 - \int_1^t \frac{\ln s}{1-s} ds$$
$$= \frac{1}{6}\pi^2 - \text{dilog}(t)$$

then we get the identity of interest

$$\frac{1}{6}\pi^2 - \text{dilog}(t) = \int_0^\infty \frac{1}{(\lambda+1)} \ln\left(\frac{t+\lambda}{\lambda}\right) d\lambda,\ t > 0.$$

Motivated by the above representations, we define the *$\mathcal{D}$-logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by

$$\mathcal{DL}og(w,\mu)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda), \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t > 0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{DL}og(w)(t) := \int_0^\infty w(\lambda) \ln\left(\frac{\lambda+t}{\lambda}\right) d\lambda. \tag{1.6}$$

Obviously,

$$\mathcal{DL}og(w,\mu)(t) = \int_0^\infty w(\lambda) \ln\left(1+\frac{t}{\lambda}\right) d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda)\left[\ln(\lambda+t) - \ln(\lambda)\right] d\mu(\lambda)$$

and one can use either of these representations when is needed.

If we use the *$\mathcal{D}$-logarithmic transform* for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$ we have

$$\mathcal{DL}og(w_{\ell^{r-1}})(t) = t^r,\ t \geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \text{dilog}(t),\ t \geq 0. \tag{1.7}$$

In the recent paper [3] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\mu(\lambda),\ s > 0, \tag{1.8}$$

for a continuous and positive function $w(\lambda)$, $\lambda > 0$,where μ is a positive measure on $(0,\infty)$ and the integral (2.3) exists for all $s > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s} d\lambda,\ s > 0. \tag{1.9}$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [3].

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t \mathcal{D}(w,\mu)(s)\,ds := \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s}ds\right) w(\lambda)\,d\mu(\lambda) = \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right) d\mu(\lambda) \tag{1.10}$$

for $t>0$, which provides the equality of interest

$$\mathcal{DL}og(w,\mu)(t) = \int_0^t \mathcal{D}(w,\mu)(s)\,ds,\ t>0, \tag{1.11}$$

provided that the integral on the right side exists for all $t>0$.

In the recent paper we obtained the following result:

Theorem 2. *For all A, $B>0$ we have the identity:*

$$\begin{aligned}&\mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\\ &= \int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\ &\quad\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda).\end{aligned}$$

Corollary 1. *If $B\geq A>0$, then $\mathcal{DL}og(w,\mu)(B)\geq\mathcal{DL}og(w,\mu)(A)$, namely $\mathcal{DL}og(w,\mu)(\cdot)$ is operator monotone on $(0,\infty)$.*

Remark 1. *Since, by (1.7),*

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \operatorname{dilog}(t),\ t\geq 0$$

and $\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)$ is operator monotone, then the function $-\operatorname{dilog}$ is operator monotone on $(0,\infty)$.

In this paper, we show among others that the function $\mathcal{DL}og(w,\mu)(\cdot)$ is operator concave on $(0,\infty)$. Upper and lower bounds for the Jensen's difference

$$\mathcal{DL}og(w,\mu)\left(\frac{A+B}{2}\right) - \frac{1}{2}\left[\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B)\right]$$

in the case when $0<\rho\leq A\leq\sigma$, $0<\varsigma\leq B\leq\tau$ and $0<n\leq B-A\leq N$ are given. Some examples for integral transforms $\mathcal{DL}og(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

We start with the following elementary identity that give a simple proof for the fact that the function $f(t)=t^{-1}$ is operator convex on $(0,\infty)$, see for instance [8, p. 8]:

Lemma 1. *For any U, $V>0$ we have*

$$\begin{aligned}&\frac{U^{-1}+V^{-1}}{2} - \left(\frac{U+V}{2}\right)^{-1}\\ &= \frac{\left(U^{-1}-V^{-1}\right)\left(U^{-1}+V^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)}{2}\geq 0.\end{aligned} \tag{2.1}$$

If more assumptions are made for the operators U and V, then one can obtain the following lower and upper bounds:

Corollary 2. *Assume that* $0<\alpha\leq U\leq\beta$ *and* $0<\gamma\leq V\leq\delta$ *for some constants* α, β, γ, δ. *Then*

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^{2}\leq\frac{U^{-1}+V^{-1}}{2}-\left(\frac{U+V}{2}\right)^{-1} \tag{2.2}$$
$$\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^{2}.$$

Proof. We have $\beta^{-1}\leq U^{-1}\leq\alpha^{-1}$ and $\delta^{-1}\leq V^{-1}\leq\gamma^{-1}$, which gives

$$\beta^{-1}+\delta^{-1}\leq U^{-1}+V^{-1}\leq\alpha^{-1}+\gamma^{-1}$$

namely

$$\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\leq\left(U^{-1}+V^{-1}\right)^{-1}\leq\left(\beta^{-1}+\delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(U^{-1}-V^{-1}\right)$ and dividing by 2, we get

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^{2}\leq\frac{\left(U^{-1}-V^{-1}\right)\left(U^{-1}+V^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)}{2}$$
$$\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^{2}.$$

□

A continuous function $g:\mathcal{S}\mathrm{A}_I(H)\to\mathrm{B}(H)$ is said to be *Gâteaux differentiable* in $U\in\mathcal{SA}_I(H)$, the class of selfadjoint operators on I, along the direction $V\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_U(V):=\lim_{s\to 0}\frac{g(U+sV)-g(U)}{s}\in\mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B\in\mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in U and we can write $g\in\mathcal{G}(U)$. If this is true for any U in an open set $\mathcal{S}$ from $\mathcal{S}\mathrm{A}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $U,V\in\mathcal{S}\mathrm{A}_I(H)$ we consider the segment of selfadjoint operators

$$[U,V]:=\{(1-t)U+tV\mid t\in[0,1]\}.$$

We observe that $U,V\in[U,V]$ and $[U,V]\subset\mathcal{S}\mathrm{A}_I(H)$.

We have the following gradient inequalities, see for instance [2]:

Lemma 2. *Let* f *be an operator convex function on* I *and* $U,V\in\mathcal{SA}_I(H)$, *with* $U\neq V$. *If* $f\in\mathcal{G}([U,V])$, *then*

$$\nabla_V f(V-U)\geq f(V)-f(U)\geq\nabla_U f(V-U). \tag{2.4}$$

Let $T,S>0$. The function $f(t)=t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=-T^{-1}ST^{-1} \tag{2.5}$$

for $T,S>0$.

Using (2.5) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1} \tag{2.6}$$

for all C, $D > 0$.

If

$$m \leq D - C \leq M$$

for some constants m, M, then

$$mD^{-2} \leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1} \leq MC^{-2}$$

and by (2.6) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.7}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2} \tag{2.8}$$

Corollary 3. *Assume that* $0 < \alpha \leq U \leq \beta$, $0 < \gamma \leq V \leq \delta$ *and* $0 < m \leq V - U \leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4} \\ &\leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^2 \leq \frac{U^{-1}+V^{-1}}{2} - \left(\frac{U+V}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}. \end{aligned} \tag{2.9}$$

Proof. From (2.8) we have

$$0 < \frac{m}{\delta^2} \leq U^{-1} - V^{-1} \leq \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \leq \left(U^{-1} - V^{-1}\right)^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.9). □

Remark 2. *If the positive operators* U, V *are separated, namely* $0 < \alpha \leq U \leq \beta < \gamma \leq V \leq \delta$ *for some constants* α, β, γ, δ, *then obviously* $0 < \gamma - \beta \leq V - U \leq \delta - \alpha$

and by (2.9) for $m=\gamma-\beta$ and $M=\delta-\alpha$, we get

$$
\begin{aligned}
(2.10)\qquad 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4} \le \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^2 \\
&\le \frac{U^{-1}+V^{-1}}{2}-\left(\frac{U+V}{2}\right)^{-1} \\
&\le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(U^{-1}-V^{-1}\right)^2 \le \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}.
\end{aligned}
$$

We have:

Lemma 3. *For all A, $B>0$ and $a>0$, then*

$$
\begin{aligned}
(2.11)\qquad & \ln\left(1+\frac{1}{a}\frac{A+B}{2}\right)-\frac{1}{2}\left[\ln\left(1+\frac{1}{a}A\right)+\ln\left(1+\frac{1}{a}B\right)\right] \\
&= \frac{1}{2}\int_0^\infty\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}-\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right) \\
&\times\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}+\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right)^{-1} \\
&\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}-\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right)d\lambda \\
&\ge 0
\end{aligned}
$$

Proof. Observe, by (1.4), that

$$
\begin{aligned}
\ln s &= \int_0^\infty \frac{(s-1)\,d\lambda}{(\lambda+1)(\lambda+s)} = \int_0^\infty \frac{(s+\lambda-\lambda-1)\,d\lambda}{(\lambda+1)(\lambda+s)} \\
&= \int_0^\infty\left[\frac{s+\lambda}{(\lambda+1)(\lambda+s)}-\frac{\lambda+1}{(\lambda+1)(\lambda+s)}\right]d\lambda \\
&= \int_0^\infty\left(\frac{1}{\lambda+1}-\frac{1}{\lambda+s}\right)d\lambda = \int_0^\infty\left[\frac{1}{\lambda+1}-(\lambda+s)^{-1}\right]d\lambda
\end{aligned}
$$

for $s>0$.

We have

$$
\begin{aligned}
&\ln\left(1+\frac{1}{a}\frac{A+B}{2}\right)-\frac{1}{2}\left[\ln\left(1+\frac{1}{a}A\right)+\ln\left(1+\frac{1}{a}B\right)\right] \\
&=\int_0^\infty\left[\frac{1}{\lambda+1}-\left(\lambda+1+\frac{1}{a}\frac{A+B}{2}\right)^{-1}\right]d\lambda \\
&-\frac{1}{2}\int_0^\infty\left[\frac{1}{\lambda+1}-\left(\lambda+1+\frac{1}{a}A\right)^{-1}\right]d\lambda \\
&-\frac{1}{2}\int_0^\infty\left[\frac{1}{\lambda+1}-\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right]d\lambda \\
&=\int_0^\infty\left\{\frac{1}{2}\left[\left(\lambda+1+\frac{1}{a}A\right)^{-1}+\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right]\right. \\
&\left.-\left(\lambda+1+\frac{1}{a}\frac{A+B}{2}\right)^{-1}\right\}d\lambda.
\end{aligned}
\tag{2.12}
$$

By the use of (2.1) for $U=\lambda+1+\frac{1}{a}A$ and $V=\lambda+1+\frac{1}{a}B$, then we get

$$
\begin{aligned}
&\frac{1}{2}\left[\left(\lambda+1+\frac{1}{a}A\right)^{-1}+\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right] \\
&-\left(\lambda+1+\frac{1}{a}\frac{A+B}{2}\right)^{-1} \\
&=\frac{1}{2}\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}-\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right) \\
&\times\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}+\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right)^{-1} \\
&\times\left(\left(\lambda+1+\frac{1}{a}A\right)^{-1}-\left(\lambda+1+\frac{1}{a}B\right)^{-1}\right) \\
&\geq 0
\end{aligned}
\tag{2.13}
$$

for all $\lambda\geq 0$.

By integrating inequality (2.13) over λ in $[0,\infty)$ and making use of (2.12), we derive (2.11). □

Theorem 3. *For all $A,\ B>0$ we have the representation*

$$\mathcal{DL}og\left(w,\mu\right)\left(\frac{A+B}{2}\right)-\frac{1}{2}\left[\mathcal{DL}og\left(w,\mu\right)\left(A\right)+\mathcal{DL}og\left(w,\mu\right)\left(B\right)\right] \tag{2.14}$$

$$=\frac{1}{2}\int_0^\infty\left[\int_0^\infty\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}-\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)\right.$$

$$\times\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}+\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)^{-1}$$

$$\left.\times\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}-\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)d\lambda\right]d\mu\left(u\right)$$

$$\geq 0.$$

The function $\mathcal{DL}og\left(w,\mu\right)\left(\cdot\right)$ is operator concave on $\left(0,\infty\right)$.

Proof. We have by (1.5) that

$$\mathcal{DL}og\left(w,\mu\right)\left(t\right):=\int_0^\infty w\left(u\right)\ln\left(1+\frac{t}{u}\right)d\mu\left(u\right),\ t>0$$

and by the continuous functional calculus, we have

$$\mathcal{DL}og\left(w,\mu\right)\left(\frac{A+B}{2}\right)-\frac{1}{2}\left[\mathcal{DL}og\left(w,\mu\right)\left(A\right)+\mathcal{DL}og\left(w,\mu\right)\left(B\right)\right] \tag{2.15}$$

$$=\int_0^\infty w\left(u\right)\ln\left(1+\frac{1}{u}\frac{A+B}{2}\right)d\mu\left(u\right)$$

$$-\frac{1}{2}\left[\int_0^\infty w\left(u\right)\ln\left(1+\frac{1}{u}A\right)d\mu\left(u\right)+\int_0^\infty w\left(u\right)\ln\left(1+\frac{1}{u}B\right)d\mu\left(u\right)\right]$$

$$=\int_0^\infty w\left(u\right)\left\{\ln\left(1+\frac{1}{u}\frac{A+B}{2}\right)\right.$$

$$\left.-\frac{1}{2}\left[\ln\left(1+\frac{1}{u}A\right)+\ln\left(1+\frac{1}{u}B\right)\right]\right\}d\mu\left(u\right)$$

By integrating (2.11) we have

$$\int_0^\infty w\left(u\right)\left\{\ln\left(1+\frac{1}{u}\frac{A+B}{2}\right)\right. \tag{2.16}$$

$$\left.-\frac{1}{2}\left[\ln\left(1+\frac{1}{u}A\right)+\ln\left(1+\frac{1}{u}B\right)\right]\right\}d\mu\left(u\right)$$

$$=\frac{1}{2}\int_0^\infty\left[\int_0^\infty\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}-\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)\right.$$

$$\times\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}+\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)^{-1}$$

$$\left.\times\left(\left(\lambda+1+\frac{1}{u}A\right)^{-1}-\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right)d\lambda\right]d\mu\left(u\right)$$

$$\geq 0.$$

By (2.15) and (2.16) we derive (2.14). □

Corollary 4. *Let* $0<\rho\leq A\leq\sigma,\ 0<\varsigma\leq B\leq\tau$ *and* $0<n\leq B-A\leq N,$ *then*

$$
\begin{aligned}
(2.17)\qquad & 0<\frac{1}{4}n^{2}L_{w,\mu}\left(\rho,\varsigma,\tau\right) \\
& \leq\mathcal{D}\mathcal{L}og\left(w,\mu\right)\left(\frac{A+B}{2}\right)-\frac{1}{2}\left[\mathcal{D}\mathcal{L}og\left(w,\mu\right)\left(A\right)+\mathcal{D}\mathcal{L}og\left(w,\mu\right)\left(B\right)\right] \\
& \leq\frac{1}{4}N^{2}L_{w,\mu}\left(\sigma,\tau,\rho\right),
\end{aligned}
$$

where

$$
\begin{aligned}
(2.18)\qquad & L_{w,\mu}\left(p,q,r\right) \\
& :=\int_{0}^{\infty}\frac{w\left(u\right)}{u^{2}}\left(\int_{0}^{\infty}\frac{\left(\lambda+1+\frac{1}{u}p\right)\left(\lambda+1+\frac{1}{u}q\right)}{\left[\lambda+1+\frac{1}{u}\left(\frac{p+q}{2}\right)\right]\left(\lambda+1+\frac{1}{u}r\right)^{4}}d\lambda\right)d\mu\left(u\right) \\
& =\int_{0}^{\infty}uw\left(u\right)\left(\int_{0}^{\infty}\frac{\left(\left(\lambda+1\right)u+p\right)\left(\left(\lambda+1\right)u+q\right)}{\left[\left(\lambda+1\right)u+\left(\frac{p+q}{2}\right)\right]\left(\left(\lambda+1\right)u+r\right)^{4}}d\lambda\right)d\mu\left(u\right).
\end{aligned}
$$

Proof. Since $0<\rho\leq A\leq\sigma,\ 0<\varsigma\leq B\leq\tau$ and $0<n\leq B-A\leq N$, then

$$\lambda+1+\frac{1}{u}\rho\leq U=\lambda+1+\frac{1}{u}A\leq\lambda+1+\frac{1}{u}\sigma,$$

$$\lambda+1+\frac{1}{u}\varsigma\leq V=\lambda+1+\frac{1}{u}B\leq\lambda+1+\frac{1}{u}\tau$$

and

$$\frac{n}{u}\leq V-U=\frac{1}{u}\left(B-A\right)\leq\frac{N}{u}$$

If we apply the inequality (2.9) for $U=\lambda+1+\frac{1}{u}A$ and $V=\lambda+1+\frac{1}{u}B$,

$$
\begin{aligned}
(2.19)\qquad & 0<\frac{1}{2}\left(\left(\lambda+1+\frac{1}{u}\rho\right)^{-1}+\left(\lambda+1+\frac{1}{u}\varsigma\right)^{-1}\right)^{-1}\frac{\left(\frac{n}{u}\right)^{2}}{\left(\lambda+1+\frac{1}{u}\tau\right)^{4}} \\
& \leq\frac{1}{2}\left[\left(\lambda+1+\frac{1}{u}A\right)^{-1}+\left(\lambda+1+\frac{1}{u}B\right)^{-1}\right] \\
& -\left(\lambda+1+\frac{1}{u}\frac{A+B}{2}\right)^{-1} \\
& \leq\frac{1}{2}\left(\left(\lambda+1+\frac{1}{u}\sigma\right)^{-1}+\left(\lambda+1+\frac{1}{u}\tau\right)^{-1}\right)^{-1}\frac{\left(\frac{N}{u}\right)^{2}}{\left(\lambda+1+\frac{1}{u}\rho\right)^{4}}
\end{aligned}
$$

for all $\lambda,u>0$.

If we take the integral over $\lambda > 0$, then we get

$$0<\frac{1}{2}n^{2}\int_{0}^{\infty}\left(\left(\lambda+1+\frac{1}{u}\rho\right)^{-1}+\left(\lambda+1+\frac{1}{u}\varsigma\right)^{-1}\right)^{-1} \times\frac{1}{u^{2}\left(\lambda+1+\frac{1}{u}\tau\right)^{4}}d\lambda \le\ln\left(1+\frac{1}{u}\frac{A+B}{2}\right)-\frac{1}{2}\left[\ln\left(1+\frac{1}{u}A\right)+\ln\left(1+\frac{1}{u}B\right)\right] \le\frac{1}{2}N^{2}\int_{0}^{\infty}\left(\left(\lambda+1+\frac{1}{u}\sigma\right)^{-1}+\left(\lambda+1+\frac{1}{u}\tau\right)^{-1}\right)^{-1} \times\frac{1}{u^{2}\left(\lambda+1+\frac{1}{u}\rho\right)^{4}}d\lambda. \tag{2.20}$$

Observe that

$$\left(\left(\lambda+1+\frac{1}{u}\rho\right)^{-1}+\left(\lambda+1+\frac{1}{u}\varsigma\right)^{-1}\right)^{-1} =\frac{\left(\lambda+1+\frac{1}{u}\rho\right)\left(\lambda+1+\frac{1}{u}\varsigma\right)}{2\left(\lambda+1\right)+\frac{1}{u}\left(\rho+\varsigma\right)}$$

and

$$\left(\left(\lambda+1+\frac{1}{u}\sigma\right)^{-1}+\left(\lambda+1+\frac{1}{u}\tau\right)^{-1}\right)^{-1} =\frac{\left(\lambda+1+\frac{1}{u}\sigma\right)\left(\lambda+1+\frac{1}{u}\tau\right)}{2\left(\lambda+1\right)+\frac{1}{u}\left(\sigma+\tau\right)}$$

and by (2.20) we get

$$0<\frac{1}{4}n^{2}\int_{0}^{\infty}\frac{\left(\lambda+1+\frac{1}{u}\rho\right)\left(\lambda+1+\frac{1}{u}\varsigma\right)}{u^{2}\left[\lambda+1+\frac{1}{u}\left(\frac{\rho+\varsigma}{2}\right)\right]\left(\lambda+1+\frac{1}{u}\tau\right)^{4}}d\lambda \le\ln\left(1+\frac{1}{u}\frac{A+B}{2}\right)-\frac{1}{2}\left[\ln\left(1+\frac{1}{u}A\right)+\ln\left(1+\frac{1}{u}B\right)\right] \le\frac{1}{4}N^{2}\int_{0}^{\infty}\frac{\left(\lambda+1+\frac{1}{u}\sigma\right)\left(\lambda+1+\frac{1}{u}\tau\right)}{u^{2}\left[\lambda+1+\frac{1}{u}\left(\frac{\sigma+\tau}{2}\right)\right]\left(\lambda+1+\frac{1}{u}\rho\right)^{4}}d\lambda.$$

By multiplying with $w(u) \geq 0$ and integrate, then we get (2.17). □

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] S.S. Dragomir, Reverses of operator Féjer's inequalities, *Tokyo J. Math.*, Vol. **44**, NO. 1, 2021. DOI: 10.3836/tjm/1502179330

[3] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 65, 15 pp. [https://rgmia.org/papers/v23/v23a65.pdf].

[4] S. S. Dragomir, Operator monotonicity of the D-logarithmic integral transform for positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 116, 17 pp. [https://rgmia.org/papers/v23/v23a116.pdf].

[5] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.
[6] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.
[7] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.
[8] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[9] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.
[10] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.
[11] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.
[12] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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