OPERATOR CONVEXITY OF LERCH TRANSFORM

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a positive measure μ on $[0,\infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\operatorname{Sp}(T)\subset(-1,1)$, by

$$\Phi_\mu(T,s,v):=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(e^t-T\right)^{-1}d\mu(t),$$

for $\operatorname{Re} v>0$ and $\operatorname{Re} s>0$, provided that the integral exists.

In this paper we show among others that, the function $\Phi_\mu(\cdot,s,v)$ is operator convex on $(-1,1)$ for $v>0$ and $s>0$. Several lower and upper bounds when $-1<\alpha\leq A\leq\beta<1$, $-1<\gamma\leq B\leq\delta<1$ and $0<m\leq A-B\leq M$ for some constants α, β, γ, δ, m, M, are given. Examples for Polylogarithm function are also provided.

1. Introduction

The *Lerch transcendent* function is given by the series

$$\Phi(z,s,\alpha):=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s},\ \ |z|<1,\ \ \alpha\neq 0,-1,-2,... \tag{1.1}$$

see for instance [3, Section 1.11, p. 27] or [1, Section 25.14]. This function, defined by Mathias Lerch in 1887 in his paper [5], includes as special cases of the parameters; the Hurwitz, Riemann zeta functions and the polylogarithms, among others. Therefore the transcendent has applications ranging from number theory to physics.

The *Hurwitz zeta* function, formally defined for complex arguments s with $\operatorname{Re}(s)>1$ and α with $\operatorname{Re}(\alpha)>0$ by

$$\zeta(s,\alpha):=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}$$

is a special case, given by

$$\zeta(s,\alpha)=\Phi(1,s,\alpha).$$

For $\alpha=1$ we have the *Riemann zeta* function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The *polylogarithm* function $\operatorname{Li}(s,z)$ is defined by a power series in z, which is also a *Dirichlet series* in s:

$$\operatorname{Li}(z,s):=\sum_{n=1}^{\infty}\frac{z^n}{n^s}=z\Phi(z,s,1). \tag{1.2}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator convex functions, Operator inequalities, Logarithmic operator inequalities. Lerch transcendent function.

RGMIA Res. Rep. Coll. 23 (2020), Art. 135, 12 pp. Received 18/11/20

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$; it can be extended to $|z| \geq 1$ by the process of analytic continuation. The special case $s = 1$ involves the ordinary natural logarithm, $\operatorname{Li}(z, 1) = -\ln(1 - z)$, while the special cases $s = 2$ and $s = 3$ are called the *dilogarithm* (also referred to as *Spence's function*) and *trilogarithm* respectively.

The *Legendre chi* function is a special case, given by

$$\chi_s(z) = 2^{-s} z \Phi\left(z^2, s, 1/2\right). \tag{1.3}$$

The *Legendre chi* function is a special function whose Taylor series is also a Dirichlet series, given by

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}. \tag{1.4}$$

The following integral representations are valid [3, p. 27]

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt \tag{1.5}$$

for $\operatorname{Re} v > 0$ and either $|z| \leq 1$, $z \neq 1$, $\operatorname{Re} s > 0$ or $z = 1$, $\operatorname{Re} s > 1$. Here $\Gamma(\cdot)$ is Euler's Gama function

$$\Gamma(s) := \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re} s > 0.$$

For a selfadjoint operator T with $\operatorname{Sp}(T) \subset (-1, 1)$ and by utilising the continuous functional calculus for selfadjoint operators, we define the transform

$$\Phi(T, s, v) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} dt \tag{1.6}$$

for $\operatorname{Re} v > 0$ and $\operatorname{Re} s > 0$.

We can also define the related transforms

$$\operatorname{Li}(T, s) := T\Phi(T, s, 1) \text{ and } \chi_s(T) := 2^{-s} T\Phi\left(T^2, s, 1/2\right) \tag{1.7}$$

for a selfadjoint operator T with $\operatorname{Sp}(T) \subset (-1, 1)$ and $\operatorname{Re} s > 0$.

Now, for a positive measure μ on $[0, \infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\operatorname{Sp}(T) \subset (-1, 1)$, by

$$\Phi_\mu(T, s, v) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} d\mu(t), \tag{1.8}$$

for $\operatorname{Re} v > 0$ and $\operatorname{Re} s > 0$, provided that the integral exists. When μ is Lebesgue measure, then we recapture (1.5).

In this paper we show among others that, the function $\Phi_\mu(\cdot, s, v)$ is operator convex on $(-1, 1)$ for $v > 0$ and $s > 0$. Several lower and upper bounds when $-1 < \alpha \leq A \leq \beta < 1$, $-1 < \gamma \leq B \leq \delta < 1$ and $0 < m \leq A - B \leq M$ for some constants α, β, γ, δ, m, M, are given. Examples for Polylogarithm function are also provided.

2. Main Results

The following operator quasi-subadditivity property holds:

Theorem 1. *Assume that* A, B *are selfadjoint with* $\operatorname{Sp}(A), \operatorname{Sp}(B), \operatorname{Sp}(A+B) \subset (-1,1)$, *then for all* $v>0$ *and* $s>0$,

$$\Phi_\mu(A+B,s,v) \le \Phi_\mu(A,s,v) + \Phi_\mu(B,s,v) + 2\frac{\partial}{\partial z}\Phi(A+B,s,v), \tag{2.1}$$

where $\frac{\partial}{\partial z}\Phi$ *is the partial derivative of* Φ *in the first variable.*

If $s>1$, *then*

$$\Phi_\mu(A+B,s,v) \le \Phi_\mu(A,s,v) + \Phi_\mu(B,s,v) + 2\frac{\partial}{\partial z}\Phi(1,s,v). \tag{2.2}$$

Proof. For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B), \operatorname{Sp}(A+B) \subset (-1,1)$, we have

$$\begin{aligned} &\Phi_\mu(A+B,s,v) - \Phi_\mu(A,s,v) - \Phi_\mu(B,s,v) \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t} \\ &\times \left[\left(e^t - A - B\right)^{-1} - \left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}\right] d\mu(t). \end{aligned} \tag{2.3}$$

Consider

$$K_t(A,B) := \left(e^t - A - B\right)^{-1} - \left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}, \ t \in [0,\infty).$$

Therefore

$$\begin{aligned} &\left(e^t - A - B\right) K_t(A,B) \left(e^t - A - B\right) \\ &= e^t - A - B - \left(e^t - A - B\right)\left(e^t - A\right)^{-1}\left(e^t - A - B\right) \\ &- \left(e^t - A - B\right)\left(e^t - B\right)^{-1}\left(e^t - A - B\right) \\ &= e^t - A - B - \left(e^t - A - B\right)\left(1 - \left(e^t - A\right)^{-1} B\right) \\ &- \left(e^t - A - B\right)\left(1 - \left(e^t - B\right)^{-1} A\right) \\ &= e^t - A - B - \left(e^t - A - B\right) + \left(e^t - A - B\right)\left(e^t - A\right)^{-1} B \\ &- \left(e^t - A - B\right) + \left(e^t - B - A\right)\left(e^t - B\right)^{-1} A \\ &= 1 - B\left(e^t - A\right)^{-1} B - \left(e^t - A - B\right) \\ &+ 1 - A\left(e^t - B\right)^{-1} A \\ &= 2 - e^t + A + B - B\left(e^t - A\right)^{-1} B - A\left(e^t - B\right)^{-1} A, \end{aligned}$$

which gives that

$$\begin{aligned} L_t(A,B) &:= \left(e^t - A - B\right) K_t(A,B)\left(e^t - A - B\right) - 2 \\ &= -e^t + A + B - B\left(e^t - A\right)^{-1} B - A\left(e^t - B\right)^{-1} A, \ t \ge 0. \end{aligned}$$

Observe that $B\left(e^t - A\right)^{-1} B \ge 0$, $A\left(e^t - B\right)^{-1} A \ge 0$ and $-e^t + A + B \le 0$, $t \ge 0$. Therefore $L_t(A,B) \le 0$, $t \ge 0$, which shows that

$$\left(e^t - A - B\right) K_t(A,B)\left(e^t - A - B\right) \le 2, \ t \ge 0.$$

If we multiply this inequality both sides by $\left(e^t - A - B\right)^{-1}$, we get

$$K_t(A,B) \le 2\left(e^t - A - B\right)^{-2}, \ t \ge 0.$$

If we multiply this by $\frac{1}{\Gamma(s)}t^{s-1}e^{-(v-1)t}$ and integrate, then we get by (2.3) that

$$\begin{aligned} &\Phi_{\mu}(A+B,s,v)-\Phi_{\mu}(A,s,v)-\Phi_{\mu}(B,s,v) \\ &\leq 2\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\left(e^{t}-A-B\right)^{-2}dt. \end{aligned} \tag{2.4}$$

Observe that, by (1.5)

$$\frac{\partial}{\partial z}\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{\left(e^{t}-z\right)^{2}}dt,$$

which give that

$$\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\left(e^{t}-A-B\right)^{-2}dt=\frac{\partial}{\partial z}\Phi(A+B,s,v).$$

By utlising (2.4), we derive (2.1).

Since $A+B<1$, then $e^{t}-(A+B)>e^{t}-1>0$, which implies that $\left(e^{t}-(A+B)\right)^{-1}<\left(e^{t}-1\right)^{-1}$. Therefore

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\left(e^{t}-A-B\right)^{-2}dt \\ &\leq\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\left(e^{t}-1\right)^{-2}dt=\frac{\partial}{\partial z}\Phi(1,s,v) \end{aligned}$$

provided $s>1$.

This proves the required inequality (2.2). □

We also have the following convexity result:

Theorem 2. *Assume that A, B are selfadjoint with* $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset(-1,1)$, *then for all $v>0$ and $s>0$,*

$$\begin{aligned} 0&\leq\frac{\Phi_{\mu}(A,s,v)+\Phi_{\mu}(B,s,v)}{2}-\Phi_{\mu}\left(\frac{A+B}{2},s,v\right) \\ &=\frac{1}{2}\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\left(\left(e^{t}-A\right)^{-1}-\left(e^{t}-B\right)^{-1}\right) \\ &\times\left(\left(e^{t}-A\right)^{-1}+\left(e^{t}-B\right)^{-1}\right)^{-1}\left(\left(e^{t}-A\right)^{-1}-\left(e^{t}-B\right)^{-1}\right)d\mu(t). \end{aligned} \tag{2.5}$$

The function $\Phi_{\mu}(\cdot,s,v)$ is operator convex on $(-1,1)$.

Proof. We start with the following elementary identity that give a simple proof for the fact that the function $f(t)=t^{-1}$ is operator convex on $(0,\infty)$, see for instance [4, p. 8]:

$$\begin{aligned} &\frac{C^{-1}+D^{-1}}{2}-\left(\frac{C+D}{2}\right)^{-1} \\ &=\frac{\left(C^{-1}-D^{-1}\right)\left(C^{-1}+D^{-1}\right)^{-1}\left(C^{-1}-D^{-1}\right)}{2}\geq 0 \end{aligned} \tag{2.6}$$

for any C, $D>0$.

If we take $C = e^t - A > 0$, $D = e^t - B > 0$, $t \in [0, \infty)$, then we get

$$\begin{aligned}(2.7)\qquad &\frac{(e^t - A)^{-1} + (e^t - B)^{-1}}{2} - \left(e^t - \frac{A+B}{2}\right)^{-1} \\ &= \frac{1}{2}\left(\left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}\right)\left(\left(e^t - A\right)^{-1} + \left(e^t - B\right)^{-1}\right)^{-1} \\ &\times \left(\left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}\right) \\ &\geq 0.\end{aligned}$$

By multiplying with $\frac{1}{\Gamma(s)} t^{s-1} e^{-(v-1)t}$ and integrating, we get

$$\begin{aligned}&\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left[\frac{(e^t - A)^{-1} + (e^t - B)^{-1}}{2} - \left(e^t - \frac{A+B}{2}\right)^{-1}\right] d\mu(t) \\ &= \frac{1}{2}\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(\left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}\right) \\ &\times \left(\left(e^t - A\right)^{-1} + \left(e^t - B\right)^{-1}\right)^{-1} \left(\left(e^t - A\right)^{-1} - \left(e^t - B\right)^{-1}\right) d\mu(t) \\ &\geq 0\end{aligned}$$

which proves the representation (2.5).

The operator convexity follows by Jensen's inequality (2.5) and the continuity of the function $\Phi_\mu(\cdot, s, v)$ in the first variable. □

If more assumptions are made for the operators A and B, then one can obtain the following lower and upper bounds:

Lemma 1. *Assume that* $0 < \alpha \leq A \leq \beta$ *and* $0 < \gamma \leq B \leq \delta$ *for some constants* α, β, γ, δ. *Then*

$$\begin{aligned}(2.8)\qquad \frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 &\leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2.\end{aligned}$$

Proof. We have $\beta^{-1} \leq A^{-1} \leq \alpha^{-1}$ and $\delta^{-1} \leq B^{-1} \leq \gamma^{-1}$, which gives

$$\beta^{-1} + \delta^{-1} \leq A^{-1} + B^{-1} \leq \alpha^{-1} + \gamma^{-1}$$

namely

$$\left(\alpha^{-1} + \gamma^{-1}\right)^{-1} \leq \left(A^{-1} + B^{-1}\right)^{-1} \leq \left(\beta^{-1} + \delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(A^{-1} - B^{-1}\right)$ and dividing by 2, we get

$$\begin{aligned}\frac{1}{2}\left(\alpha^{-1} + \gamma^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 &\leq \frac{\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)}{2} \\ &\leq \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2.\end{aligned}$$

□

A continuous function $g:\mathcal{SA}_I(H)\rightarrow\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$, the class of selfadjoint operators on I, along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B):=\lim_{s\rightarrow 0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H).$$

If the limit (2.4) exists for all $B\in\mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B\in\mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that A, $B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if, [4]

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following gradient inequalities, see for instance [2]:

Lemma 2. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then*

$$\nabla_B f(B-A)\geq f(B)-f(A)\geq\nabla_A f(B-A). \tag{2.9}$$

Let T, $S>0$. The function $f(t)=t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\rightarrow 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=-T^{-1}ST^{-1} \tag{2.10}$$

for T, $S>0$.

Using (2.10) for the operator convex function $f(t)=t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1}\geq D^{-1}-C^{-1}\geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1}\leq C^{-1}-D^{-1}\leq C^{-1}(D-C)C^{-1} \tag{2.11}$$

for all C, $D>0$.

If

$$m\leq D-C\leq M$$

for some constants m,M, then

$$mD^{-2}\leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1}\leq MC^{-2}$$

and by (2.11) we derive

$$mD^{-2}\leq C^{-1}-D^{-1}\leq MC^{-2}. \tag{2.12}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2} \tag{2.13}$$

Corollary 1. *Assume that* $0 < \alpha \leq A \leq \beta$, $0 < \gamma \leq B \leq \delta$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4} \\ &\leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}. \end{aligned} \tag{2.14}$$

Proof. From (2.13) we have

$$0 < \frac{m}{\delta^2} \leq A^{-1} - B^{-1} \leq \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \leq \left(A^{-1}-B^{-1}\right)^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.3) we get (2.14). □

Remark 1. *If the positive operators* A, B *are separated, namely* $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$ *for some constants* α, β, γ, δ, *then obviously* $0 < \gamma - \beta \leq B - A \leq \delta - \alpha$ *and by (2.14) for* $m = \gamma - \beta$ *and* $M = \delta - \alpha$, *we get*

$$\begin{aligned} 0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4} \leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \\ &\leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}. \end{aligned} \tag{2.15}$$

We have the following upper and lower bounds:

Theorem 3. *Assume that* $-1 < \alpha \leq A \leq \beta < 1$, $-1 < \gamma \leq B \leq \delta < 1$ *and* $0 < m \leq A - B \leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned} 0 &< \frac{1}{4}m^2\Lambda_{s,\nu,\mu}(\beta,\delta,\gamma) \\ &\leq \frac{\Phi_\mu(A,s,v)+\Phi_\mu(B,s,v)}{2} - \Phi_\mu\left(\frac{A+B}{2},s,v\right) \\ &\leq \frac{1}{4}M^2\Lambda_{s,\nu,\mu}(\alpha,\gamma,\beta), \end{aligned} \tag{2.16}$$

where

$$\Lambda_{s,\nu,\mu}(p,q,r) := \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}\left(e^t-p\right)\left(e^t-q\right)}{\left(e^t-\frac{p+q}{2}\right)\left(e^t-r\right)^4}d\mu(t). \tag{2.17}$$

Proof. We have

$$0<e^{t}-\beta\leq e^{t}-A\leq e^{t}-\alpha \text{ and } 0<e^{t}-\delta\leq e^{t}-B\leq e^{t}-\gamma$$

and by (2.14) we get

$$\begin{aligned} 0 &< \frac{1}{2}\left(\left(e^{t}-\beta\right)^{-1}+\left(e^{t}-\delta\right)^{-1}\right)^{-1}\frac{m^{2}}{\left(e^{t}-\gamma\right)^{4}} \\ &\leq \frac{\left(e^{t}-A\right)^{-1}+\left(e^{t}-B\right)^{-1}}{2}-\left(e^{t}-\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\left(e^{t}-\alpha\right)^{-1}+\left(e^{t}-\gamma\right)^{-1}\right)^{-1}\frac{M^{2}}{\left(e^{t}-\beta\right)^{4}}. \end{aligned} \tag{2.18}$$

Observe that

$$\left(\left(e^{t}-\beta\right)^{-1}+\left(e^{t}-\delta\right)^{-1}\right)^{-1}=\frac{\left(e^{t}-\beta\right)\left(e^{t}-\delta\right)}{2e^{t}-(\beta+\delta)}$$

and

$$\left(\left(e^{t}-\alpha\right)^{-1}+\left(e^{t}-\gamma\right)^{-1}\right)^{-1}=\frac{\left(e^{t}-\alpha\right)\left(e^{t}-\gamma\right)}{2e^{t}-(\alpha+\gamma)}$$

for $t\geq 0$.

From (2.18) we then get

$$\begin{aligned} 0 &< \frac{1}{4}m^{2}\frac{\left(e^{t}-\beta\right)\left(e^{t}-\delta\right)}{\left(e^{t}-\frac{\beta+\delta}{2}\right)\left(e^{t}-\gamma\right)^{4}} \\ &\leq \frac{\left(e^{t}-A\right)^{-1}+\left(e^{t}-B\right)^{-1}}{2}-\left(e^{t}-\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{4}M^{2}\frac{\left(e^{t}-\alpha\right)\left(e^{t}-\gamma\right)}{\left(e^{t}-\frac{\alpha+\gamma}{2}\right)\left(e^{t}-\beta\right)^{4}}. \end{aligned}$$

By multiplying with $\frac{1}{\Gamma(s)}t^{s-1}e^{-(v-1)t}$ and integrating, we get

$$\begin{aligned} 0 &< \frac{1}{4}m^{2}\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\frac{\left(e^{t}-\beta\right)\left(e^{t}-\delta\right)}{\left(e^{t}-\frac{\beta+\delta}{2}\right)\left(e^{t}-\gamma\right)^{4}}d\mu(t) \\ &\leq \frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t} \\ &\quad\times\left[\frac{\left(e^{t}-A\right)^{-1}+\left(e^{t}-B\right)^{-1}}{2}-\left(e^{t}-\frac{A+B}{2}\right)^{-1}\right]d\mu(t) \\ &\leq \frac{1}{4}M^{2}\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\frac{\left(e^{t}-\alpha\right)\left(e^{t}-\gamma\right)}{\left(e^{t}-\frac{\alpha+\gamma}{2}\right)\left(e^{t}-\beta\right)^{4}}d\mu(t). \end{aligned}$$

This is equivalent to (2.16). □

The upper and lower bounds from Theorem 3 are difficult to compute. However simpler lower and upper bounds may be provided as in the following two corollaries.

Corollary 2. *With the assumptions of Theorem 3, we have*

$$0<\frac{1}{24}m^{2}\frac{(1-\beta)(1-\delta)}{\left(1-\frac{\beta+\delta}{2}\right)}\frac{\partial^{3}}{\partial z^{3}}\Phi_{\mu}(\gamma,s,v)\leq\frac{\Phi_{\mu}(A,s,v)+\Phi_{\mu}(B,s,v)}{2}-\Phi_{\mu}\left(\frac{A+B}{2},s,v\right). \tag{2.19}$$

Proof. Let $a,\,b\in\mathbb{R}$. Consider

$$g(x)=\frac{(x-b)(x-a)}{x-\frac{a+b}{2}},\ x\neq\frac{a+b}{2}.$$

We have

$$\begin{aligned}g'(x)&=\frac{2\left(x-\frac{a+b}{2}\right)^{2}-(x-b)(x-a)}{\left(x-\frac{a+b}{2}\right)^{2}}\\&=\frac{2x^{2}-2(a+b)x+2\left(\frac{a+b}{2}\right)^{2}-x^{2}+(a+b)x-ab}{\left(x-\frac{a+b}{2}\right)^{2}}\\&=\frac{x^{2}-(a+b)x+\left(\frac{a+b}{2}\right)^{2}+\left(\frac{a+b}{2}\right)^{2}-ab}{\left(x-\frac{a+b}{2}\right)^{2}}\\&=\frac{\left(x-\frac{a+b}{2}\right)^{2}+\frac{1}{4}(b-a)^{2}}{\left(x-\frac{a+b}{2}\right)^{2}},\end{aligned}$$

which shows that g is increasing on $(-\infty,\frac{a+b}{2})$ and on $\left(\frac{a+b}{2},\infty\right)$.

Therefore

$$\frac{(e^{t}-\beta)(e^{t}-\delta)}{\left(e^{t}-\frac{\beta+\delta}{2}\right)}\geq\frac{(1-\beta)(1-\delta)}{1-\frac{\beta+\delta}{2}}>0,\ t\geq0.$$

We then obtain,

$$\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}e^{-(v-1)t}\frac{(e^{t}-\beta)(e^{t}-\delta)}{\left(e^{t}-\frac{\beta+\delta}{2}\right)(e^{t}-\gamma)^{4}}d\mu(t)\geq\frac{(1-\beta)(1-\delta)}{\left(1-\frac{\beta+\delta}{2}\right)\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{(e^{t}-\gamma)^{4}}d\mu(t). \tag{2.20}$$

Observe that, by (1.6), we have

$$\frac{\partial}{\partial z}\Phi_{\mu}(z,s,v)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{(e^{t}-z)^{2}}dt,$$

$$\frac{\partial^{2}}{\partial z^{2}}\Phi_{\mu}(z,s,v)=\frac{2}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{(e^{t}-z)^{3}}dt$$

and

$$\frac{\partial^{3}}{\partial z^{3}}\Phi_{\mu}(z,s,v)=\frac{6}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{(e^{t}-z)^{4}}dt.$$

By (2.20) we get

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\frac{(e^t-\beta)(e^t-\delta)}{\left(e^t-\frac{\beta+\delta}{2}\right)(e^t-\gamma)^4}d\mu(t)$$
$$\geq \frac{(1-\beta)(1-\delta)}{6\left(1-\frac{\beta+\delta}{2}\right)}\frac{\partial^3}{\partial z^3}\Phi_\mu(\gamma,s,v).$$

By utilising the first inequality in (2.16), we deduce the desired lower bound (2.19). □

Corollary 3. *With the assumptions of Theorem 3 and if* $v>1$, *we have*

$$\frac{\Phi_\mu(A,s,v)+\Phi_\mu(B,s,v)}{2}-\Phi_\mu\left(\frac{A+B}{2},s,v\right)$$
$$\leq \frac{1}{24}M^2\left[\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v-1)-\frac{\alpha+\gamma}{2}\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v)\right]. \tag{2.21}$$

Proof. Using the elementary inequality

$$ab\leq\frac{1}{4}(b+a)^2,\ a,\ b\in\mathbb{R},$$

we get

$$\left(e^t-\alpha\right)\left(e^t-\gamma\right)\leq\left(e^t-\frac{\alpha+\gamma}{2}\right)^2.$$

This implies that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\frac{(e^t-\alpha)(e^t-\gamma)}{\left(e^t-\frac{\alpha+\gamma}{2}\right)(e^t-\beta)^4}d\mu(t) \tag{2.22}$$
$$\leq\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(e^t-\frac{\alpha+\gamma}{2}\right)\frac{1}{(e^t-\beta)^4}d\mu(t)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-2)t}}{(e^t-\beta)^4}d\mu(t)-\frac{\alpha+\gamma}{2}\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{(e^t-\beta)^4}d\mu(t).$$

Since

$$\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{(e^t-z)^4}dt=\frac{1}{6}\frac{\partial^3}{\partial z^3}\Phi_\mu(z,s,v),\ z>0,$$

hence

$$\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-2)t}}{(e^t-\beta)^4}d\mu(t)=\frac{1}{6}\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v-1)$$

and

$$\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(v-1)t}}{(e^t-\beta)^4}d\mu(t)=\frac{1}{6}\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v).$$

Then by (2.22) we get

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\frac{(e^t-\alpha)(e^t-\gamma)}{\left(e^t-\frac{\alpha+\gamma}{2}\right)(e^t-\beta)^4}d\mu(t)$$
$$\leq\frac{1}{6}\left[\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v-1)-\frac{\alpha+\gamma}{2}\frac{\partial^3}{\partial z^3}\Phi_\mu(\beta,s,v)\right]$$

for $v>1$ and $s>0$.

By making use of the second inequality in (2.16) we obtain the desired result (2.21). □

3. Examples for Polylogarithm

Observe that

$$\operatorname{Li}(T,s)=T\Phi(T,s,1)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}T\left(e^t-T\right)^{-1}dt \tag{3.1}$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(T-e^t+e^t\right)\left(e^t-T\right)^{-1}dt$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left[e^t\left(e^t-T\right)^{-1}-1\right]dt$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left[\left(e^t-T\right)^{-1}-e^{-t}\right]dt$$

for $\operatorname{Re}s>0$ and T with $\operatorname{Sp}(T)\subset(-1,1)$.

Assume that A, B are selfadjoint with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset(-1,1)$, then for all $s>0$,

$$\frac{\operatorname{Li}(A,s)+\operatorname{Li}(B,s)}{2}-\operatorname{Li}\left(\frac{A+B}{2},s\right)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left[\frac{\left(e^t-A\right)^{-1}+\left(e^t-B\right)^{-1}}{2}-\left(e^t-\frac{A+B}{2}\right)^{-1}\right]dt.$$

By utilising the identity (2.7), we derive

$$\frac{\operatorname{Li}(A,s)+\operatorname{Li}(B,s)}{2}-\operatorname{Li}\left(\frac{A+B}{2},s\right)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left[\frac{\left(e^t-A\right)^{-1}+\left(e^t-B\right)^{-1}}{2}-\left(e^t-\frac{A+B}{2}\right)^{-1}\right]dt$$
$$=\frac{1}{2}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left(\left(e^t-A\right)^{-1}-\left(e^t-B\right)^{-1}\right)\left(\left(e^t-A\right)^{-1}+\left(e^t-B\right)^{-1}\right)^{-1}$$
$$\times\left(\left(e^t-A\right)^{-1}-\left(e^t-B\right)^{-1}\right)dt$$
$$\geq 0,$$

for $s>0$, which shows that $\operatorname{Li}(\cdot,s)$ *is operator convex on* $(-1,1)$.

Assume that $-1<\alpha\leq A\leq\beta<1$, $-1<\gamma\leq B\leq\delta<1$ and $0<m\leq A-B\leq M$ for some constants α, β, γ, δ, m, M. Then, like in Theorem 3

$$0<\frac{1}{4}m^2\Lambda_s(\beta,\delta,\gamma) \tag{3.2}$$
$$\leq\frac{\operatorname{Li}(A,s)+\operatorname{Li}(B,s)}{2}-\operatorname{Li}\left(\frac{A+B}{2},s\right)$$
$$\leq\frac{1}{4}M^2\Lambda_s(\alpha,\gamma,\beta),$$

where

$$\Lambda_s(p,q,r):=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^t\left(e^t-p\right)\left(e^t-q\right)}{\left(e^t-\frac{p+q}{2}\right)\left(e^t-r\right)^4}dt \tag{3.3}$$

and $s>0$.

For $z\in(-1,1)$, we have

$$\operatorname{Li}(z,s)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{t}\left[\left(e^{t}-z\right)^{-1}-e^{-t}\right]dt.$$

Observe that

$$\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{t}}{\left(e^{t}-z\right)^{4}}dt=\frac{1}{6}\frac{\partial^3}{\partial z^3}\operatorname{Li}(z,s).$$

Like in Corollary 2, we have for $s>0$, that

$$\begin{aligned} 0 &< \frac{1}{24}m^2\frac{(1-\beta)(1-\delta)}{\left(1-\frac{\beta+\delta}{2}\right)}\frac{\partial^3}{\partial z^3}\operatorname{Li}(z,s) \\ &\le \frac{\operatorname{Li}(A,s)+\operatorname{Li}(B,s)}{2}-\operatorname{Li}\left(\frac{A+B}{2},s\right), \end{aligned} \tag{3.4}$$

provided $-1<A\le\beta<1$, $-1<B\le\delta<1$ and $0<m\le A-B\le M$ for some constants α, β, γ, δ, m, M.

References

[1] T. M. Apostol, Zeta and related functions, in: F. W. F. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, National Institute of Standards and Technology, Cambridge, Washington, DC, 2010, pp. 601–616. Available online in http://dlmf.nist.gov/25.

[2] S.S. Dragomir, Reverses of operator Féjer's inequalities, *Tokyo J. Math.*, Vol. **44**, NO. 1, 2021. DOI: 10.3836/tjm/1502179330

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[4] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[5] M. Lerch, Note sur la fonction $\mathfrak{k}(w,x,s)=\sum_{k=0}^{\infty}e^{2\pi ikx}(w+k)^{-s}$, *Acta Math.* **11** (1887) 19–24.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.