ON THE SUBADDITIVITY OF OPERATOR MONOTONE FUNCTIONS FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Assume that f is an operator monotone function on $[0,\infty)$ and A, $B>0$. In this paper we show among other that, if $AB+BA\geq k$, k a real number, then

$$\begin{aligned}&k\left[f(A+B)-f(0)-(A+B)f'(A+B)\right](A+B)^{-2}\\&\leq f(A)+f(B)-f(A+B)-f(0),\end{aligned}$$

which generalizes and improves a recent result of Moslehian and Najafi. In particular we obtain that

$$k(1-r)(A+B)^{r-2}\leq A^{r}+B^{r}-(A+B)^{r}$$

for $r\in(0,1]$.

If if $A+B\leq K$, then the reverse inequality also holds

$$\begin{aligned}&f(A)+f(B)-f(A+B)-f(0)\\&\leq K^{2}\left[f(A+B)-f(0)-(A+B)f'(A+B)\right](A+B)^{-2},\end{aligned}$$

which gives the power inequality

$$A^{r}+B^{r}-(A+B)^{r}\leq K^{2}(1-r)(A+B)^{r-2}.$$

Some integral inequalities are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq0$) if $\langle Tx,x\rangle\geq0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r}=\frac{\sin(r\pi)}{\pi}t\int_{0}^{\infty}\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_{0}^{u}\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

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By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.4}$$

The interested reader may find several examples of monotone operator functions in [2], [3], [4], [9] and [10].

Assume that $A,\ B \geq 0$. In the recent paper [8], Moslehian and Najafi showed that $AB+BA$ is positive if and only if the following *operator subadditivity property* holds

$$f(A+B) \leq f(A) + f(B) \tag{1.5}$$

for *all nonnegative operator monotone functions f* on $[0,\infty)$. For some interesting consequences of this result see [8].

In this paper we show among other that, if $AB+BA \geq k$, k a real number, then for any operator monotone function f on $[0,\infty)$,

$$\begin{aligned} &k\left[f(A+B) - f(0) - (A+B) f'(A+B)\right](A+B)^{-2} \\ &\leq f(A) + f(B) - f(A+B) - f(0), \end{aligned}$$

which generalizes and improves the result of Moslehian and Najafi (1.5). In particular we obtain that

$$k(1-r)(A+B)^{r-2} \leq A^r + B^r - (A+B)^r$$

for $r \in (0,1]$.

If $A+B \leq K$, then the reverse inequality also holds

$$\begin{aligned} &f(A) + f(B) - f(A+B) - f(0) \\ &\leq K^2\left[f(A+B) - f(0) - (A+B) f'(A+B)\right](A+B)^{-2}, \end{aligned}$$

which gives the power inequality

$$A^r + B^r - (A+B)^r \leq K^2 (1-r)(A+B)^{r-2}.$$

Some integral inequalities are also provided.

2. Main Results

We start with the following lemma of interest in itself:

Lemma 1. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). Then for all A, $B > 0$ we have*

$$\begin{aligned} &f(A)+f(B)-f(A+B)-f(0) \\ &= \int_0^\infty \lambda (A+B+\lambda)^{-1} \left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB\right] \\ &\times (A+B+\lambda)^{-1} d\mu(\lambda) \\ &+ \int_0^\infty \lambda (A+B+\lambda)^{-1} (BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda). \end{aligned} \tag{2.1}$$

Proof. For A, $B > 0$ and $\lambda \geq 0$, define

$$K_\lambda := (A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}. \tag{2.2}$$

We have

$$\begin{aligned} &f(A)+f(B)-f(A+B)-f(0) \\ &= f(0) + bA + \int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) \\ &+ f(0) + bA + \int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) \\ &- f(0) + b(A+B) - \int_0^\infty \lambda (A+B)(A+B+\lambda)^{-1} d\mu(\lambda) - f(0) \\ &= \int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) + \int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) \\ &- \int_0^\infty \lambda (A+B)(A+B+\lambda)^{-1} d\mu(\lambda). \end{aligned} \tag{2.3}$$

For $T > 0$, we have that

$$\begin{aligned} \int_0^\infty \lambda T (T+\lambda)^{-1} d\mu(\lambda) &= \int_0^\infty \lambda (T+\lambda-\lambda)(T+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty \lambda \left(1 - \lambda (T+\lambda)^{-1}\right) d\mu(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} &\int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) + \int_0^\infty \lambda A (A+\lambda)^{-1} d\mu(\lambda) \\ &- \int_0^\infty \lambda (A+B)(A+B+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty \lambda \left(1 - \lambda (A+\lambda)^{-1}\right) d\mu(\lambda) + \int_0^\infty \lambda \left(1 - \lambda (B+\lambda)^{-1}\right) d\mu(\lambda) \\ &- \int_0^\infty \lambda \left(1 - \lambda (A+B+\lambda)^{-1}\right) d\mu(\lambda) \\ &= \int_0^\infty \lambda (1 - \lambda K_\lambda) d\mu(\lambda). \end{aligned} \tag{2.4}$$

For A, $B>0$ and $\lambda\geq 0$, define

$$W_{\lambda}:=1-\lambda K_{\lambda},$$

where K_{λ} is given in (2.2).

We have successively

$$\begin{aligned}
&(A+B+\lambda)W_{\lambda}(A+B+\lambda)\\
&=(A+B+\lambda)(1-\lambda K_{\lambda})(A+B+\lambda)\\
&=(A+B+\lambda)^{2}-\lambda(A+B+\lambda)K_{\lambda}(A+B+\lambda)\\
&=(A+B+\lambda)(A+B+\lambda)\\
&-\lambda\left[B(A+\lambda)^{-1}B+A(B+\lambda)^{-1}A+2(A+B)+\lambda\right]\\
&=A^{2}+BA+\lambda A+AB+B^{2}+\lambda B+\lambda A+\lambda B+\lambda^{2}\\
&-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A-2\lambda(A+B)-\lambda^{2}\\
&=A^{2}+B^{2}+BA+AB-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A\\
&=A(B+\lambda)^{-1}(B+\lambda)A-\lambda A(B+\lambda)^{-1}A\\
&+B(A+\lambda)^{-1}(A+\lambda)B-\lambda B(A+\lambda)^{-1}B+BA+AB\\
&=A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB,
\end{aligned}$$

therefore

$$\begin{aligned}
W_{\lambda}&=(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right]\\
&\times(A+B+\lambda)^{-1}.
\end{aligned}\tag{2.5}$$

By (2.3) and (2.4) we have

$$\begin{aligned}
&f(A)+f(B)-f(A+B)-f(0)\\
&=\int_{0}^{\infty}\lambda W_{\lambda}d\mu(\lambda)\\
&=\int_{0}^{\infty}\lambda(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right]\\
&\times(A+B+\lambda)^{-1}d\mu(\lambda)\\
&+\int_{0}^{\infty}\lambda(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)
\end{aligned}$$

and the identity (2.1) is proved. □

Theorem 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.3). Then for all A, $B>0$ we have*

$$\begin{aligned}
&\int_{0}^{\infty}\lambda(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)\\
&\leq f(A)+f(B)-f(A+B)-f(0)\\
&\leq\int_{0}^{\infty}\lambda(A+B+\lambda)^{-1}(A+B)^{2}(A+B+\lambda)^{-1}d\mu(\lambda).
\end{aligned}\tag{2.6}$$

Proof. Since A, $B > 0$, hence

$$0 < (B+\lambda)^{-1} B \le 1 \text{ and } 0 < (A+\lambda)^{-1} A \le 1$$

and by multiplying both sides with A and B respectively, we get

$$0 < A(B+\lambda)^{-1} BA \le A^2 \text{ and } 0 < B(A+\lambda)^{-1} AB \le B^2$$

for $\lambda \ge 0$.

If we multiply both sides by $(A+B+\lambda)^{-1}$ with $\lambda \ge 0$, we obtain

$$\begin{aligned}
&0 < (A+B+\lambda)^{-1} A(B+\lambda)^{-1} BA(A+B+\lambda)^{-1} \\
&\quad \le (A+B+\lambda)^{-1} A^2 (A+B+\lambda)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&0 < (A+B+\lambda)^{-1} B(A+\lambda)^{-1} AB(A+B+\lambda)^{-1} \\
&\quad \le (A+B+\lambda)^{-1} B^2 (A+B+\lambda)^{-1}
\end{aligned}$$

where $\lambda \ge 0$.

If we add these inequality, multiply by $\lambda \ge 0$ and integrate, then we get

$$\begin{aligned}
&0 \le \int_0^\infty \lambda (A+B+\lambda)^{-1} \left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB\right] \\
&\times (A+B+\lambda)^{-1} d\mu(\lambda) \\
&\le \int_0^\infty \lambda (A+B+\lambda)^{-1} \left(A^2+B^2\right)(A+B+\lambda)^{-1}.
\end{aligned}$$

Moreover, if we add to these inequalities the same quantity

$$\int_0^\infty \lambda (A+B+\lambda)^{-1} (BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda),$$

then we obtain

$$\begin{aligned}
&\int_0^\infty \lambda (A+B+\lambda)^{-1} (BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda) \\
&\le \int_0^\infty \lambda (A+B+\lambda)^{-1} \left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB\right] \\
&\times (A+B+\lambda)^{-1} d\mu(\lambda) \\
&+ \int_0^\infty \lambda (A+B+\lambda)^{-1} (BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda) \\
&\le \int_0^\infty \lambda (A+B+\lambda)^{-1} \left(A^2+B^2\right)(A+B+\lambda)^{-1} \\
&+ \int_0^\infty \lambda (A+B+\lambda)^{-1} (BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda) \\
&= \int_0^\infty \lambda (A+B+\lambda)^{-1} (A+B)^2 (A+B+\lambda)^{-1}
\end{aligned}$$

and by employing the identity (2.1), we derive (2.6). ☐

Corollary 1. *With the assumptions of Theorem 2 and if* $AB + BA \geq k$, k *a real number, then*

$$k\left[f(A+B) - f(0) - (A+B) f'(A+B)\right](A+B)^{-2} \leq f(A) + f(B) - f(A+B) - f(0). \tag{2.7}$$

If $AB + BA \geq k \geq 0$, *then*

$$0 \leq k\left[f(A+B) - f(0) - (A+B) f'(A+B)\right](A+B)^{-2} \leq f(A) + f(B) - f(A+B) - f(0). \tag{2.8}$$

Proof. If $AB+BA \geq k$, then by multiplying both sides by $(A+B+\lambda)^{-1}$ for $\lambda \geq 0$, we get

$$(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1} \geq k(A+B+\lambda)^{-2}.$$

If we multiply by $\lambda \geq 0$ and integrate, then we obtain

$$k\int_0^\infty \lambda (A+B+\lambda)^{-2} d\mu(\lambda) \leq \int_0^\infty \lambda (A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda). \tag{2.9}$$

If we take the derivative over t in (1.3), then we get

$$\frac{f(t) - f(0)}{t} = b + \int_0^\infty \frac{\lambda}{t+\lambda} d\mu(\lambda), \ t > 0. \tag{2.10}$$

Taking the derivative over t in (2.9), we get

$$\frac{tf'(t) - f(t) + f(0)}{t^2} = -\int_0^\infty \frac{\lambda}{(t+\lambda)^2} d\mu(\lambda), \ t > 0.$$

This gives that

$$\begin{aligned}&(f(A+B) - f(0) - (A+B) f'(A+B))(A+B)^{-2} \\ &= \int_0^\infty \lambda (A+B+\lambda)^{-2} d\mu(\lambda) \geq 0\end{aligned}$$

and by (2.9) we derive (2.7).

The inequality (2.8) follows by that fact that

$$(f(A+B) - f(0) - (A+B) f'(A+B))(A+B)^{-2} \geq 0.$$

□

Corollary 2. *With the assumptions of Theorem 2 and if* $(0 <)\ A+B \leq K$, *then*

$$f(A) + f(B) - f(A+B) - f(0) \leq K^2\left[f(A+B) - f(0) - (A+B) f'(A+B)\right](A+B)^{-2}. \tag{2.11}$$

Proof. Since $0 < A+B \leq K$, hence $(A+B)^2 \leq K^2$. If we multiply this inequality both sides by $(A+B+\lambda)^{-1}$, we get

$$(A+B+\lambda)^{-1}(A+B)^2(A+B+\lambda)^{-1} \leq K^2(A+B+\lambda)^{-2}$$

for $\lambda \geq 0$.

If we multiply by $\lambda \geq 0$ and integrate, then we get

$$\int_0^\infty \lambda (A+B+\lambda)^{-1} (A+B)^2 (A+B+\lambda)^{-1} \mu(\lambda) \leq K^2 \int_0^\infty \lambda (A+B+\lambda)^{-2} \mu(\lambda)$$

and by the second inequality in (2.6) we deduce the desired result (2.11). □

The symmetrized product of two operators $A, B \in B(H)$ is defined by $S(A,B) = AB + BA$. In general, the symmetrized product of two operators A, B is not positive (see for instance [9]). Also Gustafson [5] showed that if $0 \leq m \leq A \leq M$ and $0 \leq n \leq B \leq N$, then we have the lower bound

$$S(A,B) \geq 2mn - \frac{1}{4}(M-m)(N-n) =: k, \tag{2.12}$$

which can take positive or negative values depending on the parameters m, M, n, N.

Corollary 3. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$. If $0 \leq m \leq A \leq M$ and $0 \leq n \leq B \leq N$, then we have the upper and lower bounds*

$$\begin{aligned} &\left[2mn - \frac{1}{4}(M-m)(N-n)\right] \\ &\times \left[f(A+B) - f(0) - (A+B) f'(A+B)\right] (A+B)^{-2} \\ &\leq f(A) + f(B) - f(A+B) - f(0) \\ &\leq (M+N)^2 \left[f(A+B) - f(0) - (A+B) f'(A+B)\right] (A+B)^{-2}. \end{aligned} \tag{2.13}$$

3. Some Integral Inequalities

We have the following integral inequality.

Proposition 1. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and that C, $D > 0$ such that $CD + DC \geq k$. Then we have the integral inequalities*

$$\begin{aligned} &\frac{1}{2} k \left[f(C+D) - f(0) - (C+D) f'(C+D)\right] (C+D)^{-2} \\ &\leq \int_0^1 f((1-t)C + tD)\, dt - \frac{f(C+D) + f(0)}{2}. \end{aligned} \tag{3.1}$$

If $CD + DC \geq 0$, then

$$\frac{f(C+D) + f(0)}{2} \leq \int_0^1 f((1-t)C + tD)\, dt. \tag{3.2}$$

Proof. Let $t \in [0,1]$ and put $A = (1-t)C + tD$, $B = tC + (1-t)D$. Then

$$\begin{aligned} AB &= ((1-t)C + tD)(tC + (1-t)D) \\ &= (1-t)tC^2 + t^2 DC + (1-t)^2 CD + t(1-t) D^2 \end{aligned}$$

and

$$\begin{aligned} BA &= (tC + (1-t)D)((1-t)C + tD) \\ &= t(1-t)C^2 + (1-t)^2 DC + t^2 CD + (1-t) tD^2 \end{aligned}$$

for $t\in[0,1]$.

Therefore

$$\begin{aligned}&AB+BA\\&=2t(1-t)C^2+\left[(1-t)^2+t^2\right](CD+DC)+2(1-t)tD^2\\&=2t(1-t)C^2+\left(1+2t^2-2t\right)(CD+DC)+2(1-t)tD^2\\&=2t(1-t)\left(C^2-CD-DC+D^2\right)+CD+DC\\&=2t(1-t)(C-D)^2+CD+DC\geq k\end{aligned}$$

for $t\in[0,1]$.

From (2.8) we have

$$\begin{aligned}&k\left[f(C+D)-f(0)-(C+D)f'(C+D)\right](C+D)^{-2}\\&\leq f((1-t)C+tD)+f(tC+(1-t)D)-f(C+D)-f(0)\end{aligned}\tag{3.3}$$

for $t\in[0,1]$.

Taking the integral and observing that

$$\int_0^1 f((1-t)C+tD)\,dt=\int_0^1 f(tC+(1-t)D)\,dt,$$

then by (3.3) we obtain (3.1). □

Proposition 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and that C, $D>0$ such that $C+D\leq K$. Then*

$$\begin{aligned}&\int_0^1 f((1-t)C+tD)\,dt-\frac{f(C+D)+f(0)}{2}\\&\leq\frac{1}{2}K^2\left[f(C+D)-f(0)-(C+D)f'(C+D)\right](C+D)^{-2}.\end{aligned}\tag{3.4}$$

Proof. The proof follows from (2.11) by a similar argument to the one from the proof of Proposition 1. □

Further, we have

Proposition 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and that C, $D>0$ such that $CD+DC\geq 0$. Then we have the integral inequalities*

$$\int_0^1 f((1-t)C+tD)\,dt+f(0)\leq\int_0^1 f(tC)\,dt+\int_0^1 f(tD)\,dt\tag{3.5}$$

Proof. If $CD+DC\geq 0$, then $t(1-t)CD+t(1-t)DC\geq 0$ and by (2.8) for $A=(1-t)C$ and $B=tD$, we get

$$0\leq f((1-t)C)+f(tD)-f((1-t)C+tD)-f(0),$$

namely

$$f((1-t)C+tD)+f(0)\leq f((1-t)C)+f(tD),$$

for all $t\in[0,1]$.

If we take the integral and observe that

$$\int_0^1 f((1-t)C)\,dt=\int_0^1 f(sC)\,ds,$$

then we obtain the desired result (3.5). □

Proposition 4. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and that* $C,\ D>0$ *such that* $C,\ D\leq K$, *then*

$$\begin{aligned}(3.6)\qquad &\int_0^1 f(tC)\,dt+\int_0^1 f(tD)\,dt-\int_0^1 f((1-t)C+tD)\,dt-f(0)\\ &\leq K^2\left[\int_0^1 f((1-t)C+tD)((1-t)C+tD)^{-2}\,dt\right.\\ &-f(0)\int_0^1((1-t)C+tD)^{-2}\,dt\\ &\left.-\int_0^1((1-t)C+tD)^{-1}f'((1-t)C+tD)\,dt\right].\end{aligned}$$

Proof. By (2.11) for $A=(1-t)C$, $B=tD$, we get

$$\begin{aligned}&f((1-t)C)+f(tD)-f((1-t)C+tD)-f(0)\\ &\leq K^2\left[f((1-t)C+tD)-f(0)-((1-t)C+tD)f'((1-t)C+tD)\right]\\ &\times((1-t)C+tD)^{-2}\end{aligned}$$

and by integrating this inequality, we derive (3.6). □

4. Some Examples

Consider the function $f(t)=(t+a)^r$ for $a\geq 0$, $r\in(0,1]$ and $t\in[0,\infty)$. Assume that $A,\ B>0$.

If $AB+BA\geq k$, k a real number, then

$$\begin{aligned}(4.1)\qquad &k\left[(A+B+a)^r-a^r-r(A+B)(A+B+a)^{r-1}\right](A+B)^{-2}\\ &\leq(A+a)^r+(B+a)^r-(A+B+a)^r-a^r.\end{aligned}$$

For $a=0$, we derive

$$(4.2)\qquad k(1-r)(A+B)^{r-2}\leq A^r+B^r-(A+B)^r.$$

If $0\leq m\leq A\leq M$ and $0\leq n\leq B\leq N$, then by (4.1) we get

$$\begin{aligned}&\left[2mn-\frac{1}{4}(M-m)(N-n)\right]\\ &\times\left[(A+B+a)^r-a^r-r(A+B)(A+B+a)^{r-1}\right](A+B)^{-2}\\ &\leq(A+a)^r+(B+a)^r-(A+B+a)^r-a^r.\end{aligned}$$

For $a=0$, we derive

$$\left[2mn-\frac{1}{4}(M-m)(N-n)\right](1-r)(A+B)^{r-2}\leq A^r+B^r-(A+B)^r.$$

If $AB+BA\geq 0$, then by (4.1) we derive

$$(A+B+a)^r+a^r\leq(A+a)^r+(B+a)^r.$$

If $A+B\leq K$, then by (2.11) we get

$$(A+a)^r+(B+a)^r-(A+B+a)^r-a^r$$
$$\leq K^2\left[(A+B+a)^r-a^r-r(A+B)(A+B+a)^{r-1}\right](A+B)^{-2}$$

for $a\geq 0$ and, in particular,

$$A^r+B^r-(A+B)^r\leq K^2(1-r)(A+B)^{r-2}. \tag{4.3}$$

Consider the operator monotone function $f(t)=-(t+a)^{-p}$, for $a>0$, $p\in(0,1]$ and $t\in[0,\infty)$. Assume that $A,\ B>0$.

If $AB+BA\geq k$, k a real number, then by (2.7)

$$k\left[a^{-p}-(A+B+a)^{-p}-p(A+B)(A+B+a)^{-p-1}\right](A+B)^{-2}$$
$$\leq(A+B+a)^{-p}+a^{-p}-(B+a)^{-p}-(A+a)^{-p}.$$

For $p=1$, we get

$$k\left[a^{-1}-(A+B+a)^{-1}-(A+B)(A+B+a)^{-2}\right](A+B)^{-2}$$
$$\leq(A+B+a)^{-1}+a^{-1}-(B+a)^{-1}-(A+a)^{-1}.$$

If $A+B\leq K$, then by (2.11)

$$(A+B+a)^{-p}+a^{-p}-(B+a)^{-p}-(A+a)^{-p}$$
$$\leq K^2\left[a^{-p}-(A+B+a)^{-p}-p(A+B)(A+B+a)^{-p-1}\right](A+B)^{-2}$$

and for $p=1$,

$$(A+B+a)^{-1}+a^{-1}-(B+a)^{-1}-(A+a)^{-1}$$
$$\leq K^2\left[a^{-1}-(A+B+a)^{-1}-(A+B)(A+B+a)^{-2}\right](A+B)^{-2}.$$

Consider the operator convex function $f(t)=\ln(t+a)$, for $a>0$ and $t\in[0,\infty)$. Assume that $A,\ B>0$.

If $AB+BA\geq k$, k a real number, then by (2.7),

$$k\left[\ln(A+B+a)-\ln a-(A+B)(A+B+a)^{-1}\right](A+B)^{-2}$$
$$\leq\ln(A+a)+\ln(B+a)-\ln(A+B+a)-\ln a.$$

For $a=1$, we get

$$k\left[(A+B)^{-2}\ln(A+B+1)-(A+B)^{-1}(A+B+1)^{-1}\right] \tag{4.4}$$
$$\leq\ln(A+1)+\ln(B+1)-\ln(A+B+1).$$

If $A+B\leq K$, then by (2.11),

$$\ln(A+a)+\ln(B+a)-\ln(A+B+a)-\ln a$$
$$\leq K^2\left[\ln(A+B+a)-\ln a-(A+B)(A+B+a)^{-1}\right](A+B)^{-2}.$$

For $a=1$, we derive

$$\ln(A+1)+\ln(B+1)-\ln(A+B+1) \tag{4.5}$$
$$\leq K^2\left[(A+B)^{-2}\ln(A+B+1)-(A+B)^{-1}(A+B+1)^{-1}\right].$$

If $A,\ B>0$ such that $AB+BA\geq k$, then by Proposition 2.1 for the power function $f(t)=t^{r}$, $r\in(0,1]$,

$$\frac{1}{2}k(1-r)(A+B)^{r-2}\leq\int_{0}^{1}((1-t)A+tB)^{r}dt-\frac{1}{2}(A+B)^{r}. \tag{4.6}$$

If $AB+BA\geq 0$, then

$$\frac{1}{2}(A+B)^{r}\leq\int_{0}^{1}((1-t)A+tB)^{r}dt\left(\leq\left(\frac{A+B}{2}\right)^{r}\right). \tag{4.7}$$

The second inequality in (4.7) follows by the operator concavity of the power function $f(t)=t^{r}$, $r\in(0,1]$ and holds for any $A,\ B\geq 0$.

If $A,\ B>0$ such that $A+B\leq K$, then by (3.4)

$$\int_{0}^{1}((1-t)A+tB)^{r}dt-\frac{1}{2}(A+B)^{r}\leq\frac{1}{2}K^{2}(1-r)(A+B)^{r-2}. \tag{4.8}$$

If $A,\ B>0$ such that $AB+BA\geq 0$, then by (3.5)

$$\left(\frac{A^{r}+B^{r}}{2}\leq\right)\int_{0}^{1}((1-t)A+tB)^{r}dt\leq\frac{A^{r}+B^{r}}{r+1}. \tag{4.9}$$

The first inequality in (4.9) follows by the operator concavity of the power function and holds for any $A,\ B\geq 0$.

If $A,\ B>0$ such that $A,\ B\leq K$, then

$$\frac{A^{r}+B^{r}}{r+1}-\int_{0}^{1}((1-t)A+tB)^{r}dt\leq K^{2}(1-r)\int_{0}^{1}((1-t)A+tB)^{r-2}dt. \tag{4.10}$$

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