ON THE SUPERADDITIVITY OF OPERATOR CONVEX FUNCTIONS FOR POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Assume that f is an operator convex function on $[0,\infty)$ and A, $B>0$. In this paper we show among other that, if $AB+BA\geq k$, for k a real number, then

$$k\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}$$
$$\leq f(A+B)+f(0)-f(A)-f(B).$$

In particular we obtain that

$$kr(A+B)^{r-1}\leq(A+B)^{r+1}-A^{r+1}-B^{r+1}$$

for $r\in(0,1]$. We also have the logarithmic inequality

$$k(A+B)^{-1}\leq(A+B)\ln(A+B)-A\ln A-B\ln(B).$$

If $A+B\leq K$, then the reverse inequality holds

$$f(A+B)+f(0)-f(A)-f(B)$$
$$\leq K^{2}\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2},$$

which gives the power inequality

$$(A+B)^{r+1}-A^{r+1}-B^{r+1}\leq K^{2}r(A+B)^{r-1}.$$

We also have the logarithmic inequality

$$(A+B)\ln(A+B)-A\ln A-B\ln(B)\leq K^{2}(A+B)^{-1}.$$

Some integral inequalities are also provided.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_{0}^{\infty}\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [9], see for instance [1, p. 144-145]:

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator inequalities, Löwner-Heinz inequality, Logarithmic operator inequalities, Integral inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 137, 14 pp. Received 18/11/20

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.2}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.3}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t)=f(0)+f'_+(0)t+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.4}$$

*where $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (**??**) holds.*

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [6] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

Assume that A, $B\geq 0$. In the recent paper [10], Moslehian and Najafi showed that $AB+BA$ is positive if and only if the following *operator subadditivity property* holds

$$f(A+B)\leq f(A)+f(B) \tag{1.5}$$

for all *nonnegative operator monotone functions* f on $[0,\infty)$. For some interesting consequences of this result see [10].

In this paper we show among other that, if f is operator convex on $[0,\infty)$, A, $B>0$ with $AB+BA\geq k$, for k a real number, then

$$\begin{aligned}&k\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}\\&\leq f(A+B)+f(0)-f(A)-f(B).\end{aligned}$$

In particular we obtain that

$$kr(A+B)^{r-1}\leq(A+B)^{r+1}-A^{r+1}-B^{r+1}$$

for $r\in(0,1]$. We also have the logarithmic inequality

$$k(A+B)^{-1}\leq(A+B)\ln(A+B)-A\ln A-B\ln(B).$$

If $A+B\leq K$, then the reverse inequality holds

$$\begin{aligned}&f\left(A+B\right)+f\left(0\right)-f\left(A\right)-f\left(B\right)\\&\leq K^{2}\left[\left(A+B\right)f'\left(A+B\right)-f\left(A+B\right)+f\left(0\right)\right]\left(A+B\right)^{-2},\end{aligned}$$

which gives the power inequality

$$\left(A+B\right)^{r+1}-A^{r+1}-B^{r+1}\leq K^{2}r\left(A+B\right)^{r-1}.$$

We also have the logarithmic inequality

$$\left(A+B\right)\ln\left(A+B\right)-A\ln A-B\ln\left(B\right)\leq K^{2}\left(A+B\right)^{-1}.$$

Some integral inequalities are also provided.

2. Main Results

We have the following identity of interest:

Lemma 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_{+}\left(0\right)\in\mathbb{R}$ and has the representation (1.4), then for all A, $B>0$,*

$$\begin{aligned}&f\left(A+B\right)+f\left(0\right)-f\left(A\right)-f\left(B\right)-c\left(AB+BA\right)\\&=\int_{0}^{\infty}\lambda^{2}\left(A+B+\lambda\right)^{-1}\\&\times\left[A\left(B+\lambda\right)^{-1}BA+B\left(A+\lambda\right)^{-1}AB\right]\left(A+B+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&+\int_{0}^{\infty}\lambda^{2}\left(A+B+\lambda\right)^{-1}\left(BA+AB\right)\left(A+B+\lambda\right)^{-1}d\mu\left(\lambda\right).\end{aligned}\tag{2.1}$$

Proof. For $t>0$ we have

$$\begin{aligned}\mathcal{C}(w,\mu)\left(t\right)&:=\int_{0}^{\infty}\lambda t^{2}\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}\lambda\left(t+\lambda-\lambda\right)^{2}\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}\lambda\left[\left(t+\lambda\right)^{2}-2\lambda\left(t+\lambda\right)+\lambda^{2}\right]\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}\lambda\left[\left(t+\lambda\right)-2\lambda+\lambda^{2}\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}\lambda\left[t-\lambda+\lambda^{2}\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right).\end{aligned}\tag{2.2}$$

Assume that A, $B>0$. Define

$$K_{\lambda}:=\left(A+\lambda\right)^{-1}+\left(B+\lambda\right)^{-1}-\left(A+B+\lambda\right)^{-1},$$

where $\lambda\geq0$.

Therefore

$$\begin{aligned}
(2.3)\qquad & (A+B+\lambda) K_\lambda (A+B+\lambda) \\
&= (A+B+\lambda)(A+\lambda)^{-1}(A+B+\lambda) \\
&+ (A+B+\lambda)(B+\lambda)^{-1}(A+B+\lambda) - A - B - \lambda \\
&= \left(1 + B(A+\lambda)^{-1}\right)(A+\lambda+B) \\
&+ \left(A(B+\lambda)^{-1} + 1\right)(A+B+\lambda) - A - B - \lambda \\
&= A + \lambda + B + B + B(A+\lambda)^{-1} B \\
&+ A(B+\lambda)^{-1} A + A + A + B + \lambda - A - B - \lambda \\
&= B(A+\lambda)^{-1} B + A(B+\lambda)^{-1} A + 2(A+B) + \lambda =: L_\lambda.
\end{aligned}$$

If $A, B, \lambda > 0$, then $L_\lambda \geq 0$, and by multiplying both sides of (2.3) with $(A+B+\lambda)^{-1}$ we get

$$K_\lambda = (A+B+\lambda)^{-1} L_\lambda (A+B+\lambda)^{-1}.$$

Further, define for $\lambda > 0$

$$W_\lambda := 1 - \lambda K_\lambda.$$

Then

$$\begin{aligned}
& (A+B+\lambda) W_\lambda (A+B+\lambda) \\
&= (A+B+\lambda)(1-\lambda K_\lambda)(A+B+\lambda) \\
&= (A+B+\lambda)^2 - \lambda (A+B+\lambda) K_\lambda (A+B+\lambda) \\
&= (A+B+\lambda)(A+B+\lambda) \\
&- \lambda \left[B(A+\lambda)^{-1} B + A(B+\lambda)^{-1} A + 2(A+B) + \lambda\right]
\end{aligned}$$

$$\begin{aligned}
&= A^2 + BA + \lambda A + AB + B^2 + \lambda B + \lambda A + \lambda B + \lambda^2 \\
&- \lambda B(A+\lambda)^{-1} B - \lambda A(B+\lambda)^{-1} A - 2\lambda(A+B) - \lambda^2 \\
&= A^2 + B^2 + BA + AB - \lambda B(A+\lambda)^{-1} B - \lambda A(B+\lambda)^{-1} A \\
&= A(B+\lambda)^{-1}(B+\lambda) A - \lambda A(B+\lambda)^{-1} A \\
&+ B(A+\lambda)^{-1}(A+\lambda) B - \lambda B(A+\lambda)^{-1} B \\
&+ BA + AB = A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB + BA + AB,
\end{aligned}$$

which implies that

$$\begin{aligned}
W_\lambda &= (A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB + BA + AB\right] \\
&\times (A+B+\lambda)^{-1}.
\end{aligned}$$

We also have the representation

$$\begin{aligned}
(2.4)\quad & \mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A+B) \\
&=\int_0^\infty \lambda\left[A-\lambda+\lambda^2(A+\lambda)^{-1}\right]d\mu(\lambda) \\
&+\int_0^\infty w(\lambda)\left[B-\lambda+\lambda^2(B+\lambda)^{-1}\right]d\mu(\lambda) \\
&-\int_0^\infty \lambda\left[A+B-\lambda+\lambda^2(A+B+\lambda)^{-1}\right]d\mu(\lambda) \\
&=\int_0^\infty \lambda\left[\lambda^2(A+\lambda)^{-1}+\lambda^2(B+\lambda)^{-1}-\lambda-\lambda^2(A+B+\lambda)^{-1}\right]d\mu(\lambda) \\
&=\int_0^\infty \lambda^3\left[(A+\lambda)^{-1}+(B+\lambda)^{-1}-\lambda^{-1}-(A+B+\lambda)^{-1}\right]d\mu(\lambda) \\
&=\int_0^\infty \lambda^3\left(K_\lambda-\lambda^{-1}\right)d\mu(\lambda).
\end{aligned}$$

Put

$$\begin{aligned}
Y_\lambda &:= K_\lambda-\lambda^{-1}=\lambda^{-1}(\lambda K_\lambda-1)=-\lambda^{-1}(1-\lambda K_\lambda)=-\lambda^{-1}W_\lambda \\
&=-\lambda^{-1}(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right] \\
&\times(A+B+\lambda)^{-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A+B) \\
&=\int_0^\infty \lambda^3 Y_\lambda d\mu(\lambda) \\
&=-\int_0^\infty \lambda^2(A+B+\lambda)^{-1} \\
&\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right](A+B+\lambda)^{-1}d\mu(\lambda),
\end{aligned}$$

which gives the following identity of interest:

$$\begin{aligned}
(2.5)\quad & \mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B) \\
&=\int_0^\infty \lambda^2(A+B+\lambda)^{-1} \\
&\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right](A+B+\lambda)^{-1}d\mu(\lambda) \\
&=\int_0^\infty \lambda^2(A+B+\lambda)^{-1} \\
&\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}d\mu(\lambda) \\
&+\int_0^\infty \lambda^2(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).
\end{aligned}$$

Finally, by (1.4) and (2.5) we derive

$$\begin{aligned}
&f(A+B)+f(0)-f(A)-f(B)\\
&=f(0)+f'_{+}(0)(A+B)+c(A+B)^{2}+\mathcal{C}(w,\mu)(A+B)\\
&+f(0)-f(0)-f'_{+}(0)A-cA^{2}-\mathcal{C}(w,\mu)(A)\\
&-f(0)-f'_{+}(0)B-cB^{2}-\mathcal{C}(w,\mu)(B)\\
&=c(AB+BA)+\mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B)\\
&=c(AB+BA)+\int_{0}^{\infty}\lambda^{2}(A+B+\lambda)^{-1}\\
&\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}d\mu(\lambda)\\
&+\int_{0}^{\infty}\lambda^{2}(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda),
\end{aligned}$$

which proves (2.1). □

Theorem 3. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_{+}(0)\in\mathbb{R}$ and has the representation (1.4), then for all A, $B>0$,*

$$\begin{aligned}
&\int_{0}^{\infty}\lambda^{2}(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)\\
&\leq f(A+B)+f(0)-f(A)-f(B)-c(AB+BA)\\
&\leq\int_{0}^{\infty}\lambda^{2}(A+B+\lambda)^{-1}(A+B)^{2}(A+B+\lambda)^{-1}.
\end{aligned}\tag{2.6}$$

Proof. Since A, $B>0$, hence

$$0<(B+\lambda)^{-1}B\leq 1\text{ and }0<(A+\lambda)^{-1}A\leq 1$$

and by multiplying both sides with A and B respectively, we get

$$0<A(B+\lambda)^{-1}BA\leq A^{2}\text{ and }0<B(A+\lambda)^{-1}AB\leq B^{2}$$

for $\lambda\geq 0$.

If we multiply both sides by $(A+B+\lambda)^{-1}$ with $\lambda\geq 0$, then we obtain

$$\begin{aligned}
0&<(A+B+\lambda)^{-1}A(B+\lambda)^{-1}BA(A+B+\lambda)^{-1}\\
&\leq(A+B+\lambda)^{-1}A^{2}(A+B+\lambda)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
0&<(A+B+\lambda)^{-1}B(A+\lambda)^{-1}AB(A+B+\lambda)^{-1}\\
&\leq(A+B+\lambda)^{-1}B^{2}(A+B+\lambda)^{-1}
\end{aligned}$$

for $\lambda\geq 0$.

If we add these inequality, multiply by $\lambda^{2}\geq 0$ and integrate, then we get

$$\begin{aligned}
0&\leq\int_{0}^{\infty}\lambda^{2}(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right]\\
&\times(A+B+\lambda)^{-1}d\mu(\lambda)\\
&\leq\int_{0}^{\infty}\lambda(A+B+\lambda)^{-1}\left(A^{2}+B^{2}\right)(A+B+\lambda)^{-1}.
\end{aligned}$$

Moreover, if we add to these inequalities the same quantity

$$\int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\mu(\lambda),$$

then we obtain

$$\begin{aligned}
&\int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\mu(\lambda) \\
&\le \int_0^\infty \lambda^2 (A+B+\lambda)^{-1} \left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB\right] \\
&\times (A+B+\lambda)^{-1} d\mu(\lambda) \\
&+ \int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\mu(\lambda) \\
&\le \int_0^\infty \lambda^2 (A+B+\lambda)^{-1} \left(A^2+B^2\right) (A+B+\lambda)^{-1} \\
&+ \int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\mu(\lambda) \\
&= \int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (A+B)^2 (A+B+\lambda)^{-1}.
\end{aligned}$$

By utilising the identity (2.1) we deduce the desired result (2.6). □

Corollary 1. *With the assumptions of Theorem 3 and if $AB+BA \ge k$, for k a real number, then*

$$\begin{aligned}
&k\left[(A+B) f'(A+B) - f(A+B) + f(0)\right] (A+B)^{-2} \\
&\le c(AB+BA-k) \\
&+ k\left[(A+B) f'(A+B) - f(A+B) + f(0)\right] (A+B)^{-2} \\
&\le f(A+B) + f(0) - f(A) - f(B).
\end{aligned} \tag{2.7}$$

If $AB+BA \ge k \ge 0$, then

$$\begin{aligned}
0 &\le ck \le k\left[(A+B) f'(A+B) - f(A+B) + f(0)\right] (A+B)^{-2} \\
&\le c(AB+BA-k) \\
&+ k\left[(A+B) f'(A+B) - f(A+B) + f(0)\right] (A+B)^{-2} \\
&\le f(A+B) + f(0) - f(A) - f(B).
\end{aligned} \tag{2.8}$$

Proof. Since $AB+BA \ge k$, hence by multiplying both sides by $(A+B+\lambda)^{-1}$ for $\lambda \ge 0$, we get

$$(A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} \ge k (A+B+\lambda)^{-2}$$

for $\lambda \ge 0$.

If we multiply by $\lambda^2 \ge 0$ and integrate, then we get

$$\begin{aligned}
&\int_0^\infty \lambda^2 (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\mu(\lambda) \\
&\ge k \int_0^\infty \lambda^2 (A+B+\lambda)^{-2} d\mu(\lambda).
\end{aligned} \tag{2.9}$$

From (1.4) we get

$$\frac{f(t)-f(0)}{t}-f_{+}'(0)-ct=\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda),$$

for $t>0$.

If we take the derivative over t in this equality, then we obtain

$$\begin{aligned}\frac{tf'(t)-f(t)+f(0)}{t^2}-c&=\int_0^\infty \frac{\lambda(t+\lambda)-t\lambda}{(t+\lambda)^2}d\mu(\lambda)\\&=\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2}d\mu(\lambda)\geq 0\end{aligned}$$

for $t>0$.

This shows that

$$\begin{aligned}0&\leq\int_0^\infty \lambda^2(A+B+\lambda)^{-2}d\mu(\lambda)\\&=[(A+B)f'(A+B)-f(A+B)+f(0)](A+B)^{-2}-c.\end{aligned}\tag{2.10}$$

By (2.9) we get

$$\begin{aligned}&\int_0^\infty \lambda^2(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)\\&\geq k[(A+B)f'(A+B)-f(A+B)+f(0)](A+B)^{-2}-ck.\end{aligned}\tag{2.11}$$

From the first inequality in (2.6), we get

$$\begin{aligned}&\int_0^\infty \lambda^2(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)\\&+c(AB+BA)\\&\leq f(A+B)+f(0)-f(A)-f(B)\end{aligned}$$

and by (2.11) we derive

$$\begin{aligned}&k[(A+B)f'(A+B)-f(A+B)+f(0)](A+B)^{-2}-ck\\&+c(AB+BA)\\&\leq f(A+B)+f(0)-f(A)-f(B),\end{aligned}$$

which proves the second inequality in (2.7).

The first inequality follows by the fact that $c\geq 0$ and $AB+BA\geq k$.

Since, by (2.10), we have

$$[(A+B)f'(A+B)-f(A+B)+f(0)](A+B)^{-2}\geq c\geq 0,$$

hence for $k\geq 0$ we derive the first two inequalities in (2.8) as well. □

Corollary 2. *With the assumptions of Theorem 3 and if* $(0<)\, A+B\leq K$, *then*

$$\begin{aligned} &f(A+B)+f(0)-f(A)-f(B) \\ &\leq K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2} \\ &-c\left(K^2-AB-BA\right) \\ &\leq K^2\left\{\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-\frac{1}{2}c\right\} \\ &\leq K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}. \end{aligned} \tag{2.12}$$

Proof. From the second inequality in (2.6) we have

$$\begin{aligned} &f(A+B)+f(0)-f(A)-f(B) \\ &\leq c(AB+BA) \\ &+\int_0^\infty \lambda^2 (A+B+\lambda)^{-1}(A+B)^2(A+B+\lambda)^{-1}\,d\mu(\lambda). \end{aligned} \tag{2.13}$$

Since $0<A+B\leq K$, then $(A+B)^2\leq K^2$, which, as above, implies that

$$\begin{aligned} &\int_0^\infty \lambda^2 (A+B+\lambda)^{-1}(A+B)^2(A+B+\lambda)^{-1}\,d\mu(\lambda) \\ &\leq K^2\int_0^\infty \lambda^2(A+B+\lambda)^{-2}\,d\mu(\lambda) \\ &= K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-cK^2. \end{aligned}$$

Therefore

$$\begin{aligned} &c(AB+BA) \\ &+\int_0^\infty \lambda^2 (A+B+\lambda)^{-1}(A+B)^2(A+B+\lambda)^{-1}\,d\mu(\lambda) \\ &\leq K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-cK^2 \\ &+c(AB+BA) \\ &= K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2} \\ &-c\left(K^2-AB-BA\right). \end{aligned}$$

By utilising (2.13) we derive the first inequality in (2.12).

Observe that for any $0<A+B\leq K$ we have

$$AB+BA\leq\frac{1}{2}(A+B)^2\leq\frac{1}{2}K^2.$$

Then

$$\begin{aligned} &K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-cK^2 \\ &+c(AB+BA) \\ &\leq K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-cK^2+\frac{1}{2}K^2c \\ &= K^2\left\{\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}-\frac{1}{2}c\right\} \\ &\leq K^2\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2} \end{aligned}$$

and the last part of (2.12) is proved. □

The symmetrized product of two operators $A, B \in B(H)$ is defined by $S(A,B) = AB+BA$. In general, the symmetrized product of two operators A, B is not positive (see for instance [11]). Also Gustafson [7] showed that if $0 \le m \le A \le M$ and $0 \le n \le B \le N$, then we have the lower bound

$$S(A,B) \ge 2mn - \frac{1}{4}(M-m)(N-n) =: k, \tag{2.14}$$

which can take positive or negative values depending on the parameters m, M, n, N.

Corollary 3. *With the assumptions of Theorem 3 and if* $0 \le m \le A \le M$ *and* $0 \le n \le B \le N$, *then*

$$\begin{aligned} &\left[2mn - \frac{1}{4}(M-m)(N-n)\right] \\ &\times \left[(A+B) f'(A+B) - f(A+B) + f(0)\right](A+B)^{-2} \\ &\le f(A+B) + f(0) - f(A) - f(B). \end{aligned} \tag{2.15}$$

3. Some Integral Inequalities

We have the following integral inequality.

Proposition 1. *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$ *and that* C, $D > 0$ *such that* $CD + DC \ge k$. *Then we have the integral inequalities*

$$\begin{aligned} &k\left[(C+D) f'(C+D) - f(C+D) + f(0)\right](C+D)^{-2} \\ &\le \frac{f(C+D) + f(0)}{2} - \int_0^1 f((1-t)C + tD)\,dt. \end{aligned} \tag{3.1}$$

If $CD + DC \ge 0$, *then*

$$\int_0^1 f((1-t)C + tD)\,dt \le \frac{f(C+D) + f(0)}{2}. \tag{3.2}$$

Proof. Let $t \in [0,1]$ and put $A = (1-t)C + tD$, $B = tC + (1-t)D$. Then

$$\begin{aligned} AB &= ((1-t)C + tD)(tC + (1-t)D) \\ &= (1-t)tC^2 + t^2DC + (1-t)^2CD + t(1-t)D^2 \end{aligned}$$

and

$$\begin{aligned} BA &= (tC + (1-t)D)((1-t)C + tD) \\ &= t(1-t)C^2 + (1-t)^2DC + t^2CD + (1-t)tD^2 \end{aligned}$$

for $t \in [0,1]$.

Therefore

$$\begin{aligned}
&AB+BA \\
&=2t\left(1-t\right)C^{2}+\left[\left(1-t\right)^{2}+t^{2}\right]\left(CD+DC\right)+2\left(1-t\right)tD^{2} \\
&=2t\left(1-t\right)C^{2}+\left(1+2t^{2}-2t\right)\left(CD+DC\right)+2\left(1-t\right)tD^{2} \\
&=2t\left(1-t\right)\left(C^{2}-CD-DC+D^{2}\right)+CD+DC \\
&=2t\left(1-t\right)\left(C-D\right)^{2}+CD+DC\geq k
\end{aligned}$$

for $t\in[0,1]$.

From (2.7) we have

$$\begin{aligned}
&k\left[\left(C+D\right)f'\left(C+D\right)-f\left(C+D\right)+f\left(0\right)\right]\left(C+D\right)^{-2} \\
&\leq f\left(C+D\right)+f\left(0\right)-f\left(\left(1-t\right)C+tD\right)-f\left(tC+\left(1-t\right)D\right)
\end{aligned} \tag{3.3}$$

for $t\in[0,1]$.

Taking the integral and observing that

$$\int_{0}^{1}f\left(\left(1-t\right)C+tD\right)dt=\int_{0}^{1}f\left(tC+\left(1-t\right)D\right)dt,$$

then by (3.3) we obtain (3.1). □

Proposition 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and that $C,\ D>0$ such that $C+D\leq K$. Then*

$$\begin{aligned}
&\frac{f\left(C+D\right)+f\left(0\right)}{2}-\int_{0}^{1}f\left(\left(1-t\right)C+tD\right)dt \\
&\leq\frac{1}{2}K^{2}\left[\left(C+D\right)f'\left(C+D\right)-f\left(C+D\right)+f\left(0\right)\right]\left(C+D\right)^{-2}.
\end{aligned} \tag{3.4}$$

Proof. The proof follows from (2.12) by a similar argument to the one from the proof of Proposition 1. □

Further, we have

Proposition 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and that $C,\ D>0$ such that $CD+DC\geq0$. Then we have the integral inequalities*

$$\int_{0}^{1}f\left(tC\right)dt+\int_{0}^{1}f\left(tD\right)dt\leq\int_{0}^{1}f\left(\left(1-t\right)C+tD\right)dt+f\left(0\right). \tag{3.5}$$

Proof. If $CD+DC\geq0$, then $t\left(1-t\right)CD+t\left(1-t\right)DC\geq0$ and by (2.8) for $A=\left(1-t\right)C$ and $B=tD$, we get

$$0\leq f\left(\left(1-t\right)C+tD\right)+f\left(0\right)-f\left(\left(1-t\right)C\right)-f\left(tD\right),$$

namely

$$f\left(\left(1-t\right)C\right)+f\left(tD\right)\leq f\left(\left(1-t\right)C+tD\right)+f\left(0\right),$$

for all $t\in\left[0,1\right]$.

If we take the integral and observe that

$$\int_{0}^{1}f\left(\left(1-t\right)C\right)dt=\int_{0}^{1}f\left(sC\right)ds,$$

then we obtain the desired result (3.5). □

Proposition 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and that C, $D>0$ such that C, $D\leq K$, then*

$$\begin{aligned}
&\int_0^1 f\left(\left(1-t\right)C+tD\right)dt+f\left(0\right)-\int_0^1 f\left(tC\right)dt-\int_0^1 f\left(tD\right)dt \\
&\leq K^2\left[f\left(0\right)\int_0^1\left(\left(1-t\right)C+tD\right)^{-2}dt\right. \\
&+\int_0^1\left(\left(1-t\right)C+tD\right)^{-1}f'\left(\left(1-t\right)C+tD\right)dt \\
&\left.-\int_0^1 f\left(\left(1-t\right)C+tD\right)\left(\left(1-t\right)C+tD\right)^{-2}dt\right].
\end{aligned} \tag{3.6}$$

Proof. By (2.12) for $A=\left(1-t\right)C$, $B=tD$, we get

$$\begin{aligned}
&f\left(\left(1-t\right)C+tD\right)+f\left(0\right)-f\left(\left(1-t\right)C\right)-f\left(tD\right) \\
&\leq K^2\left[f\left(0\right)+\left(\left(1-t\right)C+tD\right)f'\left(\left(1-t\right)C+tD\right)-f\left(\left(1-t\right)C+tD\right)\right] \\
&\times\left(\left(1-t\right)C+tD\right)^{-2}
\end{aligned}$$

and by integrating this inequality, we derive (3.6). □

4. Some Examples

For $r\in[0,1]$ the function $f\left(t\right)=t^{r+1}$ is operator convex on $[0,\infty)$. If A, $B>0$ with $AB+BA\geq k$, for k a real number, then by (2.7)

$$kr\left(A+B\right)^{r-1}\leq\left(A+B\right)^{r+1}-A^{r+1}-B^{r+1}. \tag{4.1}$$

If $AB+BA\geq 0$, then

$$A^{r+1}+B^{r+1}\leq\left(A+B\right)^{r+1}. \tag{4.2}$$

If $A+B\leq K$, then by (2.12),

$$\left(A+B\right)^{r+1}-A^{r+1}-B^{r+1}\leq K^2 r\left(A+B\right)^{r-1}. \tag{4.3}$$

For $p\in[0,1]$, $a>0$, the function $f\left(t\right)=\left(t+a\right)^{-p}$ is operator convex on $[0,\infty)$. Then by (2.7) we get

$$\begin{aligned}
&k\left[a^{-p}-\left(p+1\right)\left(A+B+a\right)^{-p}\right]\left(A+B\right)^{-2} \\
&\leq\left(A+B+a\right)^{-p}+a^{-p}-\left(A+a\right)^{-p}-\left(B+a\right)^{-p},
\end{aligned} \tag{4.4}$$

provided A, $B>0$ with $AB+BA\geq k$.

If $AB+BA\geq 0$, then

$$\left(A+a\right)^{-p}+\left(B+a\right)^{-p}\leq\left(A+B+a\right)^{-p}+a^{-p}. \tag{4.5}$$

If $A+B\leq K$, then by (2.12) we get

$$\begin{aligned}
&\left(A+B+a\right)^{-p}+a^{-p}-\left(A+a\right)^{-p}-\left(B+a\right)^{-p} \\
&\leq K^2\left[a^{-p}-\left(p+1\right)\left(A+B+a\right)^{-p}\right]\left(A+B\right)^{-2}.
\end{aligned} \tag{4.6}$$

Let $\varepsilon>0$ and consider the function $f(t)=(t+\varepsilon)\ln(t+\varepsilon)$. This function is operator convex on $[0,\infty)$ and by (2.7) we have

$$\begin{aligned}&k\left[(A+B)\left[\ln(A+B+\varepsilon)+1\right]-(A+B+\varepsilon)\ln(A+B+\varepsilon)+\varepsilon\ln\varepsilon\right]\\&\times(A+B)^{-2}\\&\leq(A+B+\varepsilon)\ln(A+B+\varepsilon)+\varepsilon\ln\varepsilon-(A+\varepsilon)\ln(A+\varepsilon)-(B+\varepsilon)\ln(B+\varepsilon),\end{aligned}$$

provided that $A,\,B>0$ with $AB+BA\geq k$, for k a real number.

By taking the limit over $\varepsilon\to 0+$ we derive

$$k(A+B)^{-1}\leq(A+B)\ln(A+B)-A\ln A-B\ln(B), \tag{4.7}$$

provided that $A,\,B>0$ with $AB+BA\geq k$.

If $AB+BA\geq 0$, then

$$A\ln A+B\ln(B)\leq(A+B)\ln(A+B). \tag{4.8}$$

By making use of (2.12) we obtain for $A+B\leq K$ that

$$(A+B)\ln(A+B)-A\ln A-B\ln(B)\leq K^2(A+B)^{-1}. \tag{4.9}$$

If $A,\,B>0$ such that $AB+BA\geq k$, then by Proposition 2.3 for the power function $f(t)=t^{r+1}$, $r\in(0,1]$,

$$\frac{1}{2}kr(A+B)^{r-1}\leq\frac{1}{2}(A+B)^{r+1}-\int_0^1((1-t)A+tB)^{r+1}dt. \tag{4.10}$$

If $AB+BA\geq 0$, then

$$\left(\left(\frac{A+B}{2}\right)^r\leq\right)\int_0^1((1-t)A+tB)^{r+1}dt\leq\frac{1}{2}(A+B)^{r+1}. \tag{4.11}$$

The first inequality in (4.11) follows by the operator convexity of the power function $f(t)=t^{r+1}$, $r\in(0,1]$ and holds for any $A,\,B\geq 0$.

If $A,\,B>0$ such that $A+B\leq K$, then by (3.4)

$$\frac{1}{2}(A+B)^{r+1}-\int_0^1((1-t)A+tB)^{r+1}dt\leq\frac{1}{2}K^2t(A+B)^{r-1}. \tag{4.12}$$

If $A,\,B>0$ such that $AB+BA\geq 0$, then by (3.5)

$$\frac{A^{r+1}+B^{r+1}}{r+2}\leq\int_0^1((1-t)A+tB)^{r+1}dt\left(\leq\frac{A^{r+1}+B^{r+1}}{2}\right). \tag{4.13}$$

The second inequality in (4.13) follows by the operator convexity of the power function and holds for any $A,\,B\geq 0$.

If $A,\,B>0$ such that $A,\,B\leq K$, then by (3.6)

$$\int_0^1((1-t)A+tB)^{r+1}-\frac{A^{r+1}+B^{r+1}}{r+2}dt\leq K^2r\int_0^1((1-t)A+tB)^{r-1}dt. \tag{4.14}$$

We also have the logarithmic inequalities

$$\begin{aligned}&\frac{1}{2}k(A+B)^{-1}\\&\leq\frac{1}{2}(A+B)\ln(A+B)-\int_0^1((1-t)A+tB)\ln((1-t)A+tB)\,dt,\end{aligned} \tag{4.15}$$

provided that $A,\,B>0$ with $AB+BA\geq k$.

If $AB+BA\geq 0$, then

$$\left(\frac{1}{2}(A+B)\ln\left(\frac{A+B}{2}\right)\leq\right)\int_0^1((1-t)A+tB)\ln((1-t)A+tB)\,dt$$
$$\leq\frac{1}{2}(A+B)\ln(A+B).$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.