UPPER BOUNDS FOR THE OPERATOR CONVEXITY DIFFERENCE IN TERMS OF THE FIRST DERIVATIVE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, the convex set of selfadjoint operators with spectra in I. If $A \neq B$ and f, as an operator function, is Gâteaux differentiable on

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\},$$

then

$$0 \leq (1-t) f(A) + t f(B) - f((1-t)A + tB) \leq t(1-t)[\nabla f_B(B-A) - \nabla f_A(B-A)]$$

for all $t \in [0,1]$, where ∇f is the Gâteaux derivative of f.

Some examples for operator convex functions on $[0,\infty)$ such as the power function, logarithmic function and positive operators are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. By $A \geq B$ we understand that $A - B \geq 0$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [5] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

We have the following representation of operator convex functions [1, p. 147]:

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Theorem 1. *A function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *with* $f'_{+}(0)\in\mathbb{R}$ *if and only if it has the representation*

$$f(t)=f(0)+f'_{+}(0)\,t+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $c\geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that*

$$\int_{0}^{\infty}\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty \tag{1.2}$$

holds.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_{0}^{\infty}\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.3}$$

Let f be an operator convex function on I. For $A,\ B\in\mathcal{SA}_{I}(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{SA}_{\varphi(I)}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{1.4}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle \varphi_{(A,B)}(t)\,x,x\right\rangle=\left\langle f((1-t)A+tB)\,x,x\right\rangle. \tag{1.5}$$

We have the following basic fact, see for instance [4]:

Lemma 1. *Let* f *be an operator convex function on* I. *For any* $A,\ B\in\mathcal{SA}_{I}(H)$, $\varphi_{(A,B)}$ *is well defined and convex in the operator order. For any* $A,\ B\in\mathcal{SA}_{I}(H)$ *and* $x\in H$ *the function* $\varphi_{(A,B);x}$ *is convex in the usual sense on* $[0,1]$.

A continuous function $g:\mathcal{SA}_{I}(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_{I}(H)$ along the direction $B\in\mathcal{SA}(H)$, the class of all selfadjoint operators on H, if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_{A}(B):=\lim_{s\to 0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{1.6}$$

If the limit (1.6) exists for all $B\in\mathcal{SA}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_{I}(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A,\ B\in\mathcal{SA}_{I}(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that $A,\ B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_{I}(H)$.

The following result also holds, see for instance [4]:

Lemma 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A),\ t \in (0,1). \tag{1.7}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{1.8}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{1.9}$$

In the recent paper [3], see also [4] for the weighted version, we obtained the following reverses of Hermite-Hadamard inequalities:

Theorem 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \end{aligned}$$

and

Theorem 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned}$$

For recent papers on operator Hermite-Hadamard inequalities, see [6]-[9] and [11]-[15].

In this paper we obtain, among others, the following upper bound for the *operator convexity difference*

$$\begin{aligned} 0 &\leq (1-t)f(A) + tf(B) - f((1-t)A+tB) \\ &\leq t(1-t)\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \end{aligned}$$

for all $t \in [0,1]$, where f is an operator convex function on I, A, $B \in \mathcal{SA}_I(H)$ and ∇f is the Gâteaux derivative of f.

Some examples for operator convex functions on $[0,\infty)$ and positive operators are also given. The following logarithmic inequality is valid

$$\begin{aligned} 0 &\leq \ln((1-t)A+tB) - (1-t)\ln(A) - tf(B) \\ &\leq t(1-t)\int_0^\infty \Big[(s1_H+A)^{-1}(B-A)(s1_H+A)^{-1} \\ &\quad - (s1_H+B)^{-1}(B-A)(s1_H+B)^{-1}\Big]ds, \end{aligned} \tag{1.10}$$

for A, $B > 0$ and $t \in [0,1]$. Also, we have

$$\begin{aligned} 0 &\leq (1-t)A^{-1} + tB^{-1} - ((1-t)A+tB)^{-1} \\ &\leq t(1-t)\left[A^{-1}(B-A)A^{-1} - B^{-1}(B-A)B^{-1}\right] \end{aligned} \tag{1.11}$$

for all A, $B>0$ and $t\in[0,1]$.

2. General Results

We have the following result for general convex functions [2]:

Lemma 3. *Let $\varphi : I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I, $a,b\in\mathring{I}$, the interior of I, with $a<b$ and $\nu\in[0,1]$. Then*

$$\begin{aligned}(2.1)\qquad &\nu(1-\nu)(b-a)\left[\varphi'_+((1-\nu)a+\nu b)-\varphi'_-((1-\nu)a+\nu b)\right]\\ &\leq(1-\nu)\varphi(a)+\nu\varphi(b)-\varphi((1-\nu)a+\nu b)\\ &\leq\nu(1-\nu)(b-a)\left[\varphi'_-(b)-\varphi'_+(a)\right].\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.2)\qquad \frac{1}{4}(b-a)\left[\varphi'_+\left(\frac{a+b}{2}\right)-\varphi'_-\left(\frac{a+b}{2}\right)\right]&\leq\frac{\varphi(a)+\varphi(b)}{2}-\varphi\left(\frac{a+b}{2}\right)\\ &\leq\frac{1}{4}(b-a)\left[\varphi'_-(b)-\varphi'_+(a)\right].\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.2).

Proof. The case $\nu=0$ or $\nu=1$ reduces to equality in (2.1).

Since φ is convex on I it follows that the function is differentiable on $\mathring{I}$ except a countably number of points, the lateral derivatives $\varphi'_\pm$ exists in each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $\varphi'_-\leq\varphi'_+$ on $\mathring{I}$.

For any $x,y\in\mathring{I}$ we have for the Lebesgue integral

$$(2.3)\qquad \varphi(x)=\varphi(y)+\int_y^x\varphi'(s)\,ds=\varphi(y)+(x-y)\int_0^1\varphi'((1-t)y+tx)\,dt.$$

Assume that $a<b$ and $\nu\in(0,1)$. By (2.3) we have

$$\begin{aligned}(2.4)\qquad &\varphi((1-\nu)a+\nu b)\\ &=\varphi(a)+\nu(b-a)\int_0^1\varphi'((1-t)a+t((1-\nu)a+\nu b))\,dt\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad &\varphi((1-\nu)a+\nu b)\\ &=\varphi(b)-(1-\nu)(b-a)\int_0^1\varphi'((1-t)b+t((1-\nu)a+\nu b))\,dt.\end{aligned}$$

If we multiply (2.4) by $1-\nu$, (2.4) by ν and add the obtained equalities, then we get

$$\begin{aligned}\varphi((1-\nu)a+\nu b)&=(1-\nu)\varphi(a)+\nu\varphi(b)\\ &\quad+(1-\nu)\nu(b-a)\int_0^1\varphi'((1-t)a+t((1-\nu)a+\nu b))\,dt\\ &\quad-(1-\nu)\nu(b-a)\int_0^1\varphi'((1-t)b+t((1-\nu)a+\nu b))\,dt,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(1-\nu)\varphi(a)+\nu\varphi(b)-\varphi((1-\nu)a+\nu b) \\ =(1-\nu)\nu(b-a)\int_0^1[\varphi'((1-t)b+t((1-\nu)a+\nu b)) \\ -\varphi'((1-t)a+t((1-\nu)a+\nu b))]\,dt.\end{aligned} \tag{2.6}$$

That is an equality of interest in itself.

Since $a<b$ and $\nu\in(0,1)$, then $(1-\nu)a+\nu b\in(a,b)$ and

$$(1-t)a+t((1-\nu)a+\nu b)\in[a,(1-\nu)a+\nu b]$$

while

$$(1-t)b+t((1-\nu)a+\nu b)\in[(1-\nu)a+\nu b,b]$$

for any $t\in[0,1]$.

By the monotonicity of the derivative we have

$$\varphi'_+((1-\nu)a+\nu b)\le\varphi'((1-t)b+t((1-\nu)a+\nu b))\le\varphi'_-(b) \tag{2.7}$$

and

$$\varphi'_+(a)\le\varphi'((1-t)a+t((1-\nu)a+\nu b))\le\varphi'_-((1-\nu)a+\nu b) \tag{2.8}$$

for any $t\in[0,1]$.

By integrating the inequalities (2.7) and (2.8) we get

$$\varphi'_+((1-\nu)a+\nu b)\le\int_0^1\varphi'((1-t)b+t((1-\nu)a+\nu b))\,dt\le\varphi'_-(b)$$

and

$$\varphi'_+(a)\le\int_0^1\varphi'((1-t)a+t((1-\nu)a+\nu b))\,dt\le\varphi'_-((1-\nu)a+\nu b),$$

which implies that

$$\begin{aligned}\varphi'_+((1-\nu)a+\nu b)-\varphi'_-((1-\nu)a+\nu b)\le\int_0^1\varphi'((1-t)b+t((1-\nu)a+\nu b))\,dt \\ -\int_0^1\varphi'((1-t)a+t((1-\nu)a+\nu b))\,dt\le\varphi'_-(b)-\varphi'_+(a).\end{aligned}$$

Making use of the equality (2.6) we the obtain the desired result (2.1).

If we consider the convex function $\varphi:[a,b]\to\mathbb{R}$, $\varphi(x)=\left|x-\frac{a+b}{2}\right|$, then we have $\varphi'_+\left(\frac{a+b}{2}\right)=1$, $\varphi'_-\left(\frac{a+b}{2}\right)=-1$ and by replacing in (2.2) we get in all terms the same quantity $\frac{1}{2}(b-a)$ which show that the constant $\frac{1}{4}$ is best possible in both inequalities from (2.2). □

Remark 1. *If the function $\varphi:I\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any a, $b\in\mathring{I}$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}0\le(1-\nu)\varphi(a)+\nu\varphi(b)-\varphi((1-\nu)a+\nu b) \\ \le\nu(1-\nu)(b-a)[\varphi'(b)-\varphi'(a)].\end{aligned} \tag{2.9}$$

The case of functions defined on the unit interval $[0,1]$ is as follows:

Corollary 1. *Assume that* $\varphi : [0,1] \to \mathbb{R}$ *is convex on* $[0,1]$ *with* $\varphi'_-(1)$, $\varphi'_+(0)$ *finite. Then for all* $\nu \in [0,1]$

$$\begin{aligned} \nu(1-\nu)\left[\varphi'_+(\nu) - \varphi'_-(\nu)\right] &\le (1-\nu)\varphi(0) + \nu\varphi(1) - \varphi(\nu) \\ &\le \nu(1-\nu)\left[\varphi'_-(1) - \varphi'_+(0)\right]. \end{aligned} \tag{2.10}$$

In particular, we have

$$\begin{aligned} \frac{1}{4}\left[\varphi'_+\left(\frac{1}{2}\right) - \varphi'_-\left(\frac{1}{2}\right)\right] &\le \frac{\varphi(0)+\varphi(1)}{2} - \varphi\left(\frac{1}{2}\right) \\ &\le \frac{1}{4}\left[\varphi'_-(1) - \varphi'_+(0)\right]. \end{aligned} \tag{2.11}$$

The constant $\frac{1}{4}$ *is best possible in both inequalities from (2.11).*

We have the following result:

Theorem 4. *Let* f *be an operator convex function on* I *and* $A, B \in \mathcal{SA}_I(H)$, *with* $A \neq B$. *If* $f \in \mathcal{G}([A,B])$, *then for all* $t \in [0,1]$,

$$\begin{aligned} 0 &\le (1-t) f(A) + t f(B) - f((1-t)A + tB) \\ &\le t(1-t)\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \\ &\le \frac{1}{4}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned} \tag{2.12}$$

In particular,

$$0 \le \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right) \le \frac{1}{4}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \tag{2.13}$$

Proof. For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$. Then by (2.10),

$$\begin{aligned} 0 &\le (1-t)\varphi_{(A,B);x}(0) + t\varphi_{(A,B);x}(1) - \varphi_{(A,B);x}(\nu) \\ &\le t(1-t)\left[\varphi'_{(A,B);x-}(1) - \varphi'_{(A,B);x+}(0)\right] \end{aligned} \tag{2.14}$$

for $x \in H$ and $t \in [0,1]$.

This is equivalent to, via Lemma 2,

$$\begin{aligned} 0 &\le (1-t)\langle f(A)x,x\rangle + t\langle f(B)x,x\rangle - \langle f((1-t)A+tB)x,x\rangle \\ &\le t(1-t)\left[\langle \nabla f_A(B-A)x,x\rangle - \langle \nabla f_A(B-A)x,x\rangle\right] \end{aligned}$$

for $x \in H$ and $t \in [0,1]$, which is in the operator order, the first and second inequalities in (2.12). □

3. Some Inequalities for Positive Operators

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping:

$$
\begin{aligned}
\mathcal{C}(w,\mu)(t) &:= \int_0^\infty w(\lambda)\, t^2 (t+\lambda)^{-1} d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)(t+\lambda-\lambda)^2 (t+\lambda)^{-1} d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[(t+\lambda)^2 - 2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1} d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[(t+\lambda) - 2\lambda + \lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda),
\end{aligned} \tag{3.1}
$$

for $t>0$.

We have the following representation of the Fréchet derivative $D(\mathcal{C}(w,\mu))$ as a function of positive selfadjoint operators:

Lemma 4. *For all* $U>0$,

$$
\begin{aligned}
&D(\mathcal{C}(w,\mu))(U)(V) \\
&= \int_0^\infty w(\lambda)\left\{V-\lambda^2 (U+\lambda)^{-1} V (U+\lambda)^{-1}\right\} d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)(U+\lambda)^{-1}\left[UVU+\lambda(VU+UV)\right](U+\lambda)^{-1} d\mu(\lambda)
\end{aligned} \tag{3.2}
$$

for all $V \in \mathcal{SA}(H)$, *the class of all selfadjoint operators on* H.

Proof. Let $U>0$ and $V\in\mathcal{SA}(H)$. By the definition of $\mathcal{C}(w,\mu)$ and by (3.1) and we have for t in a small open interval around 0 that

$$
\mathcal{C}(w,\mu)(U+tV) = \int_0^\infty w(\lambda)\left[U+tV-\lambda+\lambda^2(U+tV+\lambda)^{-1}\right] d\mu(\lambda).
$$

Then

$$
\begin{aligned}
&\mathcal{C}(w,\mu)(U+tV) - \mathcal{C}(w,\mu)(U) \\
&= \int_0^\infty w(\lambda)\left[U+tV-\lambda+\lambda^2(U+tV+\lambda)^{-1}\right] d\mu(\lambda) \\
&- \int_0^\infty w(\lambda)\left[U-\lambda+\lambda^2(U+\lambda)^{-1}\right] d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(U+tV+\lambda)^{-1}-(U+\lambda)^{-1}\right]\right\} d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(U+tV+\lambda)^{-1}(-tV)(U+\lambda)^{-1}\right]\right\} d\mu(\lambda) \\
&= t\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(U+tV+\lambda)^{-1}V(U+\lambda)^{-1}\right]\right\} d\mu(\lambda).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(3.3)\qquad & D\left(\mathcal{C}\left(w,\mu\right)\right)\left(U\right)\left(V\right)\\
&=\lim_{t\to 0}\frac{\mathcal{C}(w,\mu)\left(U+tV\right)-\mathcal{C}(w,\mu)\left(U\right)}{t}\\
&=\lim_{t\to 0}\int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left[\left(U+tV+\lambda\right)^{-1}V\left(U+\lambda\right)^{-1}\right]\right\}d\mu\left(\lambda\right)\\
&=\int_0^\infty w\left(\lambda\right)\left\{V-\lambda^2\left(U+\lambda\right)^{-1}V\left(U+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right)
\end{aligned}
$$

for $U>0$ and $V\in\mathcal{SA}\left(H\right)$, which proves the first identity in (3.2).

Define for $\lambda\geq 0$,

$$U_\lambda:=V-\lambda^2\left(U+\lambda\right)^{-1}V\left(U+\lambda\right)^{-1}.$$

If we multiply U_λ both sides by $U+\lambda$, then we get

$$
\begin{aligned}
\left(U+\lambda\right)U_\lambda\left(U+\lambda\right)&=\left(U+\lambda\right)V\left(U+\lambda\right)-\lambda^2V\\
&=\left(UV+\lambda V\right)\left(U+\lambda\right)-\lambda^2V\\
&=UVU+\lambda VU+\lambda UV+\lambda^2V-\lambda^2V\\
&=UVU+\lambda\left(VU+UV\right).
\end{aligned}
$$

If we multiply both sides by $\left(U+\lambda\right)^{-1}$ we get

$$U_\lambda=\left(U+\lambda\right)^{-1}\left[UVU+\lambda\left(VU+UV\right)\right]\left(U+\lambda\right)^{-1},$$

which, by (3.3), implies the second representation in (3.1). □

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex on $[0,\infty)$ and has the representation (1.1), then*

$$
\begin{aligned}
(3.4)\qquad & D\left(f\right)\left(U\right)\left(V\right)\\
&=f'_+\left(0\right)V+c\left(UV+VU\right)\\
&+\int_0^\infty\lambda\left\{V-\lambda^2\left(U+\lambda\right)^{-1}V\left(U+\lambda\right)^{-1}\right\}d\mu\left(\lambda\right)\\
&=f'_+\left(0\right)V+c\left(UV+VU\right)\\
&+\int_0^\infty\lambda\left(U+\lambda\right)^{-1}\left[UVU+\lambda\left(VU+UV\right)\right]\left(U+\lambda\right)^{-1}d\mu\left(\lambda\right)
\end{aligned}
$$

for all $U>0$ and $V\in\mathcal{SA}\left(H\right)$.

Proof. For $\ell\left(t\right)=t$, we have

$$
\begin{aligned}
& D\left(f\right)\left(U\right)\left(V\right)\\
&=D\left[f\left(0\right)+f'_+\left(0\right)\ell+c\ell^2+\mathcal{C}(w,\mu)\right]\left(U\right)\left(V\right)\\
&=D\left[f\left(0\right)\right]\left(U\right)\left(V\right)+f'_+\left(0\right)D\left(\ell\right)\left(U\right)\left(V\right)+cD\left(\ell^2\right)\left(U\right)\left(V\right)\\
&+D\left[\mathcal{C}(w,\mu)\right]\left(U\right)\left(V\right)\\
&=f'_+\left(0\right)V+c\left(UV+VU\right)+D\left[\mathcal{C}(w,\mu)\right]\left(U\right)\left(V\right)
\end{aligned}
$$

and, by Lemma 4, the identity (3.4) is obtained. □

Define the kernel

$$K_{\lambda}(A,B) := (A+\lambda)^{-1}(B-A)(A+\lambda)^{-1} - (B+\lambda)^{-1}(B-A)(B+\lambda)^{-1} \tag{3.5}$$

for A, $B>0$.

We have the following upper bound for the operator convexity difference:

Theorem 5. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex on $[0,\infty)$ and has the representation (1.1), then for all A, $B>0$,*

$$\begin{aligned} 0 &\le (1-t)f(A)+tf(B)-f((1-t)A+tB) \\ &\le t(1-t)\left[2c(B-A)^2+\int_0^\infty \lambda^3 K_\lambda(A,B)\,d\mu(\lambda)\right]. \end{aligned} \tag{3.6}$$

Proof. We have for A, $B>0$ that

$$\begin{aligned} &D(f)(B)(B-A)-D(f)(A)(B-A) \\ &= f'_+(0)(B-A)+c(B(B-A)+(B-A)B) \\ &+\int_0^\infty \lambda\left\{B-A-\lambda^2(B+\lambda)^{-1}(B-A)(B+\lambda)^{-1}\right\}d\mu(\lambda) \\ &- f'_+(0)(B-A)-c(A(B-A)+(B-A)A) \\ &-\int_0^\infty \lambda\left\{B-A-\lambda^2(A+\lambda)^{-1}(B-A)(A+\lambda)^{-1}\right\}d\mu(\lambda) \\ &= c(B(B-A)+(B-A)B-A(B-A)-(B-A)A) \\ &+\int_0^\infty \lambda^3 K_\lambda(A,B)\,d\mu(\lambda). \end{aligned}$$

Observe that

$$B(B-A)+(B-A)B-A(B-A)-(B-A)A = 2(B-A)^2.$$

Therefore,

$$D(f)(B)(B-A)-D(f)(A)(B-A) = 2c(B-A)^2+\int_0^\infty \lambda^3 K_\lambda(A,B)\,d\mu(\lambda).$$

By utilising Theorem 4 for A, $B>0$, we get (3.6). □

4. Some Examples

We note that the function $f(x)=-\ln x$ is operator convex on $(0,\infty)$. The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [10, p. 155]):

$$\nabla \ln_T(S) = \int_0^\infty (s+T)^{-1}S(s+T)^{-1}\,ds,\ T,\ S>0. \tag{4.1}$$

By utilising Theorem 4 we derive the following logarithmic inequalities for all A, $B>0$ and $t\in[0,1]$,

$$0 \le \ln((1-t)A+tB)-(1-t)\ln(A)-tf(B) \le t(1-t)\int_0^\infty K_s(A,B)\,ds, \tag{4.2}$$

where

$$K_s(A,B) := (A+s)^{-1}(B-A)(A+s)^{-1}-(B+s)^{-1}(B-A)(B+s)^{-1},\ s\ge 0.$$

In particular, for all A, $B>0$,

$$0\leq\ln\left(\frac{A+B}{2}\right)-\frac{\ln(A)+\ln(B)}{2}\leq\frac{1}{4}\int_{0}^{\infty}K_{s}(A,B)\,ds. \tag{4.3}$$

The function $f(x)=x^{-1}$ is also operator convex on $(0,\infty)$, operator Gâteaux differentiable and

$$\nabla f_{T}(S)=-T^{-1}ST^{-1},\ T,S>0.$$

By utilising Theorem 4 we derive the following inequalities

$$\begin{aligned}0&\leq(1-t)A^{-1}+tB^{-1}-((1-t)A+tB)^{-1}\\&\leq t(1-t)\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right]\end{aligned} \tag{4.4}$$

for all A, $B>0$ and $t\in[0,1]$.

In particular, for all A, $B>0$,

$$\begin{aligned}0&\leq\frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1}\\&\leq\frac{1}{4}\left[A^{-1}(B-A)A^{-1}-B^{-1}(B-A)B^{-1}\right].\end{aligned} \tag{4.5}$$

From (1.3) we have for $r\in(0,1]$ the following representation

$$t^{r+1}=\frac{\sin(r\pi)}{\pi}\int_{0}^{\infty}\frac{t^{2}\lambda^{r-1}}{\lambda+t}d\lambda. \tag{4.6}$$

If we use the inequality (3.6) for $c=0$ and $d\mu(\lambda)=\frac{\sin(r\pi)}{\pi}\lambda^{r-2}d\lambda$, then we get the power inequality

$$\begin{aligned}0&\leq(1-t)A^{r+1}+tB^{r+1}-((1-t)A+tB)^{r+1}\\&\leq\frac{\sin(r\pi)}{\pi}t(1-t)\int_{0}^{\infty}\lambda^{r+1}K_{\lambda}(A,B)\,d\lambda,\end{aligned} \tag{4.7}$$

for all A, $B>0$, where the kernel $K_{\lambda}(A,B)$ is defined in (3.5).

In particular, we have for all A, $B>0$,

$$0\leq\frac{A^{r+1}+B^{r+1}}{2}-\left(\frac{A+B}{2}\right)^{r+1}\leq\frac{1}{4}\frac{\sin(r\pi)}{\pi}\int_{0}^{\infty}\lambda^{r+1}K_{\lambda}(A,B)\,d\lambda. \tag{4.8}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa