SOME NEW LIPSCHITZ TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we provide some bounds for the quantity $\|f(y)-f(x)\|$ where $f:D\subset\mathbb{C}\rightarrow\mathbb{C}$ is an analytic function on the convex domain D and x, $y\in\mathcal{B}$, a Banach algebra, with the spectra $\sigma(x)$, $\sigma(y)\subset D$. We show among others that, if x, $y\in\mathcal{B}$ with $\sigma(x)$, $\sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R,

$$\|f(y)-f(x)\|\leq\frac{R\|y-x\|\|f\|_{R,\infty}}{(R-\|y\|)(R-\|x\|)}$$

and

$$\|f(y)-f(x)\|\leq\frac{1}{2}R\|y-x\|\|f\|_{R,\infty}\times\left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\},$$

where $\|f\|_{R,\infty}:=\sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right|<\infty$ and, in general the bounds are not comparable, meaning that one can be better than the other one depending on the elements x, $y\in\mathcal{B}$.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\rightarrow[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\text{Inv}(\mathcal{B})$. If $a,b\in\text{Inv}(\mathcal{B})$ then $ab\in\text{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\rightarrow\infty}\|a^n\|^{1/n}<1$, then $1-a\in\text{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\text{Inv}(\mathcal{B})$;
(iii) $\text{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\text{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\text{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\text{Inv}(\mathcal{B})\};$$

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the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [12] and [14].

For some recent norm inequalities for functions on Banach algebras, see [6], [2] and [4]-[10].

In this paper we provide some bounds for the quantity $\|f(y) - f(x)\|$ where $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$, a Banach algebra, with the spectra $\sigma(x)$, $\sigma(y) \subset D$. We show among others that, if x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$, where $D(0,R)$ is an open disk centered in 0 and of radius R,

$$\|f(y) - f(x)\| \le \frac{R\|y-x\| \|f\|_{R,\infty}}{(R - \|y\|)(R - \|x\|)}$$

and

$$\|f(y) - f(x)\| \le \frac{1}{2} R \|y - x\| \|f\|_{R,\infty} \times \left\{ \frac{1}{2}\left[(R-\|y\|)^{-2} + (R - \|x\|)^{-2}\right] + \left(R - \left\|\frac{x+y}{2}\right\|\right)^{-2} \right\},$$

where $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi it}\right)\right| < \infty$ and, in general, the bounds are not comparable, meaning that one can be better than the other one depending on the elements x, $y \in \mathcal{B}$.

2. Main Results

We have the following simple identity:

Lemma 1. *Let a, $b \in B$ with $a \neq b$ and assume that the closed segment $[a,b] := \{(1-t)a + tb,\ t \in [0,1]\} \subset \operatorname{Inv}(\mathcal{B})$. Then we have the representation*

$$b^{-1} - a^{-1} = \int_0^1 ((1-t)a + tb)^{-1}(a-b)((1-t)a+tb)^{-1}\,dt. \tag{2.1}$$

Proof. Consider the function with values in Banach algebra B, $\varphi_{a,b} : [0,1] \to B$,

$$\varphi_{a,b}(t) = ((1-t)a + tb)^{-1}.$$

This is well defined for $t \in [0,1]$.

We recall the following simple identity for the two invertible elements a, b in B

$$a^{-1} - b^{-1} = a^{-1}(b-a)b^{-1}.$$

For $t \in (0,1)$ and h in a neighborhood of 0 such that $t + h \in (0,1)$, we have

$$\begin{aligned}
&\varphi_{a,b}(t+h) - \varphi_{a,b}(t)\\
&= ((1-(t+h))a + (t+h)b)^{-1} - ((1-t)a+tb)^{-1}\\
&= ((1-(t+h))a + (t+h)b)^{-1}((1-t)a + tb - (1-(t+h))a - (t+h)b)\\
&\times ((1-t)a+tb)^{-1}\\
&= h((1-(t+h))a + (t+h)b)^{-1}(a-b)((1-t)a+tb)^{-1}.
\end{aligned}$$

This implies, for $h \neq 0$, that

$$\begin{aligned}
&\frac{\varphi_{a,b}(t+h) - \varphi_{a,b}(t)}{h}\\
&= ((1-(t+h))a + (t+h)b)^{-1}(a-b)((1-t)a+tb)^{-1}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\varphi'_{a,b}(t) = ((1-t)a+tb)^{-1}(a-b)((1-t)a+tb)^{-1}$$

that holds for $t \in (0,1)$.

By the use of Bochner's integral, [13], for functions with values in Banach algebras, we have

$$\varphi_{a,b}(1) - \varphi_{a,b}(0) = \int_0^1 \varphi'_{a,b}(t)\,dt,$$

which is the desired equality (2.1). □

Corollary 1. *Let x, $y \in B$ with $x \neq y$ and assume that $\|x\|$, $\|y\| < 1$, then*

$$\begin{aligned}
&(1-y)^{-1} - (1-x)^{-1}\\
&= \int_0^1 (1-(1-t)x - ty)^{-1}(y-x)(1-(1-t)x-ty)^{-1}\,dt.
\end{aligned} \tag{2.2}$$

Follows by Lemma 1 by taking $b=1-y$ and $a=1-x$.

We have the following representation result for the difference $f(y)-f(x)$ for x, $y\in\mathcal{B}$.

Theorem 1. *Let $f:D\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and x, $y\in\mathcal{B}$ with $\sigma(x)$, $\sigma(y)\subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y)\subset\operatorname{ins}(\gamma)$. Then we have*

$$\begin{aligned}(2.3)\qquad f(y)-f(x)&=\frac{1}{2\pi i}\int_0^1\left(\int_\gamma f(\xi)\left(\xi-(1-t)x-ty\right)^{-1}\right.\\&\qquad\left.\times(y-x)\left(\xi-(1-t)x-ty\right)^{-1}d\xi\right)dt\\&=\frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1\left(\xi-(1-t)x-ty\right)^{-1}\right.\\&\qquad\left.\times(y-x)\left(\xi-(1-t)x-ty\right)^{-1}dt\right)d\xi.\end{aligned}$$

Proof. Using the Riesz functional calculus, we have

$$\begin{aligned}(2.4)\qquad f(y)-f(x)&=\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-y)^{-1}d\xi-\int_\gamma f(\xi)(\xi-x)^{-1}d\xi\\&=\frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi-y)^{-1}-(\xi-x)^{-1}\right]d\xi\\&=\frac{1}{2\pi i}\int_\gamma f(\xi)\xi^{-1}\left[\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\right]d\xi.\end{aligned}$$

Since $\left\|\frac{y}{\xi}\right\|$, $\left\|\frac{x}{\xi}\right\|<1$ for $\xi\in\gamma$, then we can apply Corollary 1 to get

$$\begin{aligned}&\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\\&=\int_0^1\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\left(\frac{y}{\xi}-\frac{x}{\xi}\right)\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}dt\\&=\int_0^1\xi\left(\xi-(1-t)x-ty\right)^{-1}(y-x)\left(\xi-(1-t)x-ty\right)^{-1}dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_\gamma f(\xi)\xi^{-1}\left[\left(1-\frac{y}{\xi}\right)^{-1}-\left(1-\frac{x}{\xi}\right)^{-1}\right]d\xi\\&=\int_\gamma f(\xi)\xi^{-1}\left(\int_0^1\xi\left(\xi-(1-t)x-ty\right)^{-1}(y-x)\left(\xi-(1-t)x-ty\right)^{-1}dt\right)d\xi\\&=\int_\gamma f(\xi)\left(\int_0^1\left(\xi-(1-t)x-ty\right)^{-1}(y-x)\left(\xi-(1-t)x-ty\right)^{-1}dt\right)d\xi\\&=\int_0^1\left(\int_\gamma f(\xi)\left(\xi-(1-t)x-ty\right)^{-1}(y-x)\left(\xi-(1-t)x-ty\right)^{-1}d\xi\right)dt,\end{aligned}$$

where for the last equality we used Fubini's theorem.

By utilising (2.4), we get (2.3). □

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} &\|f(y)-f(x)\| \\ &\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi| \\ &\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-2} dt\right)|d\xi| \\ &\le \frac{1}{2\pi}\|y-x\|\int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|. \end{aligned} \tag{2.5}$$

Proof. We have, by taking the norm in (2.3), that

$$\begin{aligned} &\|f(y)-f(x)\| \\ &\le \frac{1}{2\pi}\int_0^1\left(\int_\gamma |f(\xi)|\right. \\ &\left.\times\left\|(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right\||d\xi|\right)dt \\ &\le \frac{1}{2\pi}\int_0^1\left(\int_\gamma |f(\xi)|\right. \\ &\left.\times\left\|(\xi-(1-t)x-ty)^{-1}\right\|\|y-x\|\left\|(\xi-(1-t)x-ty)^{-1}\right\||d\xi|\right)dt \\ &= \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\int_0^1\left(\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi| \\ &=: \|y-x\|B(f,x,y), \end{aligned} \tag{2.6}$$

which proves the first inequality in (2.5).

We have

$$\begin{aligned} B(f,x,y) &:= \frac{1}{2\pi}\int_\gamma |f(\xi)|\int_0^1\left(\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi| \\ &= \frac{1}{2\pi}\int_\gamma |f(\xi)||\xi|^{-2}\left(\int_0^1\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^2 dt\right)|d\xi|. \end{aligned}$$

Since

$$\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\| \le (1-t)\left\|\frac{x}{\xi}\right\|+t\left\|\frac{y}{\xi}\right\| < 1-t+t=1$$

for $\xi\in\gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\le \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k}\\ &=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}\le|\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)|\,|\xi|^{-2}\left(\int_{0}^{1}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}dt\right)|d\xi|\\ &\le\int_{\gamma}|f(\xi)|\left(\int_{0}^{1}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\right)|d\xi|\end{aligned}$$

and by (2.6) we derive the second inequality in (2.5).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\ge|\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\le\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}\le\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}$$

for $\xi\in\gamma$ and $t\in[0,1]$.

Taking the integral over $t\in[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\\
&\le \int_0^1 \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}dt\\
&=-\frac{1}{\|x\|-\|y\|}\int_0^1 \frac{d}{dt}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}dt\\
&=-\frac{1}{\|x\|-\|y\|}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}\Big|_0^1\\
&=\frac{1}{\|y\|-\|x\|}\left[(|\xi|-\|y\|)^{-1}-(|\xi|-\|x\|)^{-1}\right]\\
&=\frac{1}{\|y\|-\|x\|}\frac{\|y\|-\|x\|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}=\frac{1}{(|\xi|-\|y\|)(|\xi|-\|x\|)},
\end{aligned}$$

for $\|y\|\neq\|x\|$, which proves the last part of (2.5).

If $\|y\|=\|x\|$, then we have

$$\begin{aligned}
&\int_0^1 \left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\\
&\le \int_0^1 \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|x\|)\right]^{-2}dt=(|\xi|-\|x\|)^{-2},
\end{aligned}$$

which also gives the last bound for $\|y\|=\|x\|$. □

The inequality between the first and last term in (2.5) was obtained in [6] with a different technique. Here two refinements are also provided due to the identity (2.3).

Corollary 3. *With the assumptions of Theorem 1 and if*

$$\|f\|_{\gamma,\infty}:=\sup_{\xi\in\gamma}|f(\xi)|<\infty,$$

then

$$\begin{aligned}
&\|f(y)-f(x)\| \quad (2.7)\\
&\le\frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_\gamma\left(\int_0^1\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2dt\right)|d\xi|\\
&\le\frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_\gamma\left(\int_0^1\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\right)|d\xi|\\
&\le\frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_\gamma\frac{|d\xi|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}.
\end{aligned}$$

Remark 1. *Assume that $f:D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and $x,\ y\in\mathcal{B}$ with $\sigma(x),\ \sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By taking γ parametrized by $\xi(t)=Re^{2\pi it}$ where $t\in[0,1]$, then $d\xi(t)=2\pi iRe^{2\pi it}dt$, $|d\xi(t)|=2\pi Rdt$, $|\xi|=R$ and by (2.5)* *we get*

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\leq R\|y-x\| \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| \left(\int_0^1 \left\|\left(Re^{2\pi i t}-(1-s)x-sy\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R\|y-x\| \int_0^1 (R-\|(1-s)x+sy\|)^{-2}\, ds \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| dt \\
&\leq \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| dt.
\end{aligned} \tag{2.8}$$

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi i t}\right)\right| < \infty$, *then we have the simpler inequalities*

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\leq R\|y-x\|\|f\|_{R,\infty} \int_0^1 \left(\int_0^1 \left\|\left(Re^{2\pi i t}-(1-s)x-sy\right)^{-1}\right\|^2 ds\right) dt \\
&\leq R\|y-x\|\|f\|_{R,\infty} \int_0^1 (R-\|(1-s)x+sy\|)^{-2}\, ds \\
&\leq \frac{R\|y-x\|\|f\|_{R,\infty}}{(R-\|y\|)(R-\|x\|)}.
\end{aligned} \tag{2.9}$$

We also have the following upper bounds:

Theorem 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\leq \frac{1}{2\pi}\|y-x\| \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right) |d\xi| \\
&\leq \frac{1}{2\pi}\|y-x\| \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-2} dt\right) |d\xi| \\
&\leq \frac{1}{4\pi}\|y-x\| \left\{\frac{1}{2}\int_\gamma |f(\xi)| \left[(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}\right] |d\xi| \right.\\
&\left. + \int_\gamma |f(\xi)| \left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-2} |d\xi|\right\} \\
&\leq \frac{1}{4\pi}\|y-x\| \int_\gamma |f(\xi)| \left[(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}\right] |d\xi|.
\end{aligned} \tag{2.10}$$

Proof. Let $\xi \in \gamma$. For $\alpha,\ \beta \geq 0$ with $\alpha+\beta=1$ and $t_1,\ t_2 \in [0,1]$, then for $g_\xi(t) := (|\xi|-\|(1-t)x+ty\|)^{-2}$, $t\in[0,1]$ we get

$$\begin{aligned}
&g_\xi(\alpha t_1+\beta t_2) \\
&= (|\xi|-\|(1-(\alpha t_1+\beta t_2))x+(\alpha t_1+\beta t_2)y\|)^{-2} \\
&= (|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|)^{-2}
\end{aligned}$$

By the properties of the norm, we have

$$\begin{aligned}
&\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\| \\
&\leq \alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|
\end{aligned}$$

which gives that

$$\begin{aligned}&|\xi|-\|\alpha\left[(1-t_1)x+t_1y\right]+\beta\left[(1-t_2)x+t_2y\right]\|\\&\geq|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\begin{aligned}&\left(|\xi|-\|\alpha\left[(1-t_1)x+t_1y\right]+\beta\left[(1-t_2)x+t_2y\right]\|\right)^{-1}\\&\leq\left(|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|\right)^{-1}\end{aligned}$$

giving that

$$\begin{aligned}&g_\xi\left(\alpha t_1+\beta t_2\right)\\&\leq\left(|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|\right)^{-2}\\&=\left(\alpha\left[|\xi|-\|(1-t_1)x+t_1y\|\right]+\beta\left[|\xi|-\|(1-t_2)x+t_2y\|\right]\right)^{-2}.\end{aligned}$$

By using the convexity of the function $(\cdot)^{-2}$ we have

$$\begin{aligned}&\left(\alpha\left[|\xi|-\|(1-t_1)x+t_1y\|\right]+\beta\left[|\xi|-\|(1-t_2)x+t_2y\|\right]\right)^{-2}\\&\leq\alpha\left[|\xi|-\|(1-t_1)x+t_1y\|\right]^{-2}+\beta\left[|\xi|-\|(1-t_2)x+t_2y\|\right]^{-2}\\&=\alpha g_\xi(t_1)+\beta g_\xi(t_2).\end{aligned}$$

Therefore

$$g_\xi\left(\alpha t_1+\beta t_2\right)\leq\alpha g_\xi(t_1)+\beta g_\xi(t_2),$$

which proves the convexity of g_ξ on $[0,1]$.

By using the Hermite-Hadamard type inequality, see for instance [11, p. 11], for g_ξ on $[0,1]$ we get

$$\int_0^1 g_\xi(t)\,dt\leq\frac{1}{2}\left\{\frac{1}{2}\left[g_\xi(1)+g_\xi(0)\right]+g_\xi\left(\frac{1}{2}\right)\right\}\leq\frac{1}{2}\left[g_\xi(1)+g_\xi(0)\right]$$

namely

$$\begin{aligned}&\int_0^1\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\\&\leq\frac{1}{2}\left\{\frac{1}{2}\left[\left(|\xi|-\|y\|\right)^{-2}+\left(|\xi|-\|x\|\right)^{-2}\right]+\left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\}\\&\leq\frac{1}{2}\left[\left(|\xi|-\|y\|\right)^{-2}+\left(|\xi|-\|x\|\right)^{-2}\right]\end{aligned}$$

for $\xi\in\gamma$.

By making use of this inequality, we have

$$\begin{aligned}
&\int_{\gamma}|f(\xi)|\left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-2}dt\right)|d\xi| \\
&\leq \frac{1}{4}\int_{\gamma}|f(\xi)|\left[(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}\right]|d\xi| \\
&+\frac{1}{2}\int_{\gamma}|f(\xi)|\left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-2}|d\xi| \\
&\leq \frac{1}{2}\int_{\gamma}|f(\xi)|\left[(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}\right]|d\xi|.
\end{aligned}$$

By making use of Theorem 1 we then derive the desired result (2.10). □

Corollary 4. *With the assumptions of Theorem 1 and if*

$$\|f\|_{\gamma,\infty}:=\sup_{\xi\in\gamma}|f(\xi)|<\infty,$$

then

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_{\gamma}\left(\int_0^1\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2dt\right)|d\xi| \\
&\leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_{\gamma}\left(\int_0^1(|\xi|-\|(1-t)x+ty\|)^{-2}dt\right)|d\xi| \\
&\leq \frac{1}{4\pi}\|y-x\|\|f\|_{\gamma,\infty} \\
&\times\int_{\gamma}\left[\frac{(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}}{2}+\left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right]|d\xi| \\
&\leq \frac{1}{4\pi}\|y-x\|\|f\|_{\gamma,\infty}\int_{\gamma}\left[(|\xi|-\|y\|)^{-2}+(|\xi|-\|x\|)^{-2}\right]|d\xi|.
\end{aligned}\tag{2.11}$$

Remark 2. *Assume that $f : D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and $x, y\in\mathcal{B}$ with $\sigma(x), \sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. Then by (2.10),*

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\leq R\|y-x\|\int_0^1\left|f\left(Re^{2\pi it}\right)\right|\left(\int_0^1\left\|\left(Re^{2\pi it}-(1-s)x-sy\right)^{-1}\right\|^2ds\right)dt \\
&\leq R\|y-x\|\int_0^1(R-\|(1-s)x+sy\|)^{-2}ds\int_{\gamma}\left|f\left(Re^{2\pi it}\right)\right|dt \\
&\leq \frac{1}{2}R\|y-x\|\left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\} \\
&\times\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt \\
&\leq \frac{1}{2}R\|y-x\|\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt
\end{aligned}\tag{2.12}$$

and, by (2.11),

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\le R\|y-x\|\|f\|_{R,\infty}\int_0^1\left(\int_0^1\left\|\left(Re^{2\pi it}-(1-s)x-sy\right)^{-1}\right\|^2 ds\right)dt \\
&\le R\|y-x\|\|f\|_{R,\infty}\int_0^1 (R-\|(1-s)x+sy\|)^{-2}\,ds \\
&\le \frac{1}{2}R\|y-x\|\|f\|_{R,\infty} \\
&\times\left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\} \\
&\le \frac{1}{2}R\|y-x\|\|f\|_{R,\infty}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right].
\end{aligned}\tag{2.13}$$

Remark 3. *From inequality (2.9) we have the following bound for the quantity* $\frac{1}{\|f\|_{R,\infty}}\|f(y)-f(x)\|$

$$B_{1,R}(x,y):=\frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)}$$

while from (2.13),

$$\begin{aligned}
B_{2,R}(x,y)&:=\frac{1}{2}R\|y-x\| \\
&\times\left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\}
\end{aligned}$$

where $\|x\|,\ \|y\|<R$.

By taking $R=1,\ x,\ y\in(-1,1),\ \|\cdot\|=|\cdot|$ *and doing a 3-dimensional plot for the difference* $B_{1,R}(x,y)-B_{2,R}(x,y)$ *on the box* $(x,y)\in(-1,1)\times(-1,1)$, *we observe that some time one bound is better than the other.*

The same conclusion then applies to the inequalities

$$\|f(y)-f(x)\|\le\frac{R\|y-x\|\|f\|_{R,\infty}}{(R-\|y\|)(R-\|x\|)}\tag{2.14}$$

and

$$\begin{aligned}
&\|f(y)-f(x)\| \\
&\le\frac{1}{2}R\|y-x\|\|f\|_{R,\infty} \\
&\times\left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\},
\end{aligned}\tag{2.15}$$

provided that $f:D\subset\mathbb{C}\to\mathbb{C}$ *is an analytic function on the domain* D *and* $x,\ y\in\mathcal{B}$ *with* $\sigma(x),\ \sigma(y)\subset D(0,R)\subset D$ *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R.

3. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a\in\mathcal{B}$. Assume that x, $y\in\mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R>0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}
&\int_0^1 \exp\left[R\cos(2\pi t)\right]dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned} \tag{3.1}$$

Assume that $f : D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the domain D and x, $y\in\mathcal{B}$ with $\sigma(x), \sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By (2.8) for the exponential function we get

$$\begin{aligned}
&\|\exp y - \exp y\| \\
&\le R\|y-x\| \\
&\times \int_0^1 \exp\left[R\cos(2\pi t)\right]\left(\int_0^1 \left\|\left(Re^{2\pi it} - (1-s)x - sy\right)^{-1}\right\|^2 ds\right)dt \\
&\le RI_0(R)\|y-x\|\int_0^1 \left(R - \|(1-s)x + sy\|\right)^{-2}ds \\
&\le \frac{RI_0(R)\|y-x\|}{(R-\|y\|)(R-\|x\|)}
\end{aligned} \tag{3.2}$$

and from (2.12),

$$
\begin{aligned}
&\|\exp y-\exp y\| \\
&\leq R\|y-x\| \\
&\times \int_0^1 \exp[R\cos(2\pi t)]\left(\int_0^1 \left\|\left(Re^{2\pi it}-(1-s)\,x-sy\right)^{-1}\right\|^2 ds\right) dt \\
&\leq RI_0(R)\|y-x\| \int_0^1 \left(R-\|(1-s)\,x+sy\|\right)^{-2} ds \\
&\leq \frac{1}{2} RI_0(R)\|y-x\| \\
&\times \left\{\frac{1}{2}\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-2}\right\} \\
&\leq \frac{1}{2} RI_0(R)\|y-x\|\left[(R-\|y\|)^{-2}+(R-\|x\|)^{-2}\right].
\end{aligned} \tag{3.3}
$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(a) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C},\, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|\, \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$
\begin{aligned}
f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda},\ \ \lambda \in D(0,1); \\
g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos\lambda,\ \ \lambda \in \mathbb{C}; \\
h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin\lambda,\ \ \lambda \in \mathbb{C}; \\
l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda},\ \ \lambda \in D(0,1);
\end{aligned} \tag{3.4}
$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$
\begin{aligned}
f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda},\ \ \lambda \in D(0,1); \\
g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh\lambda,\ \ \lambda \in \mathbb{C};
\end{aligned} \tag{3.5}
$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh\lambda,\ \lambda \in \mathbb{C};$$
$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.6}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 2. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \leq f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{3.7}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty}\left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \leq \lim_{n\to\infty}\sum_{j=0}^{n}\left|\alpha_j\lambda^j\right| \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}|\alpha_j||\lambda|^j = \sum_{j=0}^{\infty}|\alpha_j||\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned} &\|f(y) - f(x)\| \\ &\leq R\|y-x\| f_A(R) \int_0^1 \left(\int_0^1 \left\|\left(Re^{2\pi it} - (1-s)x - sy\right)^{-1}\right\|^2 ds\right) dt \\ &\leq R\|y-x\| f_A(R) \int_0^1 (R - \|(1-s)x + sy\|)^{-2} ds \leq \frac{Rf_A(R)\|y-x\|}{(R-\|y\|)(R-\|x\|)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
(3.9)\qquad & \|f(y)-f(x)\| \\
& \leq R\|y-x\| f_A(R) \int_0^1 \left(\int_0^1 \left\| \left(Re^{2\pi it} - (1-s)x - sy \right)^{-1} \right\|^2 ds \right) dt \\
& \leq R\|y-x\| f_A(R) \int_0^1 \left(R - \|(1-s)x+sy\| \right)^{-2} ds \\
& \leq \frac{1}{2} R\|y-x\| f_A(R) \\
& \times \left\{ \frac{1}{2}\left[(R-\|y\|)^{-2} + (R-\|x\|)^{-2} \right] + \left(R - \left\| \frac{x+y}{2} \right\| \right)^{-2} \right\}.
\end{aligned}$$

The proof follows by Remarks 1, 2 and Lemma 2. As examples, one can consider the functions f and f_A listed above.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: `sever.dragomir@vu.edu.au`
URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa