LOWER AND UPPER BOUNDS FOR THE PERTURBED SLATER'S GAP OF CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. In this paper we establish some lower and upper bounds for the perturbed Slater's gap

$$\Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f) d\mu$$

for some classes of twice differentiable convex functions Φ defined on an interval I and $v \in I$. Applications for exponential and logarithm are also given.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_-(x) \le \Phi'_+(x) \le \Phi'_-(y) \le \Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(x) \ge \Phi(a) + (x - a)\varphi(a) \text{ for any } x,\ a \in I.$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, $\Phi'_-, \Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(x) \le \varphi(x) \le \Phi'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

The following result is well known in the literature as *Slater's inequality:*

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20..

Key words and phrases. Slater's inequality, Measurable functions, Lebesgue integral, Discrete inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 14, 24 pp. Received 19/01/20

Theorem 1 (Slater, 1981, [11]). *If $\Phi : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $\varphi \in \partial\Phi$ and $f : \Omega \to I$ is μ-measurable and such that $\Phi \circ f$, $\varphi \circ f$, $f \cdot \varphi \circ f \in L(\Omega, \mu)$ then*

$$\int_\Omega \Phi \circ f d\mu \le \Phi\left(\frac{\int_\Omega f \cdot (\varphi \circ f) \, d\mu}{\int_\Omega \varphi \circ f d\mu}\right). \tag{1.1}$$

As pointed out in [10], see also [3, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\int_\Omega f \cdot (\varphi \circ f) \, d\mu}{\int_\Omega \varphi \circ f d\mu} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

For some related results, see [1], [8], [9] and [12]. Extensions for continuous functions of selfadjoint operators may be found in [4]-[6].

In the recent paper [7] we established some upper bounds for the perturbed Slater's gap for some classes of differentiable convex functions Φ defined on an interval I and $u \in I$:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior of I denoted by $\mathring{I}$. Assume that $f : \Omega \to I$ is μ-measurable and such that f, $\Phi \circ f$, $\Phi' \circ f$, $f \cdot \Phi' \circ f \in L(\Omega, \mu)$. Then for all $u \in I$,*

$$\begin{aligned} 0 &\le \Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f) \, d\mu \\ &\le \int_\Omega [\Phi' \circ f - \Phi'(u)] (f - u) \, d\mu \\ &\le \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(u)| \int_\Omega |f - u| \, d\mu, \\ \\ \left(\int_\Omega |\Phi' \circ f - \Phi'(u)|^p \, d\mu\right)^{1/p} \left(\int_\Omega |f - u|^q \, d\mu\right)^{1/q}, \\ p, \ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega |f - u| \int_\Omega |\Phi' \circ f - \Phi'(u)| \, d\mu, \end{cases} \end{aligned} \tag{1.3}$$

provided the integrals in the last term are finite.

We have the following reverse of perturbed Jensen's inequality, see also [2] for the first inequality below:

Corollary 1. *With the assumptions of Theorem 2, we have*

$$\begin{aligned} 0 &\le \Phi\left(\int_\Omega f d\mu\right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f) \, d\mu \\ &\le \int_\Omega \left[\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right] \left(f - \int_\Omega f d\mu\right) d\mu \\ &\le \begin{cases} \operatorname{essup}_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right| \int_\Omega \left|f - \int_\Omega f d\mu\right| d\mu, \\ \\ \left(\int_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right|^p d\mu\right)^{1/p} \left(\int_\Omega \left|f - \int_\Omega f d\mu\right|^q d\mu\right)^{1/q}, \\ p, \ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \operatorname{essup}_\Omega \left|f - \int_\Omega f d\mu\right| \int_\Omega \left|\Phi' \circ f - \Phi'\left(\int_\Omega f d\mu\right)\right| d\mu. \end{cases} \end{aligned} \tag{1.4}$$

We have the following reverse of perturbed Slater's inequality:

Corollary 2. *With the assumptions of Theorem 2 and if* $\int_\Omega \Phi' \circ f d\mu \neq 0$ *and the Slater's point*

$$\sigma := \frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \in I, \tag{1.5}$$

then

$$0 \leq \Phi(\sigma) - \int_\Omega \Phi \circ f d\mu \leq \int_\Omega [\Phi' \circ f - \Phi'(\sigma)]\,(f-\sigma)\, d\mu \tag{1.6}$$
$$\leq \begin{cases} \operatorname{essup}_\Omega |\Phi' \circ f - \Phi'(\sigma)| \int_\Omega |f-\sigma|\, d\mu, \\ \\ \left(\int_\Omega |\Phi' \circ f - \Phi'(\sigma)|^p\, d\mu\right)^{1/p} \left(\int_\Omega |f-\sigma|^q\, d\mu\right)^{1/q}, \\ p,\ q>1 \ \text{with}\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \operatorname{essup}_\Omega |f-\sigma| \int_\Omega |\Phi' \circ f - \Phi'(\sigma)|\, d\mu. \end{cases}$$

Motivated by the above results, we establish in this paper some new lower and upper bounds for the *perturbed Slater's gap*

$$\Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot (\Phi' \circ f)\, d\mu$$

for some classes of twice differentiable convex functions Φ defined on an interval I and $u \in I$. Applications for exponential and logarithm are also given.

2. General Results

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1. *Let* $I \subset \mathbb{R}$ *be a closed interval,* $a \in I$ *and let* n *be a positive integer. If* $g : I \longrightarrow \mathbb{C}$ *is such that the* n*-derivative* $g^{(n)}$ *is absolutely continuous on* I*, then for each* $x \in I$

$$g(x) = T_n(g; a, x) + R_n(g; a, x), \tag{2.1}$$

where $T_n(g; c, y)$ *is Taylor's polynomial, i.e.,*

$$T_n(g; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} g^{(k)}(a). \tag{2.2}$$

Note that $g^{(0)} := g$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(g; a, x) := \frac{1}{n!} \int_a^x (x-t)^n g^{(n+1)}(t)\, dt. \tag{2.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\, dt = (d-c) \int_0^1 h((1-s)c + sd)\, ds.$$

Therefore,

$$\begin{aligned}&\int_a^x g^{(n+1)}(t)(x-t)^n\,dt\\&=(x-a)\int_0^1 g^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^n\,ds\\&=(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds.\end{aligned}$$

The identity (2.1) can then be written as

$$\begin{aligned}g(x)&=\sum_{k=0}^n\frac{1}{k!}g^{(k)}(a)(x-a)^k\\&\quad+\frac{1}{n!}(x-a)^{n+1}\int_0^1 g^{(n+1)}((1-s)a+sx)(1-s)^n\,ds\end{aligned}\tag{2.4}$$

for all $x,\,a\in I$.

Theorem 3. *Let $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\,f^2,\,\Phi\circ f,\,\Phi''\circ f\in L(\Omega,\mu)$. Then for all $u\in\mathring{I}$,*

$$\begin{aligned}0&\le\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+su)(1-s)\,ds\right)\left(\int_\Omega f\,d\mu-u\right)^2\\&\le\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+su)(1-s)\,ds\right)\int_\Omega(f-u)^2\,d\mu\\&\le\Phi(u)-\int_\Omega\Phi\circ f\,d\mu-u\int_\Omega\Phi'\circ f\,d\mu+\int_\Omega f\cdot\Phi'\circ f\,d\mu\\&\le\operatorname*{essup}_{t\in\Omega}\left(\int_0^1\Phi''((1-s)f(t)+su)(1-s)\,ds\right)\int_\Omega(f-u)^2\,d\mu\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}0&\le\operatorname*{essinf}_{t\in\Omega}(f(t)-u)^2\int_\Omega\left(\int_0^1\Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\\&\le\Phi(u)-\int_\Omega\Phi\circ f\,d\mu-u\int_\Omega\Phi'\circ f\,d\mu+\int_\Omega f\cdot\Phi'\circ f\,d\mu\\&\le\operatorname*{essup}_{t\in\Omega}(f(t)-u)^2\int_\Omega\left(\int_0^1\Phi''((1-s)f+su)(1-s)\,ds\right)d\mu.\end{aligned}\tag{2.6}$$

Proof. We have from (2.4) for $n=2$ that

$$\Phi(x)=\Phi(c)+\Phi'(c)(x-c)+(x-c)^2\int_0^1\Phi''((1-s)c+sx)(1-s)\,ds$$

for all $x,\,c\in\mathring{I}$, where Φ is twice differentiable on $\mathring{I}$.

This implies that

$$\begin{aligned}\Phi(u)&=\Phi(f(t))+\Phi'(f(t))(u-f(t))\\&\quad+(u-f(t))^2\int_0^1\Phi''((1-s)f(t)+su)(1-s)\,ds\end{aligned}\tag{2.7}$$

for all $u\in\mathring{I}$ and μ-a.e. $t\in\Omega$.

By taking the integral in (2.7) we get

$$\begin{aligned}\Phi(u) &= \int_\Omega \Phi\circ f d\mu + u\int_\Omega \Phi'\circ f d\mu - \int_\Omega f\cdot\Phi'\circ f d\mu \\ &+ \int_\Omega (f-u)^2 \left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu \end{aligned} \tag{2.8}$$

for all $u\in\mathring{I}$.

We observe that for μ-a.e. $t\in\Omega$ we have

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds\right)\\ &\le \int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds\\ &\le \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds\right)\int_\Omega (f-u)^2 d\mu\\ &\le \int_\Omega (f-u)^2\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\\ &\le \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds\right)\int_\Omega (f-u)^2 d\mu,\end{aligned}$$

and by (2.8) we derive (2.5).

We also have

$$\begin{aligned}&\operatorname*{essinf}_{t\in\Omega}(f(t)-u)^2\int_\Omega\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\\ &\le \int_\Omega (f-u)^2\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\\ &\le \operatorname*{essup}_{t\in\Omega}(f(t)-u)^2\int_\Omega\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\end{aligned}$$

and by (2.8) we derive (2.5). □

Corollary 3. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}(2.9)\qquad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\int_\Omega fd\mu\right)(1-s)\,ds\right)\\ &\quad\times\left[\int_\Omega f^2 d\mu-\left(\int_\Omega fd\mu\right)^2\right]\\ &\leq \Phi\left(\int_\Omega fd\mu\right)-\int_\Omega \Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega \Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu\\ &\leq \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\int_\Omega fd\mu\right)(1-s)\,ds\right)\\ &\quad\times\left[\int_\Omega f^2 d\mu-\left(\int_\Omega fd\mu\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}(2.10)\qquad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\\ &\quad\times\int_\Omega\left(\int_0^1 \Phi''\left((1-s)\,f+s\int_\Omega fd\mu\right)(1-s)\,ds\right)d\mu\\ &\leq \Phi\left(\int_\Omega fd\mu\right)-\int_\Omega \Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega \Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu\\ &\leq \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\\ &\quad\times\int_\Omega\left(\int_0^1 \Phi''\left((1-s)\,f+s\int_\Omega fd\mu\right)(1-s)\,ds\right)d\mu.\end{aligned}$$

The inequality (2.9) follows by (2.5) on taking $u=\int_\Omega fd\mu$ and observing that

$$\int_\Omega\left(f-\int_\Omega fd\mu\right)^2 d\mu=\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2.$$

Corollary 4. *With the assumptions of Theorem 3 and if σ is the Slater point defined by (1.5), then we have*

$$\begin{aligned}(2.11)\qquad 0 &\leq \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\sigma\right)(1-s)\,ds\right)\left(\int_\Omega fd\mu-\sigma\right)^2\\ &\leq \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\sigma\right)(1-s)\,ds\right)\int_\Omega (f-\sigma)^2\,d\mu\\ &\leq \Phi\left(\frac{\int_\Omega f\cdot(\Phi'\circ f)\,d\mu}{\int_\Omega \Phi'\circ fd\mu}\right)-\int_\Omega \Phi\circ fd\mu\\ &\leq \operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+s\sigma\right)(1-s)\,ds\right)\int_\Omega (f-\sigma)^2\,d\mu\end{aligned}$$

and

$$0 \le \underset{t\in\Omega}{\operatorname{essinf}} (f(t)-\sigma)^2 \int_\Omega \left(\int_0^1 \Phi''((1-s)f+s\sigma)(1-s)\,ds\right)d\mu \tag{2.12}$$
$$\le \Phi\left(\frac{\int_\Omega f\cdot(\Phi'\circ f)\,d\mu}{\int_\Omega \Phi'\circ f d\mu}\right) - \int_\Omega \Phi\circ f d\mu$$
$$\le \underset{t\in\Omega}{\operatorname{essup}} (f(t)-\sigma)^2 \int_\Omega \left(\int_0^1 \Phi''((1-s)f+s\sigma)(1-s)\,ds\right)d\mu.$$

Corollary 5. *With the assumptions of Theorem 3 and if there exists an interval* $[m,M]\subset I$ *such that* $m\le f\le M$ μ*-a.e. on* Ω*, then*

$$0 \le \underset{t\in\Omega}{\operatorname{essinf}} \left(\int_0^1 \Phi''\left((1-s)f(t)+s\frac{m+M}{2}\right)(1-s)\,ds\right) \tag{2.13}$$
$$\times\left(\int_\Omega f d\mu - \frac{m+M}{2}\right)^2$$
$$\le \underset{t\in\Omega}{\operatorname{essinf}} \left(\int_0^1 \Phi''\left((1-s)f(t)+s\frac{m+M}{2}\right)(1-s)\,ds\right)$$
$$\times\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu$$
$$\le \Phi\left(\frac{m+M}{2}\right) - \int_\Omega \Phi\circ f d\mu - \frac{m+M}{2}\int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot\Phi'\circ f d\mu$$
$$\le \underset{t\in\Omega}{\operatorname{essup}} \left(\int_0^1 \Phi''\left((1-s)f(t)+s\frac{m+M}{2}\right)(1-s)\,ds\right)$$
$$\times\int_\Omega \left(f - \frac{m+M}{2}\right)^2 d\mu$$

and

$$0 \le \underset{t\in\Omega}{\operatorname{essinf}} \left(f(t) - \frac{m+M}{2}\right)^2 \tag{2.14}$$
$$\times\int_\Omega \left(\int_0^1 \Phi''\left((1-s)f+s\frac{m+M}{2}\right)(1-s)\,ds\right)d\mu$$
$$\le \Phi\left(\frac{m+M}{2}\right) - \int_\Omega \Phi\circ f d\mu - \frac{m+M}{2}\int_\Omega \Phi'\circ f d\mu + \int_\Omega f\cdot\Phi'\circ f d\mu$$
$$\le \underset{t\in\Omega}{\operatorname{essup}} \left(f(t) - \frac{m+M}{2}\right)^2$$
$$\times\int_\Omega \left(\int_0^1 \Phi''\left((1-s)f+s\frac{m+M}{2}\right)(1-s)\,ds\right)d\mu.$$

Remark 1. *Since* $m\le f\le M$ μ*-a.e. on* Ω *is equivalent to*

$$\left|f - \frac{m+M}{2}\right| \le \frac{1}{2}(M-m) \ \ \mu\text{-a.e.on } \Omega,$$

then from the last inequality in (2.14) we obtain

$$0\leq\Phi\left(\frac{m+M}{2}\right)-\int_{\Omega}\Phi\circ fd\mu-\frac{m+M}{2}\int_{\Omega}\Phi'\circ fd\mu+\int_{\Omega}f\cdot\Phi'\circ fd\mu \tag{2.15}$$
$$\leq\frac{1}{4}\left(M-m\right)^{2}\int_{\Omega}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)f+s\frac{m+M}{2}\right)\left(1-s\right)ds\right)d\mu.$$

Theorem 4. *Let $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\ f^{2},\ \Phi\circ f,\ \Phi''\circ f\in L\left(\Omega,\mu\right).$ Then for all $u\in\mathring{I}$,*

$$0\leq\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}\left(f-u\right)^{2}\Phi''\left(\left(1-s\right)f+su\right)d\mu\right) \tag{2.16}$$
$$\leq\Phi\left(u\right)-\int_{\Omega}\Phi\circ fd\mu-u\int_{\Omega}\Phi'\circ fd\mu+\int_{\Omega}f\cdot\Phi'\circ fd\mu$$
$$\leq\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}\left(f-u\right)^{2}\Phi''\left(\left(1-s\right)f+su\right)d\mu\right).$$

In particular,

$$0\leq\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}\left(f-\int_{\Omega}fd\mu\right)^{2}\Phi''\left(\left(1-s\right)f+s\int_{\Omega}fd\mu\right)d\mu\right) \tag{2.17}$$
$$\leq\Phi\left(\int_{\Omega}fd\mu\right)-\int_{\Omega}\Phi\circ fd\mu-\int_{\Omega}fd\mu\int_{\Omega}\Phi'\circ fd\mu+\int_{\Omega}f\cdot\Phi'\circ fd\mu$$
$$\leq\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}\left(f-\int_{\Omega}fd\mu\right)^{2}\Phi''\left(\left(1-s\right)f+s\int_{\Omega}fd\mu\right)d\mu\right)$$

and

$$0\leq\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}\left(f-\sigma\right)^{2}\Phi''\left(\left(1-s\right)f+s\sigma\right)d\mu\right) \tag{2.18}$$
$$\leq\Phi\left(\frac{\int_{\Omega}f\cdot\left(\Phi'\circ f\right)d\mu}{\int_{\Omega}\Phi'\circ fd\mu}\right)-\int_{\Omega}\Phi\circ fd\mu$$
$$\leq\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}\left(f-\sigma\right)^{2}\Phi''\left(\left(1-s\right)f+s\sigma\right)d\mu\right).$$

Proof. Using Fubini's theorem we get by (2.8) that

$$\Phi\left(u\right)=\int_{\Omega}\Phi\circ fd\mu+u\int_{\Omega}\Phi'\circ fd\mu-\int_{\Omega}f\cdot\Phi'\circ fd\mu \tag{2.19}$$
$$+\int_{\Omega}\left(f-u\right)^{2}\left(\int_{0}^{1}\Phi''\left(\left(1-s\right)f+su\right)\left(1-s\right)ds\right)d\mu$$
$$=\int_{\Omega}\Phi\circ fd\mu+u\int_{\Omega}\Phi'\circ fd\mu-\int_{\Omega}f\cdot\Phi'\circ fd\mu$$
$$+\int_{0}^{1}\left(\int_{\Omega}\left(f-u\right)^{2}\Phi''\left(\left(1-s\right)f+su\right)d\mu\right)\left(1-s\right)ds$$

for all $u\in\mathring{I}$.

Since

$$\begin{aligned}
&\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)\int_{0}^{1}(1-s)\,ds\\
&\leq\int_{0}^{1}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)(1-s)\,ds\\
&\leq\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)\int_{0}^{1}(1-s)\,ds,
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)\\
&\leq\int_{0}^{1}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)(1-s)\,ds\\
&\leq\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right),
\end{aligned}$$

hence by (2.19) we get (2.16). □

We also have:

Theorem 5. *Let $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\ f^{2},\ \Phi\circ f,\ \Phi''\circ f\in L(\Omega,\mu)$. Then*

$$\begin{aligned}
0&\leq\frac{1}{2(M-m)}\operatorname*{essinf}_{s\in[0,1]}\left[\int_{m}^{M}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)du\right]\\
&\leq\frac{1}{M-m}\int_{m}^{M}\Phi(u)\,du-\int_{\Omega}\Phi\circ f\,d\mu\\
&-\frac{m+M}{2}\int_{\Omega}\Phi'\circ f\,d\mu+\int_{\Omega}f\cdot\Phi'\circ f\,d\mu\\
&\leq\frac{1}{2(M-m)}\operatorname*{essup}_{s\in[0,1]}\left[\int_{m}^{M}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)du\right].
\end{aligned}\tag{2.20}$$

Proof. If we take the integral mean $\frac{1}{M-m}\int_{m}^{M}$ over u in (2.8) and using Fubini's theorem, then we get

$$\begin{aligned}
&\frac{1}{M-m}\int_{m}^{M}\Phi(u)\,du\\
&=\int_{\Omega}\Phi\circ f\,d\mu+\frac{m+M}{2}\int_{\Omega}\Phi'\circ f\,d\mu-\int_{\Omega}f\cdot\Phi'\circ f\,d\mu\\
&+\frac{1}{M-m}\left(\int_{m}^{M}\int_{\Omega}(f-u)^{2}\left(\int_{0}^{1}\Phi''((1-s)f+su)(1-s)\,ds\right)d\mu\right)du\\
&=\int_{\Omega}\Phi\circ f\,d\mu+\frac{m+M}{2}\int_{\Omega}\Phi'\circ f\,d\mu-\int_{\Omega}f\cdot\Phi'\circ f\,d\mu\\
&\frac{1}{M-m}\int_{0}^{1}\left[\int_{m}^{M}\left(\int_{\Omega}(f-u)^{2}\Phi''((1-s)f+su)\,d\mu\right)du\right](1-s)\,ds.
\end{aligned}\tag{2.21}$$

Observe that

$$\begin{aligned}&\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right]\int_0^1(1-s)\,ds\\&\le\int_0^1\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right](1-s)\,ds\\&\le\operatorname*{essup}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right]\int_0^1(1-s)\,ds,\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{2}\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right]\\&\le\int_0^1\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right](1-s)\,ds\\&\le\frac{1}{2}\operatorname*{essup}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega (f-u)^2\,\Phi''((1-s)f+su)\,d\mu\right)du\right],\end{aligned}$$

which, by (2.21), is equivalent to (2.20). □

3. Related Results

When the second derivative is bounded, we have the following lower and upper bounds of interest:

Theorem 6. *Let $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f:\Omega\to I$ so that $f,\ f^2,\ \Phi\circ f,\ \Phi''\circ f\in L(\Omega,\mu)$. If there exists the constants $0<\gamma<\Gamma<\infty$ such that $\gamma\le\Phi''(x)\le\Gamma$ for a.e. $x\in I$, then for all $u\in\mathring{I}$,*

$$\begin{aligned}0&\le\frac{1}{2}\gamma\left(\int_\Omega fd\mu-u\right)^2\le\frac{1}{2}\gamma\int_\Omega (f-u)^2\,d\mu\\&\le\Phi(u)-\int_\Omega\Phi\circ fd\mu-u\int_\Omega\Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu\\&\le\frac{1}{2}\Gamma\int_\Omega (f-u)^2\,d\mu\end{aligned}\tag{3.1}$$

and, in particular

$$\begin{aligned}0&\le\frac{1}{2}\gamma\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\\&\le\Phi\left(\int_\Omega fd\mu\right)-\int_\Omega\Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega\Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu\\&\le\frac{1}{2}\Gamma\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\end{aligned}\tag{3.2}$$

and

$$0 \le \frac{1}{2}\gamma \left(\int_\Omega f d\mu - \sigma \right)^2 \le \frac{1}{2}\gamma \int_\Omega (f-\sigma)^2 d\mu \tag{3.3}$$
$$\le \Phi\left(\frac{\int_\Omega f \cdot (\Phi' \circ f)\, d\mu}{\int_\Omega \Phi' \circ f d\mu} \right) - \int_\Omega \Phi \circ f d\mu \le \frac{1}{2}\Gamma \int_\Omega (f-\sigma)^2 d\mu,$$

where σ is Slater's point defined by (1.5).

If there exists an interval $[m, M] \subset I$ such that $m \le f \le M$ μ-a.e. on Ω, then we also have

$$0 \le \frac{1}{2}\gamma \left(\int_\Omega f d\mu - \frac{m+M}{2} \right)^2 \le \frac{1}{2}\gamma \int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu \tag{3.4}$$
$$\le \Phi\left(\frac{m+M}{2}\right) - \int_\Omega \Phi \circ f d\mu - \frac{m+M}{2} \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$\le \frac{1}{2}\Gamma \int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu \le \frac{1}{8}\Gamma (M-m)^2$$

and

$$0 \le \frac{1}{2}\gamma \left[\int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu + \frac{1}{12}(M-m)^2 \right] \tag{3.5}$$
$$\le \frac{1}{M-m}\int_m^M \Phi(u)\, du - \int_\Omega \Phi \circ f d\mu$$
$$- \frac{m+M}{2} \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$\le \frac{1}{2}\Gamma \left[\int_\Omega \left(f - \frac{m+M}{2} \right)^2 d\mu + \frac{1}{12}(M-m)^2 \right].$$

Proof. Inequality (3.1) follows by (2.5) and (3.4) by (2.13).

Observe that

$$\gamma \int_m^M \left(\int_\Omega (f-u)^2 d\mu \right) du \le \int_m^M \left(\int_\Omega (f-u)^2 \Phi''((1-s) f + su)\, d\mu \right) du \tag{3.6}$$
$$\le \Gamma \int_m^M \left(\int_\Omega (f-u)^2 d\mu \right) du.$$

Moreover,

$$\int_m^M \left(\int_\Omega (f-u)^2 d\mu \right) du$$
$$= \int_\Omega \left(\frac{(M-f)^3 + (f-m)^3}{3} \right) d\mu$$
$$= \frac{1}{3}(M-m) \int_\Omega \left[(M-f)^2 - (M-f)(f-m) + (f-m)^2 \right] d\mu.$$

Also

$$\begin{aligned}&(M-f)^2-(M-f)(f-m)+(f-m)^2\\&=3\left[\left(f-\frac{m+M}{2}\right)^2+\frac{1}{12}(M-m)^2\right],\end{aligned}$$

which implies that

$$\begin{aligned}&\int_\Omega\left[(M-f)^2-(M-f)(f-m)+(f-m)^2\right]d\mu\\&=3\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right].\end{aligned}$$

Therefore

$$\begin{aligned}&\int_m^M\left(\int_\Omega(f-u)^2d\mu\right)du\\&=(M-m)\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right]\end{aligned}$$

and by (3.6) we get

$$\begin{aligned}&\gamma(M-m)\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right]\\&\le\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega(f-u)^2\Phi''((1-s)f+su)\,d\mu\right)du\right]\end{aligned}$$

and

$$\begin{aligned}&\operatorname*{essup}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega(f-u)^2\Phi''((1-s)f+su)\,d\mu\right)du\right]\\&\le\Gamma(M-m)\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right].\end{aligned}$$

These imply that

$$\begin{aligned}&\frac{1}{2}\gamma\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right]\\&\le\frac{1}{2(M-m)}\operatorname*{essinf}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega(f-u)^2\Phi''((1-s)f+su)\,d\mu\right)du\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2(M-m)}\operatorname*{essup}_{s\in[0,1]}\left[\int_m^M\left(\int_\Omega(f-u)^2\Phi''((1-s)f+su)\,d\mu\right)du\right]\\&\le\frac{1}{2}\Gamma\left[\int_\Omega\left(f-\frac{m+M}{2}\right)^2d\mu+\frac{1}{12}(M-m)^2\right],\end{aligned}$$

and by (2.20) we get the desired result (3.5). □

Remark 2. *If there exists an interval* $[m, M] \subset I$ *such that* $m \leq f \leq M$ μ*-a.e. on* Ω*, then*

$$\frac{1}{2}\gamma\left[\int_{\Omega}\left(f-\frac{m+M}{2}\right)^{2}d\mu+\frac{1}{12}\left(M-m\right)^{2}\right]\geq\frac{1}{24}\gamma\left(M-m\right)^{2}$$

and

$$\begin{aligned}&\frac{1}{2}\Gamma\left[\int_{\Omega}\left(f-\frac{m+M}{2}\right)^{2}d\mu+\frac{1}{12}\left(M-m\right)^{2}\right]\\&\leq\frac{1}{2}\Gamma\left[\frac{1}{4}\left(M-m\right)^{2}+\frac{1}{12}\left(M-m\right)^{2}\right]=\frac{1}{6}\Gamma\left(M-m\right)^{2}\end{aligned}$$

and by (3.5) we get

$$\begin{aligned}0&\leq\frac{1}{24}\gamma\left(M-m\right)^{2}\leq\frac{1}{M-m}\int_{m}^{M}\Phi\left(u\right)du-\int_{\Omega}\Phi\circ fd\mu\\&-\frac{m+M}{2}\int_{\Omega}\Phi'\circ fd\mu+\int_{\Omega}f\cdot\Phi'\circ fd\mu\\&\leq\frac{1}{6}\Gamma\left(M-m\right)^{2}.\end{aligned}\tag{3.7}$$

If some monotonicity properties for the second derivative are assumed, then we have the following results as well.

Theorem 7. *Let* $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable convex function on* $\mathring{I}$ *and* $f : \Omega \to I$ *so that* f*,* f^2*,* $\Phi \circ f$*,* $\Phi'' \circ f \in L(\Omega, \mu)$*. If the second derivative* Φ'' *is monotonic nondecreasing on an interval* $[m, M] \subset \mathring{I}$*, then we have*

$$\begin{aligned}0&\leq\frac{1}{u-m}\left\{\frac{\Phi\left(u\right)-\Phi\left(m\right)}{u-m}-\Phi'\left(m\right)\right\}\left(\int_{\Omega}fd\mu-u\right)^{2}\\&\leq\frac{1}{u-m}\left\{\frac{\Phi\left(u\right)-\Phi\left(m\right)}{u-m}-\Phi'\left(m\right)\right\}\int_{\Omega}\left(f-u\right)^{2}d\mu\\&\leq\Phi\left(u\right)-\int_{\Omega}\Phi\circ fd\mu-u\int_{\Omega}\Phi'\circ fd\mu+\int_{\Omega}f\cdot\Phi'\circ fd\mu\\&\leq\frac{1}{M-u}\left\{\Phi'\left(M\right)-\frac{\Phi\left(M\right)-\Phi\left(u\right)}{M-u}\right\}\int_{\Omega}\left(f-u\right)^{2}d\mu\end{aligned}\tag{3.8}$$

for $u \in [m, M]$ *and, in particular,*

$$\begin{aligned}0&\leq\frac{1}{\int_{\Omega}fd\mu-m}\left\{\frac{\Phi\left(\int_{\Omega}fd\mu\right)-\Phi\left(m\right)}{\int_{\Omega}fd\mu-m}-\Phi'\left(m\right)\right\}\\&\times\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right]\end{aligned}\tag{3.9}$$

$$\leq \Phi\left(\int_\Omega f d\mu\right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$\leq \frac{1}{M - \int_\Omega f d\mu} \left\{ \Phi'(M) - \frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} \right\}$$
$$\times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2 \right].$$

Also

$$(3.10) \qquad 0 \leq \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\left(\int_\Omega f d\mu - \sigma\right)^2$$
$$\leq \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\int_\Omega (f-\sigma)^2 d\mu$$
$$\leq \Phi\left(\frac{\int_\Omega f \cdot (\Phi' \circ f) d\mu}{\int_\Omega \Phi' \circ f d\mu}\right) - \int_\Omega \Phi \circ f d\mu$$
$$\leq \frac{1}{M - \sigma}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma}\right\}\int_\Omega (f-\sigma)^2 d\mu.$$

If the second derivative Φ'' *is monotonic nondecreasing on an interval* $[m, M] \subset \mathring{I}$, *then we also have*

$$(3.11) \qquad 0 \leq \operatorname*{essinf}_{t\in\Omega} (f(t) - u)^2 \int_\Omega \frac{1}{f - m}\left\{\Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f - m}\right\} d\mu$$
$$\leq \Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$\leq \operatorname*{essup}_{t\in\Omega} (f(t) - u)^2 \int_\Omega \frac{1}{M - f}\left\{\frac{\Phi(M) - \Phi \circ f}{M - f} - \Phi' \circ f\right\} d\mu.$$

for $u \in [m, M]$ *and, in particular,*

$$(3.12) \qquad 0 \leq \operatorname*{essinf}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{f - m}\left\{\Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f - m}\right\} d\mu$$
$$\leq \Phi\left(\int_\Omega f d\mu\right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu$$
$$\leq \operatorname*{essup}_{t\in\Omega} \left(f(t) - \int_\Omega f d\mu\right)^2 \int_\Omega \frac{1}{M - f}\left\{\frac{\Phi(M) - \Phi \circ f}{M - f} - \Phi' \circ f\right\} d\mu.$$

Proof. First, observe that for $u,\, v\in[m,M]$ with $u\neq v$ we have

$$\begin{aligned}
(3.13)\qquad & \int_0^1 \Phi''\left((1-s)\,v+su\right)(1-s)\,ds \\
&= \frac{1}{u-v}\int_0^1 (1-s)\,d\left(\Phi'\left((1-s)\,v+su\right)\right) \\
&= \frac{1}{u-v}\left[(1-s)\,\Phi'\left((1-s)\,v+su\right)\Big|_0^1+\int_0^1 \Phi'\left((1-s)\,v+su\right)ds\right] \\
&= \frac{1}{u-v}\left\{-\Phi'(v)+\int_0^1 \Phi'\left((1-s)\,v+su\right)ds\right\} \\
&= \frac{1}{v-u}\left\{\Phi'(v)-\int_0^1 \Phi'\left((1-s)\,v+su\right)ds\right\} \\
&= \frac{1}{v-u}\left\{\Phi'(v)-\frac{\Phi(v)-\Phi(u)}{v-u}\right\}.
\end{aligned}$$

If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M]\subset\mathring{I}$, then for μ-a.e. $t\in\Omega$

$$\begin{aligned}
\int_0^1 \Phi''\left((1-s)\,m+su\right)(1-s)\,ds &\leq \int_0^1 \Phi''\left((1-s)\,f(t)+su\right)(1-s)\,ds \\
&\leq \int_0^1 \Phi''\left((1-s)\,M+su\right)(1-s)\,ds
\end{aligned}$$

and by (3.13) we get

$$\begin{aligned}
\frac{1}{u-m}\left\{\frac{\Phi(u)-\Phi(m)}{u-m}-\Phi'(m)\right\} &\leq \int_0^1 \Phi''\left((1-s)\,f(t)+su\right)(1-s)\,ds \\
&\leq \frac{1}{M-u}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi(u)}{M-u}\right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{u-m}\left\{\frac{\Phi(u)-\Phi(m)}{u-m}-\Phi'(m)\right\} \\
&\leq \operatorname*{essinf}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+su\right)(1-s)\,ds\right)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname*{essup}_{t\in\Omega}\left(\int_0^1 \Phi''\left((1-s)\,f(t)+su\right)(1-s)\,ds\right) \\
&\leq \frac{1}{M-u}\left\{\Phi'(M)-\frac{\Phi(M)-\Phi(u)}{M-u}\right\}
\end{aligned}$$

and by (2.5) we get (3.8).

If the second derivative Φ'' is monotonic nondecreasing on an interval $[m,M] \subset \mathring{I}$, then for μ-a.e. $t \in \Omega$

$$\int_0^1 \Phi''((1-s)f(t)+sm)(1-s)\,ds \le \int_0^1 \Phi''((1-s)f(t)+su)(1-s)\,ds \le \int_0^1 \Phi''((1-s)f(t)+sM)(1-s)\,ds.$$

By utilising (3.13) we get

$$\int_0^1 \Phi''((1-s)f(t)+sm)(1-s)\,ds = \frac{1}{f(t)-m}\left\{\Phi'(f(t)) - \frac{\Phi(f(t))-\Phi(m)}{f(t)-m}\right\}$$

and

$$\int_0^1 \Phi''((1-s)f(t)+sM)(1-s)\,ds = \frac{1}{M-f(t)}\left\{\frac{\Phi(M)-\Phi(f(t))}{M-f(t)} - \Phi'(f(t))\right\}$$

for μ-a.e. $t \in \Omega$.

Then

$$\int_\Omega \frac{1}{f-m}\left\{\Phi'\circ f - \frac{\Phi\circ f-\Phi(m)}{f-m}\right\}d\mu \le \int_\Omega\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu$$

and

$$\int_\Omega\left(\int_0^1 \Phi''((1-s)f+su)(1-s)\,ds\right)d\mu \le \int_\Omega \frac{1}{M-f}\left\{\frac{\Phi(M)-\Phi\circ f}{M-f} - \Phi'\circ f\right\}d\mu$$

and by (2.6) we get (3.11). □

Remark 3. *If the second derivative Φ'' is monotonic nonincreasing on an interval $[m,M] \subset \mathring{I}$, then we have*

$$\begin{aligned} (3.14)\qquad 0 &\le \frac{1}{M-u}\left\{\Phi'(M) - \frac{\Phi(M)-\Phi(u)}{M-u}\right\}\left(\int_\Omega f\,d\mu - u\right)^2 \\ &\le \frac{1}{M-u}\left\{\Phi'(M) - \frac{\Phi(M)-\Phi(u)}{M-u}\right\}\int_\Omega (f-u)^2\,d\mu \\ &\le \Phi(u) - \int_\Omega \Phi\circ f\,d\mu - u\int_\Omega \Phi'\circ f\,d\mu + \int_\Omega f\cdot\Phi'\circ f\,d\mu \\ &\le \frac{1}{u-m}\left\{\frac{\Phi(u)-\Phi(m)}{u-m} - \Phi'(m)\right\}\int_\Omega (f-u)^2\,d\mu \end{aligned}$$

for $u \in [m, M]$ and, in particular,

$$0 \leq \frac{1}{M - \int_\Omega f d\mu} \left\{ \Phi'(M) - \frac{\Phi(M) - \Phi\left(\int_\Omega f d\mu\right)}{M - \int_\Omega f d\mu} \right\} \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right] \leq \Phi\left(\int_\Omega f d\mu \right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu \leq \frac{1}{\int_\Omega f d\mu - m} \left\{ \frac{\Phi\left(\int_\Omega f d\mu\right) - \Phi(m)}{\int_\Omega f d\mu - m} - \Phi'(m) \right\} \times \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu \right)^2 \right]. \tag{3.15}$$

If in (3.14) we take $u = \sigma$, the Slater's point, then we get

$$0 \leq \frac{1}{M - \sigma} \left\{ \Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma} \right\} \left(\int_\Omega f d\mu - \sigma \right)^2 \leq \frac{1}{M - \sigma} \left\{ \Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma} \right\} \int_\Omega (f - \sigma)^2 d\mu \leq \Phi\left(\frac{\int_\Omega f \cdot (\Phi' \circ f) d\mu}{\int_\Omega \Phi' \circ f d\mu} \right) - \int_\Omega \Phi \circ f d\mu \leq \frac{1}{\sigma - m} \left\{ \frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m) \right\} \int_\Omega (f - \sigma)^2 d\mu. \tag{3.16}$$

If the second derivative Φ'' is monotonic nonincreasing on an interval $[m, M] \subset \mathring{I}$, then we also have

$$0 \leq \operatorname*{essinf}_{t \in \Omega} (f(t) - u)^2 \int_\Omega \frac{1}{M - f} \left\{ \frac{\Phi(M) - \Phi \circ f}{M - f} - \Phi' \circ f \right\} d\mu \leq \Phi(u) - \int_\Omega \Phi \circ f d\mu - u \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu \leq \operatorname*{essup}_{t \in \Omega} (f(t) - u)^2 d\mu \int_\Omega \frac{1}{f - m} \left\{ \Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f - m} \right\} d\mu \tag{3.17}$$

for $u \in [m, M]$ and, in particular,

$$0 \leq \operatorname*{essinf}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 \int_\Omega \frac{1}{M - f} \left\{ \frac{\Phi(M) - \Phi \circ f}{M - f} - \Phi' \circ f \right\} d\mu \leq \Phi\left(\int_\Omega f d\mu \right) - \int_\Omega \Phi \circ f d\mu - \int_\Omega f d\mu \int_\Omega \Phi' \circ f d\mu + \int_\Omega f \cdot \Phi' \circ f d\mu \leq \operatorname*{essup}_{t \in \Omega} \left(f(t) - \int_\Omega f d\mu \right)^2 d\mu \int_\Omega \frac{1}{f - m} \left\{ \Phi' \circ f - \frac{\Phi \circ f - \Phi(m)}{f - m} \right\} d\mu. \tag{3.18}$$

Theorem 8. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$ and $f : \Omega \to I$ so that f, f^2, $\Phi \circ f$, $\Phi'' \circ f \in L(\Omega, \mu)$. If the second derivative Φ'' is* *convex, then*

$$0 \le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\Phi''\left(\int_\Omega fd\mu\right) \tag{3.19}$$
$$\le \Phi\left(\int_\Omega fd\mu\right)-\int_\Omega \Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega \Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu$$
$$\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\int_\Omega \Phi''\circ fd\mu.$$

If the second derivative Φ'' *is concave, then*

$$0 \le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\int_\Omega \Phi''\circ fd\mu \tag{3.20}$$
$$\le \Phi\left(\int_\Omega fd\mu\right)-\int_\Omega \Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega \Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu$$
$$\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2\Phi''\left(\int_\Omega fd\mu\right).$$

Proof. From (2.10) and Fubini's theorem, we have

$$0 \le \operatorname*{essinf}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2 \tag{3.21}$$
$$\times\int_0^1\left(\int_\Omega \Phi''\left((1-s)f+s\int_\Omega fd\mu\right)d\mu\right)(1-s)\,ds$$
$$\le \Phi\left(\int_\Omega fd\mu\right)-\int_\Omega \Phi\circ fd\mu-\int_\Omega fd\mu\int_\Omega \Phi'\circ fd\mu+\int_\Omega f\cdot\Phi'\circ fd\mu$$
$$\le \operatorname*{essup}_{t\in\Omega}\left(f(t)-\int_\Omega fd\mu\right)^2$$
$$\times\int_0^1\left(\int_\Omega \Phi''\left((1-s)f+s\int_\Omega fd\mu\right)d\mu\right)(1-s)\,ds.$$

By Jensen's integral inequality, we have

$$\int_\Omega \Phi''\left((1-s)f+s\int_\Omega fd\mu\right)d\mu \ge \Phi''\left[\int_\Omega\left((1-s)f+s\int_\Omega fd\mu\right)d\mu\right]$$
$$=\Phi''\left[(1-s)\int_\Omega fd\mu+s\int_\Omega fd\mu\right]$$
$$=\Phi''\left(\int_\Omega fd\mu\right)$$

and

$$\begin{aligned}\int_{\Omega}\Phi''\left((1-s)f+s\int_{\Omega}fd\mu\right)d\mu &\le \int_{\Omega}\left[(1-s)\Phi''(f)+s\Phi''\left(\int_{\Omega}fd\mu\right)\right]d\mu\\ &=(1-s)\int_{\Omega}\Phi''\circ fd\mu+s\Phi''\left(\int_{\Omega}fd\mu\right)\\ &\le(1-s)\int_{\Omega}\Phi''\circ fd\mu+s\int_{\Omega}\Phi''\circ fd\mu\\ &=\int_{\Omega}\Phi''\circ fd\mu\end{aligned}$$

and by (3.21) we get the desired result (3.19). □

4. Some Examples for the Discrete Case

Let $\Phi:[m,M]\to\mathbb{R}$ be a twice differentiable convex function on (m,M) and $x_k\in[m,M]$, $p_k\ge 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$. If there exists the constants $0<\gamma<\Gamma<\infty$ such that $\gamma\le\Phi''(x)\le\Gamma$ for a.e. $x\in[m,M]$, then for all $v\in I$, we have by (3.1), (3.2) and (3.3) for the discrete measure

$$\begin{aligned}0 &\le \frac{1}{2}\gamma\left(\sum_{k=1}^{n}p_kx_k-u\right)^2\le\frac{1}{2}\gamma\sum_{k=1}^{n}p_k(x_k-u)^2\\ &\le\Phi(u)-\sum_{k=1}^{n}p_k\Phi(x_k)-u\sum_{k=1}^{n}p_k\Phi'(x_k)+\sum_{k=1}^{n}p_kx_k\Phi'(x_k)\\ &\le\frac{1}{2}\Gamma\sum_{k=1}^{n}p_k(x_k-u)^2\end{aligned}\tag{4.1}$$

and, in particular,

$$\begin{aligned}0 &\le \frac{1}{2}\gamma\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\\ &\le\Phi\left(\sum_{k=1}^{n}p_kx_k\right)-\sum_{k=1}^{n}p_k\Phi(x_k)\\ &\quad-\sum_{k=1}^{n}p_kx_k\sum_{k=1}^{n}p_k\Phi'(x_k)+\sum_{k=1}^{n}p_kx_k\Phi'(x_k)\\ &\le\frac{1}{2}\Gamma\left[\sum_{k=1}^{n}p_kx_k^2-\left(\sum_{i=1}^{n}p_ix_i\right)^2\right]\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}0 &\le \frac{1}{2}\gamma\left(\sum_{k=1}^{n}p_kx_k-\sigma\right)^2\le\frac{1}{2}\gamma\sum_{k=1}^{n}p_k(x_k-\sigma)^2\\ &\le\Phi\left(\frac{\sum_{k=1}^{n}p_kx_k\Phi'(x_k)}{\sum_{k=1}^{n}p_k\Phi'(x_k)}\right)-\sum_{k=1}^{n}p_k\Phi(x_k)\le\frac{1}{2}\Gamma\sum_{k=1}^{n}p_k(x_k-\sigma)^2,\end{aligned}\tag{4.3}$$

where σ is the discrete Slater's point defined by

$$\sigma := \frac{\sum_{k=1}^{n} p_k x_k \Phi'(x_k)}{\sum_{k=1}^{n} p_k \Phi'(x_k)},$$

where $\sum_{k=1}^{n} p_k \Phi'(x_k) \neq 0$.

From (3.4) and (3.5) we also have

$$\begin{aligned} (4.4)\quad 0 &\leq \frac{1}{2}\gamma\left(\sum_{k=1}^{n} p_k x_k - \frac{m+M}{2}\right)^2 \leq \frac{1}{2}\gamma\sum_{k=1}^{n} p_k\left(x_k - \frac{m+M}{2}\right)^2 \\ &\leq \Phi\left(\frac{m+M}{2}\right) - \sum_{k=1}^{n} p_k \Phi(x_k) - \frac{m+M}{2}\sum_{k=1}^{n} p_k \Phi'(x_k) + \sum_{k=1}^{n} p_k x_k \Phi'(x_k) \\ &\leq \frac{1}{2}\Gamma\sum_{k=1}^{n} p_k\left(x_k - \frac{m+M}{2}\right)^2 \leq \frac{1}{8}\Gamma(M-m)^2 \end{aligned}$$

and

$$\begin{aligned} (4.5)\quad 0 &\leq \frac{1}{2}\gamma\left[\sum_{k=1}^{n} p_k\left(x_k - \frac{m+M}{2}\right)^2 + \frac{1}{12}(M-m)^2\right] \\ &\leq \frac{1}{M-m}\int_m^M \Phi(u)\,du - \sum_{k=1}^{n} p_k \Phi(x_k) \\ &\quad - \frac{m+M}{2}\sum_{k=1}^{n} p_k \Phi'(x_k) + \sum_{k=1}^{n} p_k x_k \Phi'(x_k) \\ &\leq \frac{1}{2}\Gamma\left[\sum_{k=1}^{n} p_k\left(x_k - \frac{m+M}{2}\right)^2 + \frac{1}{12}(M-m)^2\right]. \end{aligned}$$

From the inequality (3.7) we also get

$$\begin{aligned} (4.6)\quad 0 &\leq \frac{1}{24}\gamma(M-m)^2 \leq \frac{1}{M-m}\int_m^M \Phi(u)\,du - \sum_{k=1}^{n} p_k \Phi(x_k) \\ &\quad - \frac{m+M}{2}\sum_{k=1}^{n} p_k \Phi'(x_k) + \sum_{k=1}^{n} p_k x_k \Phi'(x_k) \\ &\leq \frac{1}{6}\Gamma(M-m)^2. \end{aligned}$$

If the second derivative Φ'' is monotonic nondecreasing on an interval $[m, M] \subset \mathring{I}$, then we have by (3.9)

$$\begin{aligned} (4.7)\quad 0 &\leq \frac{1}{\sum_{k=1}^{n} p_k x_k - m}\left\{\frac{\Phi\left(\sum_{k=1}^{n} p_k x_k\right) - \Phi(m)}{\sum_{k=1}^{n} p_k x_k - m} - \Phi'(m)\right\} \\ &\quad \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \end{aligned}$$

$$\leq \Phi\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} p_k \Phi(x_k) - \sum_{k=1}^{n} p_k x_k \sum_{k=1}^{n} p_k \Phi'(x_k) + \sum_{k=1}^{n} p_k x_k \Phi'(x_k)$$

$$\leq \frac{1}{M - \sum_{k=1}^{n} p_k x_k}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi\left(\sum_{k=1}^{n} p_k x_k\right)}{M - \sum_{k=1}^{n} p_k x_k}\right\} \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right],$$

and

$$0 \leq \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\left(\sum_{k=1}^{n} p_k x_k - \sigma\right)^2 \tag{4.8}$$

$$\leq \frac{1}{\sigma - m}\left\{\frac{\Phi(\sigma) - \Phi(m)}{\sigma - m} - \Phi'(m)\right\}\sum_{k=1}^{n} p_k (x_k - \sigma)^2$$

$$\leq \Phi\left(\frac{\sum_{k=1}^{n} p_k x_k \Phi'(x_k)}{\sum_{k=1}^{n} p_k \Phi'(x_k)}\right) - \sum_{k=1}^{n} p_k \Phi(x_k)$$

$$\leq \frac{1}{M - \sigma}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi(\sigma)}{M - \sigma}\right\}\sum_{k=1}^{n} p_k (x_k - \sigma)^2,$$

while in the case of Φ'' is monotonic nonincreasing on an interval $[m, M] \subset \mathring{I}$, then we have by (3.15)

$$0 \leq \frac{1}{M - \sum_{k=1}^{n} p_k x_k}\left\{\Phi'(M) - \frac{\Phi(M) - \Phi\left(\sum_{k=1}^{n} p_k x_k\right)}{M - \sum_{k=1}^{n} p_k x_k}\right\} \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right] \tag{4.9}$$

$$\leq \Phi\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} p_k \Phi(x_k)$$

$$- \sum_{k=1}^{n} p_k x_k \sum_{k=1}^{n} p_k \Phi'(x_k) + \sum_{k=1}^{n} p_k x_k \Phi'(x_k)$$

$$\leq \frac{1}{\sum_{k=1}^{n} p_k x_k - m}\left\{\frac{\Phi\left(\sum_{k=1}^{n} p_k x_k\right) - \Phi(m)}{\sum_{k=1}^{n} p_k x_k - m} - \Phi'(m)\right\} \times \left[\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right].$$

If the second derivative Φ'' is convex, then

$$
\begin{aligned}
0 &\leq \min_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\Phi''\left(\sum_{k=1}^{n}p_kx_k\right) \\
&\leq \Phi\left(\sum_{k=1}^{n}p_kx_k\right)-\sum_{k=1}^{n}p_k\Phi\left(x_k\right) \\
&-\sum_{k=1}^{n}p_kx_k\sum_{k=1}^{n}p_k\Phi'\left(x_k\right)+\sum_{k=1}^{n}p_kx_k\Phi'\left(x_k\right) \\
&\leq \max_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\sum_{k=1}^{n}p_k\Phi''\left(x_k\right).
\end{aligned} \tag{4.10}
$$

If the second derivative Φ'' is concave, then

$$
\begin{aligned}
0 &\leq \min_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\sum_{k=1}^{n}p_k\Phi''\left(x_k\right) \\
&\leq \Phi\left(\sum_{k=1}^{n}p_kx_k\right)-\sum_{k=1}^{n}p_k\Phi\left(x_k\right) \\
&-\sum_{k=1}^{n}p_kx_k\sum_{k=1}^{n}p_k\Phi'\left(x_k\right)+\sum_{k=1}^{n}p_kx_k\Phi'\left(x_k\right) \\
&\leq \max_{k\in\{1,\dots,n\}}\left(x_k-\sum_{i=1}^{n}p_ix_i\right)^2\Phi''\left(\sum_{k=1}^{n}p_kx_k\right).
\end{aligned} \tag{4.11}
$$

We consider the exponential function $\Phi(x)=\exp x$ and $x_k\in[m,M]$, $p_k\geq 0$ for $k\in\{1,\dots,n\}$ with $\sum_{k=1}^{n}p_k=1$. For this function the Slater's point is

$$
\epsilon:=\frac{\sum_{k=1}^{n}p_kx_k\exp\left(x_k\right)}{\sum_{k=1}^{n}p_k\exp\left(x_k\right)},
$$

and by (4.3)

$$
\begin{aligned}
0 &\leq \frac{1}{2}\left(\sum_{k=1}^{n}p_kx_k-\frac{\sum_{k=1}^{n}p_kx_k\exp\left(x_k\right)}{\sum_{k=1}^{n}p_k\exp\left(x_k\right)}\right)^2\exp m \\
&\leq \frac{1}{2}\sum_{k=1}^{n}p_k\left(x_k-\frac{\sum_{i=1}^{n}p_ix_i\exp\left(x_i\right)}{\sum_{i=1}^{n}p_i\exp\left(x_i\right)}\right)^2\exp m \\
&\leq \exp\left(\frac{\sum_{k=1}^{n}p_kx_k\exp\left(x_k\right)}{\sum_{k=1}^{n}p_k\exp\left(x_k\right)}\right)-\sum_{k=1}^{n}p_k\exp\left(x_k\right) \\
&\leq \frac{1}{2}\exp M\sum_{k=1}^{n}p_k\left(x_k-\frac{\sum_{i=1}^{n}p_ix_i\exp\left(x_i\right)}{\sum_{i=1}^{n}p_i\exp\left(x_i\right)}\right)^2
\end{aligned} \tag{4.12}
$$

while by (4.10) we get

$$0 \le \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \exp\left(\sum_{k=1}^{n} p_k x_k \right) \tag{4.13}$$
$$\le \exp\left(\sum_{k=1}^{n} p_k x_k \right) - \sum_{k=1}^{n} p_k \exp(x_k) \left(1 - x_k + \sum_{i=1}^{n} p_i x_i \right)$$
$$\le \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{k=1}^{n} p_k \exp(x_k).$$

If we consider the convex function $\Phi(x) = -\ln x$, and $x_k \in [m, M]$, $p_k \ge 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then the Slater's point is

$$\ell = \frac{1}{\sum_{k=1}^{n} \frac{p_k}{x_k}}$$

and by (4.3) we get

$$0 \le \frac{1}{2M^2} \left(\sum_{k=1}^{n} p_k x_k - \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}} \right)^2 \le \frac{1}{2M^2} \sum_{k=1}^{n} p_k \left(x_k - \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}} \right)^2 \tag{4.14}$$
$$\le \sum_{k=1}^{n} p_k \ln(x_k) - \ln\left(\frac{1}{\sum_{k=1}^{n} \frac{p_k}{x_k}} \right) \le \frac{1}{2m^2} \sum_{k=1}^{n} p_k \left(x_k - \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}} \right)^2.$$

From (4.10) we also have

$$0 \le \min_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \left(\sum_{k=1}^{n} p_k x_k \right)^{-2} \tag{4.15}$$
$$\le \sum_{k=1}^{n} p_k \ln(x_k) - \ln\left(\sum_{k=1}^{n} p_k x_k \right) + \sum_{k=1}^{n} p_k x_k \sum_{k=1}^{n} \frac{p_k}{x_k} - 1$$
$$\le \max_{k\in\{1,...,n\}} \left(x_k - \sum_{i=1}^{n} p_i x_i \right)^2 \sum_{k=1}^{n} p_k x_k^{-2}.$$

References

[1] S. Abramovich, Jensen, Hölder, Minkowski, Jensen-Steffensen and Slater-Pečarić inequalities derived through N-quasiconvexity. *Math. Inequal. Appl.* **19** (2016), no. 4, 1203–1226.

[2] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), No. 3, 551–562. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suplement, Art. 12. [Online http://rgmia.org/papers/v5e/GTIILFApp.pdf].

[3] S. S. Dragomir, *Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type*, Nova Science Publishers, NY, 2004.

[4] S. S. Dragomir, Some Slater type inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Rev. Un. Mat. Argentina* **52** (2011), no. 1, 109–120.

[5] S. S. Dragomir, Some Slater's type inequalities for convex functions defined on linear spaces and applications. *Abstr. Appl. Anal.* **2012**, Art. ID 168405, 16 pp.

[6] S. S. Dragomir, Some Slater's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Toyama Math. J.* **38** (2016), 75–99.

[7] S. S. Dragomir, Some upper bounds for the perturbed Slater's gap of convex functions, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 8, 17 pp. [Online http://rgmia.org/papers/v23/v23a08.pdf].

[8] S. S. Dragomir and J. E. Pečarić, A generalization of Slater's inequality. *Zb. Rad.* (Kragujevac) No. **13** (1992), 5–9.

[9] A. M. Khan and J. Pečarić, On Slater's integral inequality. *J. Math. Inequal.* **5** (2011), no. 2, 231–241.

[10] J. Pečarić, A multidimensional generalization of Slater's inequality. *J. Approx. Theory* **44** (1985), no. 3, 292–294.

[11] M. S. Slater, A companion inequality to Jensen's inequality, *J. Approx. Theory*, **32**(1981), 160-166.

[12] G.-S. Yang, and K.-L. Tseng, Some refinements of Slater, Pečarić and Wang-Wang's inequalities. *Math. Inequal. Appl.* **6** (2003), no. 3, 437–443.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.