UPPER AND LOWER BOUNDS FOR THE OPERATOR CONVEXITY DIFFERENCE IN TERMS OF THE SECOND DERIVATIVE

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Abstract. In this paper we show among others that, if $f:[0,\infty)\to\mathbb{R}$ is operator convex on $[0,\infty)$ with $f'_{+}(0)\in\mathbb{R}$ and $A,\ B>0$, $0\le k\le(1-t)A+tB\le K$ for $t\in[0,1]$ and $m^2\le(B-A)^2\le M^2$ for some constants k, K, m, M, then

$$0\le m^2f''(K)\,t(1-t)\le(1-t)f(A)+tf(B)-f((1-t)A+tB)$$
$$\le M^2f''(k)\,t(1-t).$$

In particular, if $0<k\le A,\ B\le K$, then we have the following upper bounds

$$0\le(1-t)A^p+tB^p-((1-t)A+tB)^p\le(K-k)^2p(p-1)k^{p-2}t(1-t),$$

for $p\in[1,2]$,

$$0\le\ln((1-t)A+tB)-(1-t)\ln(A)-t\ln(B)\le\frac{(K-k)^2}{k^2}t(1-t)$$

and

$$0\le(1-t)A\ln A+tB\ln B-((1-t)A+tB)\ln((1-t)A+tB)$$
$$\le\frac{(K-k)^2}{k}t(1-t)$$

for $t\in[0,1]$.

1. Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\ge0$) if $\langle Tx,x\rangle\ge0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. By $A\ge B$ we understand that $A-B\ge0$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\le(\ge)(1-\lambda)f(A)+\lambda f(B)\qquad\text{(OC)}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\ge f(B)$ holds for any $A\ge B>0$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [5] and the references therein.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Operator convex functions, Jensen's difference, Operator power function. Operator logarithmic function.

RGMIA Res. Rep. Coll. 23 (2020), Art. 140, 16 pp. Received 04/12/20

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\le r\le 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t)=f(0)+f'_+(0)\,t+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda)\,, \tag{1.1}$$

where $c\ge 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty \tag{1.2}$$

holds.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.3}$$

Let f be an operator convex function on I. For A, $B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{SA}_{\varphi(I)}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB)\,. \tag{1.4}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle \varphi_{(A,B)}(t)\,x,x\right\rangle=\left\langle f((1-t)A+tB)\,x,x\right\rangle. \tag{1.5}$$

We have the following basic fact, see for instance [4]:

Lemma 1. *Let f be an operator convex function on I. For any A, $B\in\mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any A, $B\in\mathcal{SA}_I(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

A continuous function $g:\mathcal{SA}_I(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$ along the direction $B\in\mathcal{SA}(H)$, the class of all selfadjoint operators on H, if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B):=\lim_{s\to 0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H)\,. \tag{1.6}$$

If the limit (1.6) exists for all $B\in\mathcal{SA}(H)$, then we say that f is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators
$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$
We observe that $A, B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

The following result also holds, see for instance [4]:

Lemma 2. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A),\ t \in (0,1). \tag{1.7}$$

Also we have for the lateral derivative that

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{1.8}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{1.9}$$

In the recent paper [3], see also [4] for the weighted version, we obtained the following reverses of Hermite-Hadamard inequalities:

Theorem 2. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right] \end{aligned}$$

and

Theorem 3. *Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\begin{aligned} 0 &\leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{1}{8}\left[\nabla f_B(B-A) - \nabla f_A(B-A)\right]. \end{aligned}$$

For recent papers on operator Hermite-Hadamard inequalities, see [6]-[9] and [11]-[15].

In this paper we show among others that, if $f : [0,\infty) \to \mathbb{R}$ is operator convex on $[0,\infty)$ with $f'_+(0) \in \mathbb{R}$ and $A, B > 0$, $0 \leq k \leq (1-t)A + tB \leq K$ for $t \in [0,1]$ and $m^2 \leq (B-A)^2 \leq M^2$ for some constants k, K, m, M, then

$$\begin{aligned} 0 &\leq m^2 f''(K)\, t(1-t) \leq (1-t) f(A) + t f(B) - f((1-t)A + tB) \\ &\leq M^2 f''(k)\, t(1-t). \end{aligned}$$

In particular, if $0 < k \leq A, B \leq K$, then we have the following upper bounds

$$0 \leq (1-t)A^p + tB^p - ((1-t)A + tB)^p \leq (K-k)^2 p(p-1) k^{p-2} t(1-t),$$

for $p \in [1,2]$,

$$0 \leq \ln((1-t)A + tB) - (1-t)\ln(A) - t\ln(B) \leq \frac{(K-k)^2}{k^2} t(1-t)$$

and

$$0 \le (1-t) A \ln A + tB \ln B - ((1-t) A + tB) \ln ((1-t) A + tB) \le \frac{(K-k)^2}{k} t (1-t)$$

for $t \in [0,1]$.

2. General Results

We have the following result:

Lemma 3. *Let $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \le \varphi''(t) \le D \text{ for any } t \in \mathring{I}, \tag{2.1}$$

then

$$\frac{1}{2}\nu (1-\nu) d (b-a)^2 \le (1-\nu)\varphi(a) + \nu\varphi(b) - \varphi((1-\nu) a + \nu b) \le \frac{1}{2}\nu(1-\nu) D (b-a)^2 \tag{2.2}$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \le \frac{\varphi(a) + \varphi(b)}{2} - \varphi\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)^2 D, \tag{2.3}$$

for any a, $b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.3).

Proof. We consider the auxiliary function $\varphi_D : I \subset \mathbb{R} \to \mathbb{R}$ defined by $\varphi_D(x) = \frac{1}{2}Dx^2 - \varphi(x)$. The function φ_D is differentiable on $\mathring{I}$ and $\varphi_D''(x) = D - \varphi''(x) \ge 0$, showing that φ_D is a convex function on $\mathring{I}$.

By the convexity of φ_D we have for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$ that

$$\begin{aligned}
0 &\le (1-\nu)\varphi_D(a) + \nu\varphi_D(b) - \varphi_D((1-\nu) a + \nu b) \\
&= (1-\nu)\left(\frac{1}{2}Da^2 - \varphi(a)\right) + \nu\left(\frac{1}{2}Db^2 - \varphi(b)\right) \\
&- \left(\frac{1}{2}D((1-\nu) a + \nu b)^2 - \varphi_D((1-\nu) a + \nu b)\right) \\
&= \frac{1}{2}D\left[(1-\nu) a^2 + \nu b^2 - ((1-\nu) a + \nu b)^2\right] \\
&- (1-\nu)\varphi(a) - \nu\varphi(b) + \varphi_D((1-\nu) a + \nu b) \\
&= \frac{1}{2}\nu(1-\nu) D (b-a)^2 - (1-\nu)\varphi(a) - \nu\varphi(b) + \varphi_D((1-\nu) a + \nu b),
\end{aligned}$$

which implies the second inequality in (2.2).

The first inequality follows in a similar way by considering the auxiliary function $\varphi_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $\varphi_d(x) = \varphi(x) - \frac{1}{2}dx^2$ that is twice differentiable and convex on $\mathring{I}$.

If we take $\varphi(x) = x^2$, then (2.1) holds with equality for $d = D = 2$ and (2.3) reduces to an equality as well. □

Corollary 1. *Let $\varphi:[0,1]\rightarrow\mathbb{R}$ be a twice differentiable function on the interval $(0,1)$. If there exists the constants d, D such that*

$$d\leq\varphi''(t)\leq D \text{ for any } t\in(0,1), \tag{2.4}$$

then

$$\frac{1}{2}\nu(1-\nu)d\leq(1-\nu)\varphi(0)+\nu\varphi(1)-\varphi(\nu)\leq\frac{1}{2}\nu(1-\nu)D \tag{2.5}$$

for any $\nu\in[0,1]$.

In particular, we have

$$\frac{1}{8}d\leq\frac{\varphi(0)+\varphi(1)}{2}-\varphi\left(\frac{1}{2}\right)\leq\frac{1}{8}D. \tag{2.6}$$

We have the following representations of the derivatives:

Lemma 4. *Let $A,\ B\in\mathcal{SA}_I(H)$, with $A\neq B$. Assume that the operator function generated by f is twice Fréchet differentiable on the segment $[A,B]$ then we have that $f_{A,B}$ is twice differentiable on $[0,1]$,*

$$f'_{A,B}(t)=D(f)((1-t)A+tB)(B-A) \tag{2.7}$$

and

$$f''_{A,B}(t)=D^2(f)((1-t)A+tB)(B-A,B-A) \tag{2.8}$$

for $t\in(0,1)$.

We also have the lateral derivatives

$$f'_{A,B}(0+)=D(f)(A)(B-A),\ f'_{A,B}(1-)=D(f)(B)(B-A), \tag{2.9}$$

$$f''_{A,B}(0+)=D^2(f)(A)(B-A,B-A), \tag{2.10}$$

and

$$f''_{A,B}(1-)=D^2(f)(B)(B-A,B-A).$$

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq0$,

$$\begin{aligned}&\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h\rightarrow0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt}&=\lim_{h\rightarrow0}\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\lim_{h\rightarrow0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\&=D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (2.7).

Similarly,

$$\begin{aligned}
&\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right] \\
&=\frac{D(f)((1-(t+h))A+(t+h)B)(B-A)-D(f)((1-t)A+tB)(B-A)}{h} \\
&=\frac{D(f)((1-t)A+tB+h(B-A))(B-A)-D(f)((1-t)A+tB)(B-A)}{h}
\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}
\frac{d^2 f_{A,B}(t)}{dt^2}&=\lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{A,B}(t+h)}{dt}-\frac{df_{A,B}(t)}{dt}\right]\right\} \\
&=D^2(f)((1-t)A+tB)(B-A,B-A),
\end{aligned}$$

which proves (2.8).

The lateral derivatives follow in a similar way. □

Theorem 4. *Let A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$ and f an operator convex function on I. Assume that the operator function generated by f is twice Fréchet differentiable on the segment $[A,B]$ and there exists the constants $K>k\geq 0$ such that*

$$k(B-A)^2\leq D^2(f)((1-s)A+sB)(B-A,B-A)\leq K(B-A)^2 \tag{2.11}$$

for all $s\in(0,1)$.

Then for all $t\in[0,1]$ we have

$$\begin{aligned}
\frac{1}{2}kt(1-t)(B-A)^2&\leq(1-t)f(A)+tf(B)-f((1-t)A+tB) \\
&\leq\frac{1}{2}Kt(1-t)(B-A)^2.
\end{aligned}\tag{2.12}$$

Proof. For $x\in H$ we can consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle.$$

Then

$$\varphi'_{(A,B);x}(t)=\langle D(f)((1-t)A+tB)(B-A)x,x\rangle$$

and

$$\varphi''_{(A,B);x}(t)=\left\langle D^2(f)((1-t)A+tB)(B-A,B-A)x,x\right\rangle$$

for $t\in(0,1)$.

From the condition (2.11) we get

$$k\left\langle(B-A)^2x,x\right\rangle\leq\varphi''_{(A,B);x}(t)\leq K\left\langle(B-A)^2x,x\right\rangle,\ t\in(0,1).$$

By using inequality (2.5) we derive

$$\begin{aligned}
&\frac{1}{2}t(1-t)k\left\langle(B-A)^2x,x\right\rangle \\
&\leq(1-t)\varphi_{(A,B);x}(0)+t\varphi_{(A,B);x}(1)-\varphi_{(A,B);x}(t) \\
&\leq\frac{1}{2}t(1-t)K\left\langle(B-A)^2x,x\right\rangle
\end{aligned}\tag{2.13}$$

for $t\in(0,1)$.

From (2.13) we deduce that

$$\begin{aligned}&\frac{1}{2}t(1-t)k\left\langle (B-A)^2 x,x\right\rangle\\&\le (1-t)\langle f(A)x,x\rangle + t\langle f(B)x,x\rangle - \langle f((1-t)A+tB)x,x\rangle\\&\le \frac{1}{2}t(1-t)K\left\langle (B-A)^2 x,x\right\rangle\end{aligned}$$

for $t\in(0,1)$ and $x\in H$, which is equivalent to the operator inequality (2.12). □

3. Some Inequalities for Positive Operators

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping:

$$\begin{aligned}\mathcal{C}(w,\mu)(t) &:= \int_0^\infty w(\lambda)t^2(t+\lambda)^{-1}d\mu(\lambda)\\&= \int_0^\infty w(\lambda)(t+\lambda-\lambda)^2(t+\lambda)^{-1}d\mu(\lambda)\\&= \int_0^\infty w(\lambda)\left[(t+\lambda)^2-2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}d\mu(\lambda)\\&= \int_0^\infty w(\lambda)\left[(t+\lambda)-2\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda)\\&= \int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda),\end{aligned} \tag{3.1}$$

for $t>0$ we have.

We have the following representation of the Fréchet derivative $D(\mathcal{C}(w,\mu))$ as a function of positive selfadjoint operators:

Lemma 5. *For all $U>0$,*

$$\begin{aligned}&D(\mathcal{C}(w,\mu))(U)(V)\\&= \int_0^\infty w(\lambda)\left\{V-\lambda^2(U+\lambda)^{-1}V(U+\lambda)^{-1}\right\}d\mu(\lambda)\\&= \int_0^\infty w(\lambda)(U+\lambda)^{-1}[UVU+\lambda(VU+UV)](U+\lambda)^{-1}d\mu(\lambda)\end{aligned} \tag{3.2}$$

for all $V\in\mathcal{S}A(H)$, the class of all selfadjoint operators on H.

Proof. Let $U>0$ and $V\in\mathcal{S}\mathrm{A}(H)$. By the definition of $\mathcal{C}(w,\mu)$ and by (3.1) and we have for t in a small open interval around 0 that

$$\mathcal{C}(w,\mu)(U+tV) = \int_0^\infty w(\lambda)\left[U+tV-\lambda+\lambda^2(U+tV+\lambda)^{-1}\right]d\mu(\lambda).$$

Then

$$\begin{aligned}
&\mathcal{C}(w,\mu)(U+tV)-\mathcal{C}(w,\mu)(U)\\
&=\int_0^\infty w(\lambda)\left[U+tV-\lambda+\lambda^2(U+tV+\lambda)^{-1}\right]d\mu(\lambda)\\
&-\int_0^\infty w(\lambda)\left[U-\lambda+\lambda^2(U+\lambda)^{-1}\right]d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(U+tV+\lambda)^{-1}-(U+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\left\{tV+\lambda^2\left[(U+tV+\lambda)^{-1}(-tV)(U+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\
&=t\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(U+tV+\lambda)^{-1}V(U+\lambda)^{-1}\right]\right\}d\mu(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&D(\mathcal{C}(w,\mu))(U)(V)\\
&=\lim_{t\to0}\frac{\mathcal{C}(w,\mu)(U+tV)-\mathcal{C}(w,\mu)(U)}{t}\\
&=\lim_{t\to0}\int_0^\infty w(\lambda)\left\{V-\lambda^2\left[(U+tV+\lambda)^{-1}V(U+\lambda)^{-1}\right]\right\}d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)\left\{V-\lambda^2(U+\lambda)^{-1}V(U+\lambda)^{-1}\right\}d\mu(\lambda)
\end{aligned}\tag{3.3}$$

for $U>0$ and $V\in\mathcal{S}\mathrm{A}(H)$, which proves the first identity in (3.2).

Define for $\lambda\geq0$,

$$U_\lambda:=V-\lambda^2(U+\lambda)^{-1}V(U+\lambda)^{-1}.$$

If we multiply U_λ both sides by $U+\lambda$, then we get

$$\begin{aligned}
(U+\lambda)U_\lambda(U+\lambda)&=(U+\lambda)V(U+\lambda)-\lambda^2V\\
&=(UV+\lambda V)(U+\lambda)-\lambda^2V\\
&=UVU+\lambda VU+\lambda UV+\lambda^2V-\lambda^2V\\
&=UVU+\lambda(VU+UV).
\end{aligned}$$

If we multiply both sides by $(U+\lambda)^{-1}$ we get

$$U_\lambda=(U+\lambda)^{-1}[UVU+\lambda(VU+UV)](U+\lambda)^{-1},$$

which, by (3.3), implies the second representation in (3.1). □

Corollary 2. *Assume that $f : [0, \infty) \to \mathbb{R}$ is operator convex on $[0, \infty)$ and has the representation (1.1), then*

$$\begin{aligned} &D(f)(U)(V) \\ &= f'_+(0)V + c(UV + VU) \\ &+ \int_0^\infty w(\lambda)\left\{V - \lambda^2 (U+\lambda)^{-1} V (U+\lambda)^{-1}\right\} d\mu(\lambda) \\ &= f'_+(0)V + c(UV + VU) \\ &+ \int_0^\infty w(\lambda)(U+\lambda)^{-1}\left[UVU + \lambda(VU + UV)\right](U+\lambda)^{-1} d\mu(\lambda) \end{aligned} \tag{3.4}$$

for all $U > 0$ and $V \in \mathcal{S}A(H)$.

Proof. For $\ell(t) = t$, we have by (1.1) that

$$\begin{aligned} &D(f)(U)(V) \\ &= D\left[f(0) + f'_+(0)\ell + c\ell^2 + \mathcal{C}(w,\mu)\right](U)(V) \\ &= D[f(0)](U)(V) + f'_+(0)D(\ell)(U)(V) + cD\left(\ell^2\right)(U)(V) \\ &+ D[\mathcal{C}(w,\mu)](U)(V) \\ &= f'_+(0)V + c(UV + VU) + D[\mathcal{C}(w,\mu)](U)(V) \end{aligned}$$

and, by Lemma 5, the identity (3.5) is obtained. □

For the case of second Fréchet derivative $D^2(\mathcal{C}(w,\mu))$, we have the representation:

Lemma 6. *For all $U > 0$,*

$$\begin{aligned} &D^2(\mathcal{C}(w,\mu))(U)(V,V) \\ &= 2\int_0^\infty \lambda^2 w(\lambda)(\lambda+U)^{-1} V (\lambda+U)^{-1} V (\lambda+U)^{-1} d\mu(\lambda) \end{aligned} \tag{3.5}$$

for all $V \in \mathcal{S}A(H)$.

Proof. Let $U > 0$ and $V \in \mathcal{S}\mathrm{A}(H)$. We have by the properties of the Fréchet second derivative that

$$\begin{aligned} &D^2(\mathcal{C}(w,\mu))(U)(V,V) \\ &= \lim_{t\to 0} \frac{D(\mathcal{C}(w,\mu))(U+tV)(V) - D(\mathcal{C}(w,\mu))(U)(V)}{t}. \end{aligned} \tag{3.6}$$

Observe, by (3.2), that we have for t in a small open interval around 0,

$$\begin{aligned} &D(\mathcal{C}(w,\mu))(U+tV)(V) \\ &= \int_0^\infty w(\lambda)\left\{V - \lambda^2(U+tV+\lambda)^{-1} V (U+tV+\lambda)^{-1}\right\} d\mu(\lambda). \end{aligned}$$

Therefore

$$
\begin{aligned}
&D(\mathcal{C}(w,\mu))(U+tV)(V)-D(\mathcal{C}(w,\mu))(U)(V) \quad (3.7)\\
&=\int_0^\infty w(\lambda)\left\{V-\lambda^2(U+tV+\lambda)^{-1}V(U+tV+\lambda)^{-1}\right\}d\mu(\lambda)\\
&-\int_0^\infty w(\lambda)\left\{V-\lambda^2(U+\lambda)^{-1}V(U+\lambda)^{-1}\right\}d\mu(\lambda)\\
&=\int_0^\infty \lambda^2 w(\lambda)\\
&\times\left[(U+\lambda)^{-1}V(U+\lambda)^{-1}-(U+tV+\lambda)^{-1}V(U+tV+\lambda)^{-1}\right]d\mu(\lambda).
\end{aligned}
$$

Define

$$
W_{t,\lambda}:=(U+\lambda)^{-1}V(U+\lambda)^{-1}-(U+tV+\lambda)^{-1}V(U+tV+\lambda)^{-1}.
$$

If we multiply both sides of $W_{t,\lambda}$ with $\lambda+U+tV$, the we get

$$
\begin{aligned}
&(\lambda+U+tV)W_{t,\lambda}(\lambda+U+tV)\\
&=(\lambda+U+tV)(\lambda+U)^{-1}V(\lambda+U)^{-1}(\lambda+U+tV)-V\\
&=\left(1+tV(\lambda+U)^{-1}\right)V\left(1+t(\lambda+U)^{-1}V\right)-V\\
&=\left(V+tV(\lambda+U)^{-1}V\right)\left(1+t(\lambda+U)^{-1}V\right)-V\\
&=V+tV(\lambda+U)^{-1}V+tV(\lambda+U)^{-1}V\\
&+t^2V(\lambda+U)^{-1}V(\lambda+U)^{-1}V-V\\
&=2tV(\lambda+U)^{-1}V+t^2V(\lambda+U)^{-1}V(\lambda+U)^{-1}V\\
&=t\left[2V(\lambda+U)^{-1}V+tV(\lambda+U)^{-1}V(\lambda+U)^{-1}V\right].
\end{aligned}
$$

If we multiply the equality by $(\lambda+U+tV)^{-1}$ both sides, we get for $t\neq 0$

$$
\begin{aligned}
\frac{W_{t,\lambda}}{t}&=(\lambda+U+tV)^{-1}\left[2V(\lambda+U)^{-1}V+tV(\lambda+U)^{-1}V(\lambda+U)^{-1}V\right] \quad (3.8)\\
&\times(\lambda+U+tV)^{-1}.
\end{aligned}
$$

If we take the limit over $t\to 0$ in (2.7), then we get

$$
\lim_{t\to 0}\left(\frac{W_{t,\lambda}}{t}\right)=2(\lambda+U)^{-1}V(\lambda+U)^{-1}V(\lambda+U)^{-1}. \quad (3.9)
$$

Since

$$
\begin{aligned}
&\lim_{t\to 0}\frac{D(\mathcal{C}(w,\mu))(U+tV)(V)-D(\mathcal{C}(w,\mu))(U)(V)}{t}\\
&=\lim_{t\to 0}\int_0^\infty \lambda^2 w(\lambda)\frac{W_{t,\lambda}}{t}d\mu(\lambda)=\int_0^\infty \lambda^2 w(\lambda)\lim_{t\to 0}\left(\frac{W_{t,\lambda}}{t}\right)d\mu(\lambda)\\
&=2\int_0^\infty \lambda^2 w(\lambda)(\lambda+U)^{-1}V(\lambda+U)^{-1}V(\lambda+U)^{-1}d\mu(\lambda),
\end{aligned}
$$

then by (3.6) we obtain the desired representation (3.5). □

Corollary 3. *If* $0 < q \le U \le Q$ *and* $0 \le n^2 \le V^2 \le N^2$ *for some constants* q, Q, n, N, *then*

$$0 < n^2 \mathcal{C}''(w,\mu)(Q) \le D^2(\mathcal{C}(w,\mu))(U)(V,V) \le N^2 \mathcal{C}''(w,\mu)(q). \tag{3.10}$$

Proof. Since

$$0 < \lambda + q \le \lambda + U \le \lambda + Q$$

for all $\lambda \ge 0$, hence

$$0 < (\lambda + Q)^{-1} \le (\lambda + U)^{-1} \le (\lambda + q)^{-1}$$

for all $\lambda \ge 0$.

If we multiply both sides by V we get

$$0 < (\lambda + Q)^{-1} V^2 \le V(\lambda + U)^{-1} V \le (\lambda + q)^{-1} V^2.$$

Further, if we multiply both sides by $(\lambda + U)^{-1}$ we get

$$\begin{aligned} 0 &< (\lambda + Q)^{-1}(\lambda + U)^{-1} V^2 (\lambda + U)^{-1} \\ &\le (\lambda + U)^{-1} V (\lambda + U)^{-1} V (\lambda + U)^{-1} \\ &\le (\lambda + q)^{-1}(\lambda + U)^{-1} V^2 (\lambda + U)^{-1}. \end{aligned} \tag{3.11}$$

Since $0 \le n^2 \le V^2 \le N^2$, then by multiplying both sides by $(\lambda + U)^{-1}$, we get

$$n^2 (\lambda + U)^{-2} \le (\lambda + U)^{-1} V^2 (\lambda + U)^{-1} \le N^2 (\lambda + U)^{-2} \tag{3.12}$$

and since

$$0 < (\lambda + Q)^{-2} \le (\lambda + U)^{-2} \le (\lambda + q)^{-2}$$

then by (3.11) and (3.12), we get

$$0 < n^2 (\lambda + Q)^{-3} \le (\lambda + U)^{-1} V (\lambda + U)^{-1} V (\lambda + U)^{-1} \le N^2 (\lambda + q)^{-3},$$

for all $\lambda \ge 0$.

If we multiply by $\lambda^2 w(\lambda)$ and integrate, then we get

$$\begin{aligned} 0 &< n^2 \int_0^\infty \lambda^2 w(\lambda)(\lambda + Q)^{-3} d\mu(\lambda) \\ &\le \int_0^\infty \lambda^2 w(\lambda)(\lambda + U)^{-1} V (\lambda + U)^{-1} V (\lambda + U)^{-1} d\mu(\lambda) \\ &\le N^2 \int_0^\infty \lambda^2 w(\lambda)(\lambda + q)^{-3} d\mu(\lambda). \end{aligned} \tag{3.13}$$

By taking the derivative over t in (3.1) we get

$$\begin{aligned} \mathcal{C}'(w,\mu)(t) &= \int_0^\infty w(\lambda)\left[t - \lambda + \lambda^2 (t+\lambda)^{-1}\right]' d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left[1 - \lambda^2 (t+\lambda)^{-2}\right] d\mu(\lambda) \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}''(w,\mu)(t) &= \int_0^\infty w(\lambda)\left[1 - \lambda^2 (t+\lambda)^{-2}\right]' d\mu(\lambda) \\ &= 2\int_0^\infty w(\lambda)\lambda^2 (t+\lambda)^{-3} d\mu(\lambda) \end{aligned}$$

for $t > 0$.

Since
$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+Q)^{-3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(Q)$$
and
$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+q)^{-3} d\mu(\lambda) = \frac{1}{2}\mathcal{C}''(w,\mu)(q)$$
then we get, by (3.5) that,
$$0 < n^2\frac{1}{2}\mathcal{C}''(w,\mu)(Q) \le \frac{1}{2}D^2(\mathcal{C}(w,\mu))(U)(V,V) \le N^2\frac{1}{2}\mathcal{C}''(w,\mu)(q),$$
which proves the desired result. □

Theorem 5. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is operator convex on* $[0,\infty)$ *and has the representation (1.1), then*

$$\begin{aligned}(3.14)\qquad & D^2(f)(U)(V,V) \\ & = 2\left[cV^2 + \int_0^\infty \lambda^2 w(\lambda)(\lambda+U)^{-1}V(\lambda+U)^{-1}V(\lambda+U)^{-1}d\mu(\lambda)\right]\end{aligned}$$

for $U>0$ *and* $V\in\mathcal{S}A(H)$.

Proof. We have by (1.1) that

$$\begin{aligned}(3.15)\qquad & D^2(f)(U)(V,V) \\ & = D^2\left[f(0)+f'_+(0)\ell + c\ell^2 + \mathcal{C}(w,\mu)\right](U)(V,V) \\ & = D^2(f(0))(U)(V,V) + f'_+(0)D^2(\ell)(U)(V,V) \\ & + cD^2\left(\ell^2\right)(U)(V,V) + D^2(\mathcal{C}(w,\mu))(U)(V,V) \\ & = cD^2\left(\ell^2\right)(U)(V,V) + D^2(\mathcal{C}(w,\mu))(U)(V,V).\end{aligned}$$

Observe that

$$\begin{aligned}D^2\left(\ell^2\right)(U)(V,V) & = \lim_{t\to0}\frac{D\left(\ell^2\right)(U+tV)(V) - D\left(\ell^2\right)(U)(V)}{t} \\ & = \lim_{t\to0}\frac{(U+tV)V + V(U+tV) - UV - VU}{t} \\ & = 2V^2,\end{aligned}$$

then by (3.15) and (3.5) we get (3.14). □

Corollary 4. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is operator convex on* $[0,\infty)$ *and has the representation (1.1). If* $0<q\le U\le Q$ *and* $0\le n^2\le V^2\le N^2$ *for some constants* q, Q, n, N, *then*

$$\begin{aligned}(3.16)\qquad & 0 \le 2n^2 f''(Q) \le 2n^2 f''(Q) + 2c\left(V^2-n^2\right) \\ & \le D^2(f)(U)(V,V) \le 2N^2 f''(q) + 2c\left(V^2-N^2\right) \le 2N^2 f''(q).\end{aligned}$$

Proof. Since, by (1.1),
$$f(t) = f(0) + f'_+(0)t + ct^2 + \mathcal{C}(w,\mu)(t),\ t>0,$$
hence
$$\mathcal{C}''(w,\mu)(t) = f''(t) - 2c.$$
Then by (3.10) and (3.14), we get
$$0 < 2n^2\mathcal{C}''(w,\mu)(Q) \le D^2(f)(U)(V,V) - 2cV^2 \le 2N^2\mathcal{C}''(w,\mu)(q),$$

namely

$$0 < 2n^2 \left[f''(Q) - 2c\right] \le D^2(f)(U)(V,V) - 2cV^2 \le 2N^2 \left[f''(q) - 2c\right],$$

which is equivalent to

$$0 < 2n^2 \left[f''(Q) - 2c\right] + 2cV^2 \le D^2(f)(U)(V,V) \le 2N^2 \left[f''(q) - 2c\right] + 2cV^2.$$

The rest is obvious and we omit the details. □

Theorem 6. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is operator convex on* $[0,\infty)$ *with* $f'_+(0) \in \mathbb{R}$ *and* $A,\ B > 0$. *If* $0 \le k \le (1-t)A + tB \le K$ *for* $t \in [0,1]$ *and* $m^2 \le (B-A)^2 \le M^2$ *for some constants* $k,\ K,\ m,\ M,$ *then*

$$\begin{aligned}(3.17)\qquad 0 &\le m^2 f''(K)\, t(1-t) \le (1-t) f(A) + t f(B) - f((1-t)A + tB) \\ &\le M^2 f''(k)\, t(1-t)\end{aligned}$$

for $t \in [0,1]$.

If $0 \le k \le \frac{A+B}{2} \le K$, *then*

$$(3.18)\qquad 0 \le \frac{1}{4} m^2 f''(K) \le \frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \le \frac{1}{4} M^2 f''(k).$$

Proof. Since for $U := (1-t)A + tB,\ t \in [0,1]$, we have $k \le U \le K$ and $m^2 \le (B-A)^2 \le M^2$, then by (3.16) we have

$$0 \le 2m^2 f''(K) \le D^2(f)((1-t)A + tB)(B-A, B-A) \le 2M^2 f''(k).$$

By a similar argument to the one in Theorem 4 we also get

$$\begin{aligned} m^2 f''(K)\, t(1-t) &\le (1-t) f(A) + t f(B) - f((1-t)A + tB) \\ &\le M^2 f''(k)\, t(1-t)\end{aligned}$$

for $t \in [0,1]$, which proves the desired result (3.17). □

Corollary 5. *With the assumption of Theorem 6 and if* $0 \le k \le A,\ B \le K$, *then (3.17) and (3.18) are valid.*

Remark 1. *If the selfadjoint operator* T *satisfies the condition* $-u \le T \le u$, *for* $u > 0$, *then* $-u\|x\|^2 \le \langle Tx, x\rangle \le u\|x\|^2$, $x \in H$ *which is equivalent to* $|\langle Tx, x\rangle| \le u\|x\|^2$, $x \in H$ *and implies that* $\|T\| = \sup_{x\in H} |\langle Tx, x\rangle| \le u$. *For* $x \in H$ *we have* $0 \le \langle T^2 x, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \le \|T\|^2 \|x\|^2 \le u^2 \|x\|^2$, *which implies that* $0 \le T^2 \le u^2$.

Now, if $0 < k \le A,\ B \le K$, *then* $-(K-k) \le B - A \le K - k$ *which implies that* $(B-A)^2 \le (K-k)^2$. *Therefore, by (3.17) we get the following simple upper bounds*

$$(3.19)\quad 0 \le (1-t) f(A) + t f(B) - f((1-t)A + tB) \le (K-k)^2 f''(k)\, t(1-t)$$

and, in particular,

$$(3.20)\qquad 0 \le \frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2}\right) \le \frac{1}{4}(K-k)^2 f''(k).$$

4. Some Examples

Assume that $A,\ B>0$ and $0<k\leq(1-t)A+tB\leq K$, for $t\in[0,1]$ or, for simplicity, $0<k\leq A,\ B\leq K$ and $m^2\leq(B-A)^2\leq M^2$ for some constants k, K, m, M.

For $p\in[1,2]$, we consider the power function $f(t)=t^p$. This function is operator convex on $[0,\infty)$ with $f'_+(0)=0$. Also we have $f''(t)=p(p-1)t^{p-2}$, $t\in(0,\infty)$. Then by Theorem 6 we have the power inequalities

$$
\begin{aligned}
0 &\leq m^2p(p-1)K^{p-2}t(1-t)\leq(1-t)A^p+tB^p-((1-t)A+tB)^p \\
&\leq M^2p(p-1)k^{p-2}t(1-t)
\end{aligned}
\tag{4.1}
$$

for $t\in[0,1]$.

If $0<k\leq\frac{A+B}{2}\leq K$, then

$$
0\leq\frac{1}{4}m^2p(p-1)K^{p-2}\leq\frac{A^p+B^p}{2}-\left(\frac{A+B}{2}\right)^p\leq\frac{1}{4}M^2p(p-1)k^{p-2}. \tag{4.2}
$$

Let $\varepsilon>0$ and consider the function $f(t)=-\ln(t+\varepsilon)$ defined on $[0,\infty)$. We have $f'_+(0)=-\frac{1}{\varepsilon}$. Also $f''(t)=\frac{1}{(t+\varepsilon)^2}$, $t\in(0,\infty)$. Then by Theorem 6 we have the logarithmic inequalities

$$
\begin{aligned}
0 &\leq m^2\frac{1}{(K+\varepsilon)^2}t(1-t) \\
&\leq\ln((1-t)A+tB+\varepsilon)-(1-t)\ln(A+\varepsilon)-t\ln(B+\varepsilon) \\
&\leq M^2\frac{1}{(k+\varepsilon)^2}t(1-t)
\end{aligned}
$$

for $t\in[0,1]$.

By taking the limit over $\varepsilon\to0+$ we derive the inequality of interest

$$
0\leq\frac{m^2}{K^2}t(1-t)\leq\ln((1-t)A+tB)-(1-t)\ln(A)-t\ln(B)\leq\frac{M^2}{k^2}t(1-t) \tag{4.3}
$$

for $t\in[0,1]$, provided $0<k\leq(1-t)A+tB\leq K$, for $t\in[0,1]$ and $m^2\leq(B-A)^2\leq M^2$.

If $0<k\leq\frac{A+B}{2}\leq K$, then

$$
0\leq\frac{m^2}{4K^2}\leq\ln\left(\frac{A+B}{2}\right)-\frac{\ln(A)+\ln(B)}{2}\leq\frac{M^2}{4k^2}. \tag{4.4}
$$

For $0<k\leq A,\ B\leq K$, the inequalities (4.3) and (4.4) also hold.

Let $\varepsilon>0$ and consider the function $f(t)=(t+\varepsilon)\ln(t+\varepsilon)$ defined on $[0,\infty)$. We have $f'_+(0)=\ln\varepsilon+1$. Also $f''(t)=\frac{1}{t+\varepsilon}$, $t\in(0,\infty)$. Then by Theorem 6 we have the logarithmic inequalities

$$
\begin{aligned}
0 &\leq m^2\frac{1}{K+\varepsilon}t(1-t) \\
&\leq(1-t)(A+\varepsilon)\ln(A+\varepsilon)+t(B+\varepsilon)\ln(B+\varepsilon) \\
&-((1-t)A+tB+\varepsilon)\ln((1-t)A+tB+\varepsilon) \\
&\leq M^2\frac{1}{k+\varepsilon}t(1-t)
\end{aligned}
$$

for $t\in[0,1]$.

By taking the limit over $\varepsilon \to 0+$ we derive the inequality of interest

$$\begin{aligned} 0 &\leq \frac{m^2}{K} t(1-t) \\ &\leq (1-t) A \ln A + tB \ln B - ((1-t) A + tB) \ln ((1-t) A + tB) \\ &\leq \frac{M^2}{k} t(1-t) \end{aligned} \tag{4.5}$$

for $t \in [0,1]$, provided $0 < k \leq (1-t) A + tB \leq K$, for $t \in [0,1]$ and $m^2 \leq (B-A)^2 \leq M^2$.

If $0 < k \leq \frac{A+B}{2} \leq K$, then

$$0 \leq \frac{m^2}{4K} \leq \frac{A \ln A + B \ln B}{2} - \left(\frac{A+B}{2}\right) \ln \left(\frac{A+B}{2}\right) \leq \frac{M^2}{4k}. \tag{4.6}$$

For $0 < k \leq A$, $B \leq K$ we also have (4.5) and (4.6).

Now, if $0 < k \leq A$, $B \leq K$, then $-(K-k) \leq B - A \leq K - k$, which implies that $(B-A)^2 \leq (K-k)^2$. Therefore, by the above inequalities (4.1)-(4.6) we have the following upper bounds

$$0 \leq (1-t) A^p + tB^p - ((1-t) A + tB)^p \leq (K-k)^2 p (p-1) k^{p-2} t (1-t), \tag{4.7}$$

for $p \in [1,2]$,

$$0 \leq \ln ((1-t) A + tB) - (1-t) \ln (A) - t \ln (B) \leq \frac{(K-k)^2}{k^2} t (1-t) \tag{4.8}$$

and

$$\begin{aligned} 0 &\leq (1-t) A \ln A + tB \ln B - ((1-t) A + tB) \ln ((1-t) A + tB) \\ &\leq \frac{(K-k)^2}{k} t (1-t) \end{aligned} \tag{4.9}$$

for $t \in [0,1]$.

In particular, we have the following upper bounds for the *Jensen's difference*

$$0 \leq \frac{A^p + B^p}{2} - \left(\frac{A+B}{2}\right)^p \leq \frac{1}{4} (K-k)^2 p (p-1) k^{p-2}, \; p \in [1,2], \tag{4.10}$$

$$0 \leq \ln \left(\frac{A+B}{2}\right) - \frac{\ln (A) + \ln (B)}{2} \leq \frac{(K-k)^2}{4k^2}. \tag{4.11}$$

and

$$0 \leq \frac{A \ln A + B \ln B}{2} - \left(\frac{A+B}{2}\right) \ln \left(\frac{A+B}{2}\right) \leq \frac{(K-k)^2}{4k}, \tag{4.12}$$

if $0 < k \leq A$, $B \leq K$.

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