PERTURBED LIPSCHITZ TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. In this paper we provide some bounds for the quantity $\|f(y)-f(x)\|$ where $f: D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$, a Banach algebra, with the spectra $\sigma(x), \sigma(y) \subset D$. We show among others that, if $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R,

$$\begin{aligned}&\Big\| f(y)-f(x)\\&-R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it}-x\right)^{-1}(y-x)\left(Re^{2\pi it}-x\right)^{-1}dt\Big\|\\&\le \frac{R\|f\|_{R,\infty}\|y-x\|^2}{(R-\|x\|)^2(R-\|y\|)}.\end{aligned}$$

and

$$\begin{aligned}&\Big\| f(y)-f(x)\\&-R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it}-x\right)^{-1}(y-x)\left(Re^{2\pi it}-x\right)^{-1}dt\Big\|\\&\le R\|y-x\|^2\|f\|_{R,\infty}\left\{\frac{1}{2}\left[(R-\|y\|)^{-3}+(R-\|x\|)^{-3}\right]\right.\\&\left.+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\},\end{aligned}$$

where $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$ and, in general the bounds are not comparable, meaning that one can be better than the other one depending on the elements x, $y \in \mathcal{B}$.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \le \|a\|\,\|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

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(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\le\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a):=\frac{1}{2\pi i}\int_\delta f(\xi)(\xi-a)^{-1}d\xi, \tag{1.1}$$

where $\delta\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a)\subset\operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a))=f(\sigma(a)) \tag{1.2}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [13] and [15].

Let $\mathfrak{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\mathfrak{Hol}(a)$ is an algebra where if $f,g\in\mathfrak{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f)\cap D(g)$. $\mathfrak{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz Functional Calculus Theorem* [**?**, p. 201-203]:

Theorem 1. *Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$.*

(a) *The map $f\mapsto f(a)$ of $\mathfrak{Hol}(a)\to\mathcal{B}$ is an algebra homomorphism.*
(b) *If $f(z)=\sum_{k=0}^{\infty}\alpha_k z^k$ has radius of convergence $r>\nu(a)$, then $f\in\mathfrak{Hol}(a)$ and $f(a)=\sum_{k=0}^{\infty}\alpha_k a^k$.*

(c) *If* $f(z) \equiv 1$, *then* $f(a) = 1$.
(d) *If* $f(z) = z$ *for all* z, $f(a) = a$.
(e) *If* $f, f_1, ..., f_n...$ *are analytic on* G, $\sigma(a) \subset G$ *and* $f_n(z) \to f(z)$ *uniformly on compact subsets of* G, *then* $\|f_n(a) - f(a)\| \to 0$ *as* $n \to \infty$.
(f) *The Riesz Functional Calculus is unique and if* a, b *are commuting elements in* $\mathcal{B}$ *and* $f \in \mathfrak{Hol}(a)$, *then* $f(a)b = bf(a)$.

For some recent norm inequalities for functions on Banach algebras, see [6], [2] and [4]-[11].

In the recent paper [7], we provided some bounds for the quantity $\|f(y) - f(x)\|$ where $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$, a Banach algebra, with the spectra $\sigma(x), \sigma(y) \subset D$. We showed among others that, if $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R,

$$\|f(y) - f(x)\| \le \frac{R\|y-x\| \|f\|_{R,\infty}}{(R - \|y\|)(R - \|x\|)}$$

and

$$\|f(y) - f(x)\| \le \frac{1}{2} R\|y-x\| \|f\|_{R,\infty} \times \left\{ \frac{1}{2}\left[(R - \|y\|)^{-2} + (R - \|x\|)^{-2} \right] + \left(R - \left\| \frac{x+y}{2} \right\| \right)^{-2} \right\},$$

where $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left| f\left(Re^{2\pi it}\right) \right| < \infty$ and, in general the bounds are not comparable, meaning that one can be better than the other one depending on the elements $x, y \in \mathcal{B}$.

In this paper we establish among others that, if $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R,

$$\begin{aligned} &\Big\| f(y) - f(x) \\ &- R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it} - x\right)^{-1}(y-x)\left(Re^{2\pi it} - x\right)^{-1} dt \Big\| \\ &\le \frac{R\|f\|_{R,\infty}\|y-x\|^2}{(R - \|x\|)^2 (R - \|y\|)} \end{aligned}$$

and

$$\begin{aligned} &\Big\| f(y) - f(x) \\ &- R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it} - x\right)^{-1}(y-x)\left(Re^{2\pi it} - x\right)^{-1} dt \Big\| \\ &\le R\|y-x\|^2 \|f\|_{R,\infty} \left\{ \frac{1}{2}\left[(R - \|y\|)^{-3} + (R - \|x\|)^{-3} \right] \right. \\ &\left. + \left(R - \left\| \frac{x+y}{2} \right\| \right)^{-3} \right\}, \end{aligned}$$

where $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left| f\left(Re^{2\pi it}\right) \right| < \infty$ and, in general the bounds are not comparable, meaning that one can be better than the other one depending on the elements $x, y \in \mathcal{B}$.

2. Main Results

We consider the function $\ell^{-1} : \text{Inv}(\mathcal{B}) \longmapsto \text{Inv}(\mathcal{B})$, $\ell^{-1}(a) = a^{-1}$.

Lemma 1. *Let $a \in \text{Inv}(\mathcal{B})$. The function ℓ^{-1} is Fréchet differentiable in a and the Fréchet derivative is*

$$D\left(\ell^{-1}\right)(a)(v) = -a^{-1}va^{-1} \tag{2.1}$$

for all $v \in \mathcal{B}$.

Proof. Let $a \in \text{Inv}(\mathcal{B})$. Since $\text{Inv}(\mathcal{B})$ is open then for $v \in \mathcal{B}$ there exists a small interval $(-\delta, \delta)$ with $\delta > 0$ such that $a + tv \in \text{Inv}(\mathcal{B})$ for all $t \in (-\delta, \delta)$. Then

$$\begin{aligned}\ell^{-1}(a+tv) - \ell^{-1}(a) &= (a+tv)^{-1} - a^{-1} = (a+tv)^{-1}(a - a - tv)a^{-1} \\ &= -t(a+tv)^{-1}va^{-1}\end{aligned}$$

for all $t \in (-\delta, \delta)$.

For $t \in (-\delta, \delta)$, $t \neq 0$, we get

$$\frac{\ell^{-1}(a+tv) - \ell^{-1}(a)}{t} = -(a+tv)^{-1}va^{-1}$$

and by taking the limit over $t \to 0$, we get (2.1). □

Lemma 2. *Let $a \in \text{Inv}(\mathcal{B})$. The function ℓ^{-1} is twice Fréchet differentiable in a and the second Fréchet derivative is*

$$D^2\left(\ell^{-1}\right)(a)(v,u) = a^{-1}\left(va^{-1}u + ua^{-1}v\right)a^{-1} \tag{2.2}$$

for all u, $v \in \mathcal{B}$.

In particular,

$$D^2\left(\ell^{-1}\right)(a)(v,v) = 2a^{-1}va^{-1}va^{-1} \tag{2.3}$$

for all $v \in \mathcal{B}$.

Proof. Let $a \in \text{Inv}(\mathcal{B})$. Since $\text{Inv}(\mathcal{B})$ is open then for $v, u \in \mathcal{B}$ there exists a small interval $(-\delta, \delta)$ with $\delta > 0$ such that $a + tv, \in \text{Inv}(\mathcal{B})$ for all $t \in (-\delta, \delta)$. Then

$$D\left(\ell^{-1}\right)(a+tv)(u) - D\left(\ell^{-1}\right)(a)(u) = a^{-1}ua^{-1} - (a+tv)^{-1}u(a+tv)^{-1}$$

for all $t \in (-\delta, \delta)$.

Consider

$$U_t := a^{-1}ua^{-1} - (a+tv)^{-1}u(a+tv)^{-1}$$

with $t \in (-\delta, \delta)$.

If we multiply U_t both sides by $a + tv$, we get

$$\begin{aligned}(a+tv)U_t(a+tv) &= (a+tv)a^{-1}ua^{-1}(a+tv) - u \\ &= \left(u + tva^{-1}u\right)\left(1 + ta^{-1}v\right) - u \\ &= u + tva^{-1}u + tua^{-1}v + t^2va^{-1}ua^{-1}v - u \\ &= t\left(va^{-1}u + ua^{-1}v\right) + t^2va^{-1}ua^{-1}v\end{aligned}$$

for $t \in (-\delta, \delta)$.

By multiplying this equality both sides by $(a+tv)^{-1}$, we get

$$U_t = t(a+tv)^{-1}\left(va^{-1}u + ua^{-1}v + tva^{-1}ua^{-1}v\right)(a+tv)^{-1}$$

for $t \in (-\delta, \delta)$.

Therefore

$$\frac{D\left(\ell^{-1}\right)(a+tv)(u)-D\left(\ell^{-1}\right)(a)(u)}{t}$$
$$=(a+tv)^{-1}\left(va^{-1}u+ua^{-1}v+tva^{-1}ua^{-1}v\right)(a+tv)^{-1}$$

for $t\in(-\delta,\delta)$, $t\neq 0$ and by taking the limit over $t\to 0$, we get (2.2). □

Theorem 2. *Let a, $b\in B$ with $a\neq b$ and assume that the closed segment $[a,b]:=\{(1-t)a+by,\ t\in[0,1]\}\subset \operatorname{Inv}(\mathcal{B})$. Then we have the identity*

$$\begin{aligned}&b^{-1}-a^{-1}+a^{-1}(b-a)a^{-1}\\&=2\int_0^1(1-t)((1-t)a+tb)^{-1}(b-a)((1-t)a+tb)^{-1}\\&\times(b-a)((1-t)a+tb)^{-1}dt.\end{aligned}\tag{2.4}$$

Proof. We use the Taylor's type formula with integral remainder, see for instance [3, p. 112] for the general case of functions for functions defined on normed spaces and with values in Banach spaces,

$$\begin{aligned}f(x)&=f(y)+D(f)(y)(x-y)\\&+\int_0^1(1-t)D^2(f)((1-t)y+tx)(x-y,x-y)\,dt\end{aligned}\tag{2.5}$$

that holds for functions f which are of class C^2 on an open and convex subset $\mathcal{O}$ in the Banach algebra $\mathcal{B}$ and y, $x\in\mathcal{O}$.

If we write this equality for ℓ^{-1}, $\mathcal{O}=[a,b]$, $x=b$ and $y=a$, then we get

$$\begin{aligned}\ell^{-1}(b)&=\ell^{-1}(a)+D\left(\ell^{-1}\right)(a)(b-a)\\&+\int_0^1(1-t)D^2\left(\ell^{-1}\right)((1-t)a+tb)(b-a,b-a)\,dt\end{aligned}\tag{2.6}$$

and, since by Lemmas 1 and 2

$$D\left(\ell^{-1}\right)(a)(b-a)=-a^{-1}(b-a)a^{-1}$$

and

$$\begin{aligned}&D^2\left(\ell^{-1}\right)((1-t)a+tb)(b-a,b-a)\\&=2((1-t)a+tb)^{-1}(b-a)((1-t)a+tb)^{-1}(b-a)((1-t)a+tb)^{-1},\end{aligned}$$

then by (2.6) we get (2.4). □

Corollary 1. *Let x, $y\in B$ with $x\neq y$ and assume that $\|x\|$, $\|y\|<1$, then*

$$\begin{aligned}&(1-y)^{-1}-(1-x)^{-1}-(1-x)^{-1}(y-x)(1-x)^{-1}\\&=2\int_0^1(1-t)(1-(1-t)x-ty)^{-1}(y-x)(1-(1-t)x-ty)^{-1}\\&\times(y-x)(1-(1-t)x-ty)^{-1}dt.\end{aligned}\tag{2.7}$$

Proof. By taking $a = 1 - x$, $b = 1 - y$ in (2.4), we get

$$\begin{aligned}
&(1-y)^{-1} - (1-x)^{-1} - (1-x)^{-1}(y-x)(1-x)^{-1} \\
&= 2\int_0^1 (1-t)\left((1-t)(1-x) + t(1-y)\right)^{-1}(1-y-(1-x)) \\
&\times \left((1-t)(1-x) + t(1-y)\right)^{-1}((1-y)-(1-x)) \\
&\times \left((1-t)(1-x) + t(1-y)\right)^{-1} dt \\
&= 2\int_0^1 (1-t)(1-(1-t)x - ty)^{-1}(x-y)(1-(1-t)x-ty)^{-1} \\
&\times (x-y)(1-(1-t)x-ty)^{-1} dt,
\end{aligned}$$

which proves (2.7). □

Theorem 3. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then we have*

$$\begin{aligned}
(2.8)\qquad & f(y) - f(x) - \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y-x)(\xi-x)^{-1} d\xi \\
&= \frac{1}{\pi i}\int_0^1 (1-t)\left(\int_\gamma f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)\right. \\
&\left.\times (\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} d\xi\right) dt.
\end{aligned}$$

Proof. Since $\left\|\frac{y}{\xi}\right\|, \left\|\frac{x}{\xi}\right\| < 1$ for $\xi \in \gamma$ then we can apply Corollary 1 to get

$$\begin{aligned}
&\left(1-\frac{y}{\xi}\right)^{-1} - \left(1-\frac{x}{\xi}\right)^{-1} - \left(1-\frac{x}{\xi}\right)^{-1}\left(\frac{y}{\xi}-\frac{x}{\xi}\right)\left(1-\frac{x}{\xi}\right)^{-1} \\
&= 2\int_0^1 (1-t)\left(1-(1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1}\left(\frac{y}{\xi}-\frac{x}{\xi}\right)\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \\
&\times \left(\frac{y}{\xi}-\frac{x}{\xi}\right)\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} dt.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\xi(\xi-y)^{-1} - \xi(\xi-x)^{-1} - \xi(\xi-x)^{-1}(y-x)(\xi-x)^{-1} \\
&= 2\xi\int_0^1 (1-t)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} \\
&\times (y-x)(\xi-(1-t)x-ty)^{-1} dt,
\end{aligned}$$

namely

$$\begin{aligned}
&(\xi-y)^{-1} - (\xi-x)^{-1} - (\xi-x)^{-1}(y-x)(\xi-x)^{-1} \\
&= 2\int_0^1 (1-t)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} \\
&\times (y-x)(\xi-(1-t)x-ty)^{-1} dt,
\end{aligned}$$

for $\xi \in \gamma$.

If we multiply by $\frac{1}{2\pi i}f(\xi)$ and integrate on γ, then we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-y)^{-1}d\xi-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}d\xi\\
&-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi\\
&=2\frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1(1-t)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right.\\
&\left.\times(y-x)(\xi-(1-t)x-ty)^{-1}dt\right)d\xi\\
&=\frac{1}{\pi i}\int_0^1(1-t)\left(\int_\gamma f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right.\\
&\left.\times(y-x)(\xi-(1-t)x-ty)^{-1}d\xi\right)dt,
\end{aligned}$$

where for the last equality we used Fubini's theorem.

By making use of Riesz functional calculus, we derive the identity (2.8). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
(2.9)\quad&\left\|f(y)-f(x)-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi\right\|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left(\int_0^1(1-t)\left\|(\xi-(1-t)x-ty)^{-1}\right\|^3dt\right)|d\xi|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left(\int_0^1(1-t)(|\xi|-\|(1-t)x+ty\|)^{-3}dt\right)|d\xi|\\
&\le\frac{1}{2\pi}\|y-x\|^2\int_\gamma\frac{|f(\xi)||d\xi|}{(|\xi|-\|x\|)^2(|\xi|-\|y\|)}.
\end{aligned}$$

Proof. We have, by taking the norm in (2.8), that

$$\begin{aligned}
(2.10)\quad&\left\|f(y)-f(x)-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi\right\|\\
&\le\frac{1}{\pi}\int_0^1(1-t)\left(\int_\gamma\left\|f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)\right.\right.\\
&\left.\left.\times(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right\||d\xi|\right)dt\\
&\le\frac{1}{\pi}\int_0^1(1-t)\int_\gamma|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|\|y-x\|\\
&\times\left\|(\xi-(1-t)x-ty)^{-1}\right\|\|y-x\|\left\|(\xi-(1-t)x-ty)^{-1}\right\||d\xi|\Big)dt\\
&=\frac{1}{\pi}\|y-x\|^2\int_0^1(1-t)\left(\int_\gamma|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^3|d\xi|\right)dt\\
&=:\|y-x\|^2C(f,x,y).
\end{aligned}$$

We have

$$\begin{aligned}
&C(f,x,y)\\
&:=\frac{1}{\pi}\int_0^1(1-t)\left(\int_\gamma|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^3|d\xi|\right)dt\\
&=\frac{1}{\pi}\int_0^1(1-t)\left(\int_\gamma|f(\xi)||\xi|^{-3}\left\|\left(\xi-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^3|d\xi|\right)dt\\
&=\frac{1}{\pi}\int_\gamma|f(\xi)||\xi|^{-3}\left(\int_0^1(1-t)\left\|\left(\xi-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^3dt\right)|d\xi|.
\end{aligned}$$

Since

$$\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\le(1-t)\left\|\frac{x}{\xi}\right\|+t\left\|\frac{y}{\xi}\right\|<1-t+t=1$$

for $\xi\in\gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}
\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|&\le\sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^k\\
&=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\
&=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\
&=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}
\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^3\le|\xi|^3\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}
&\int_\gamma|f(\xi)||\xi|^{-3}\left(\int_0^1(1-t)\left\|\left(\xi-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^3dt\right)|d\xi|\\
&\le\int_\gamma|f(\xi)||\xi|^{-3}\left(\int_0^1(1-t)|\xi|^3\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}dt\right)|d\xi|\\
&=\int_\gamma|f(\xi)|\left(\int_0^1(1-t)\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}dt\right)|d\xi|
\end{aligned}$$

and by (2.10) we derive the second inequality in (2.9).

By the triangle inequality we have

$$\begin{aligned}|\xi| - \|(1-t)\,x + ty\| &\geq |\xi| - (1-t)\,\|x\| - t\,\|y\| \\ &= (1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|) > 0\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$(|\xi| - \|(1-t)\,x + ty\|)^{-1} \leq [(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-1}$$

and thus

$$(|\xi| - \|(1-t)\,x + ty\|)^{-3} \leq [(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-3}$$

for $\xi \in \gamma$ and $t \in [0,1]$.

By multiplying with $1-t$ and integrating over $t \in [0,1]$, we get

$$\begin{aligned}&\int_0^1 (1-t)\,(|\xi| - \|(1-t)\,x + ty\|)^{-3}\,dt \\ &\leq \int_0^1 (1-t)\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-3}\,dt \\ &= -\frac{1}{2\,(\|x\| - \|y\|)} \\ &\times \int_0^1 (1-t)\,\frac{d}{dt}\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\,dt \\ &=: I\,(x,y,\xi)\,,\end{aligned}$$

for $\|y\| \neq \|x\|$.

Using integration by parts, we get

$$\begin{aligned}&\int_0^1 (1-t)\,\frac{d}{dt}\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\,dt \\ &= (1-t)\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\Big|_0^1 \\ &+ \int_0^1 [(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\,dt \\ &= \int_0^1 [(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\,dt - (|\xi| - \|x\|)^{-2}\,.\end{aligned}$$

Also,

$$\begin{aligned}&\int_0^1 [(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-2}\,dt \\ &= -\frac{1}{\|x\| - \|y\|}\int_0^1 \frac{d}{dt}\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-1}\,dt \\ &= -\,\frac{1}{\|x\| - \|y\|}\,[(1-t)\,(|\xi| - \|x\|) + t\,(|\xi| - \|y\|)]^{-1}\Big|_0^1 \\ &= \frac{1}{\|y\| - \|x\|}\left[(|\xi| - \|y\|)^{-1} - (|\xi| - \|x\|)^{-1}\right] \\ &= \frac{1}{\|y\| - \|x\|}\,\frac{\|y\| - \|x\|}{(|\xi| - \|y\|)\,(|\xi| - \|x\|)} = \frac{1}{(|\xi| - \|y\|)\,(|\xi| - \|x\|)}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^1 (1-t)\frac{d}{dt}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}dt\\&=\frac{1}{(|\xi|-\|y\|)(|\xi|-\|x\|)}-\frac{1}{(|\xi|-\|x\|)^2}\\&=\frac{1}{(|\xi|-\|x\|)}\left(\frac{1}{|\xi|-\|y\|}-\frac{1}{|\xi|-\|x\|}\right)\\&=\frac{\|y\|-\|x\|}{(|\xi|-\|x\|)^2(|\xi|-\|y\|)}\end{aligned}$$

and

$$I(x,y,\xi)=\frac{1}{2(|\xi|-\|x\|)^2(|\xi|-\|y\|)}$$

for $\|y\|\neq\|x\|$.

This implies that

$$C(f,x,y)\leq\frac{1}{2\pi}\int_\gamma\frac{|f(\xi)|\,|d\xi|}{(|\xi|-\|x\|)^2(|\xi|-\|y\|)}.$$

By utilising (2.10) we then get (2.9).

For $\|y\|=\|x\|$ we have

$$\int_0^1(1-t)\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-3}dt=\frac{1}{2}\frac{1}{(|\xi|-\|x\|)^3}$$

and the inequality (2.9) also holds with $\|x\|$ instead of $\|y\|$. □

Corollary 3. *With the assumptions of Theorem 2 and if*

$$\|f\|_{\gamma,\infty}:=\sup_{\xi\in\gamma}|f(\xi)|<\infty,$$

then

$$\begin{aligned}&\left\|f(y)-f(x)-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi\right\|\\&\leq\frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\\&\times\int_\gamma\left(\int_0^1(1-t)\left\|(\xi-(1-t)x-ty)^{-1}\right\|^3dt\right)|d\xi|\\&\leq\frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\\&\times\int_\gamma\left(\int_0^1(1-t)(|\xi|-\|(1-t)x+ty\|)^{-3}dt\right)|d\xi|\\&\leq\frac{1}{2\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma\frac{|d\xi|}{(|\xi|-\|x\|)^2(|\xi|-\|y\|)}.\end{aligned}\tag{2.11}$$

Remark 1. *Assume that $f:D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and x, $y\in\mathcal{B}$ with $\sigma(x)$, $\sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By taking γ parametrized by $\xi(t)=Re^{2\pi it}$*

where $t \in [0,1]$, *then* $d\xi(t) = 2\pi i R e^{2\pi i t} dt$, $|d\xi(t)| = 2\pi R dt$, $|\xi| = R$ *and by (2.9) we get*

$$\begin{aligned} &\left\| f(y) - f(x) \right. \\ &\left. -R \int_0^1 e^{2\pi i t} f\left(Re^{2\pi i t}\right) \left(Re^{2\pi i t} - x\right)^{-1} (y-x) \left(Re^{2\pi i t} - x\right)^{-1} dt \right\| \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\leq 2R \|y - x\|^2 \\ &\times \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| \left(\int_0^1 (1-s) \left\| \left(Re^{2\pi i t} - (1-s)x - sy\right)^{-1} \right\|^3 ds \right) dt \\ &\leq 2R \|y-x\|^2 \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| dt \int_0^1 (1-s) \left(R - \|(1-s)x + sy\|\right)^{-3} ds \\ &\leq \frac{R\|y-x\|^2}{(R - \|x\|)^2 (R - \|y\|)} \int_0^1 \left|f\left(Re^{2\pi i t}\right)\right| dt. \end{aligned}$$

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi i t}\right)\right| < \infty$, *then we have the simpler inequalities*

$$\begin{aligned} &\left\| f(y) - f(x) \right. \\ &\left. -R \int_0^1 e^{2\pi i t} f\left(Re^{2\pi i t}\right) \left(Re^{2\pi i t} - x\right)^{-1} (y-x) \left(Re^{2\pi i t} - x\right)^{-1} dt \right\| \\ &\leq 2R \|f\|_{R,\infty} \|y-x\|^2 \\ &\times \int_0^1 \left(\int_0^1 (1-s) \left\| \left(Re^{2\pi i t} - (1-s)x - sy\right)^{-1} \right\|^3 ds \right) dt \\ &\leq 2R \|f\|_{R,\infty} \|y-x\|^2 \int_0^1 (1-s) \left(R - \|(1-s)x + sy\|\right)^{-3} ds \\ &\leq \frac{R\|f\|_{R,\infty}\|y-x\|^2}{(R - \|x\|)^2 (R - \|y\|)}. \end{aligned} \tag{2.13}$$

We also have the following alternative upper bounds:

Theorem 4. *With the assumptions of Theorem 2 we have*

$$\begin{aligned} &\left\| f(y) - f(x) - \frac{1}{2\pi i} \int_\gamma f(\xi) (\xi - x)^{-1} (y-x) (\xi - x)^{-1} d\xi \right\| \\ &\leq \frac{1}{\pi} \|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (1-t) \left\| (\xi - (1-t)x - ty)^{-1} \right\|^3 dt \right) |d\xi| \\ &\leq \frac{1}{\pi} \|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (1-t) \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt \right) |d\xi| \end{aligned} \tag{2.14}$$

$$\leq \frac{1}{2\pi}\|y-x\|^2\left\{\frac{1}{2}\int_\gamma |f(\xi)|\left[(|\xi|-\|y\|)^{-3}+(|\xi|-\|x\|)^{-3}\right]|d\xi|\right.$$
$$\left.+\int_\gamma |f(\xi)|\left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-3}|d\xi|\right\}$$
$$\leq \frac{1}{2\pi}\|y-x\|^2\int_\gamma |f(\xi)|\left[(|\xi|-\|y\|)^{-3}+(|\xi|-\|x\|)^{-3}\right]|d\xi|.$$

Proof. Let $\xi\in\gamma$. For $\alpha,\ \beta\geq 0$ with $\alpha+\beta=1$ and $t_1,\ t_2\in[0,1]$, then for $h_\xi(t):=(|\xi|-\|(1-t)x+ty\|)^{-3}$, $t\in[0,1]$ we get

$$\begin{aligned}&h_\xi(\alpha t_1+\beta t_2)\\&=(|\xi|-\|(1-(\alpha t_1+\beta t_2))x+(\alpha t_1+\beta t_2)y\|)^{-3}\\&=(|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|)^{-3}\end{aligned}$$

By the properties of the norm, we have

$$\begin{aligned}&\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|\\&\leq\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|\end{aligned}$$

which gives that

$$\begin{aligned}&|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|\\&\geq|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\begin{aligned}&(|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|)^{-1}\\&\leq(|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|)^{-1}\end{aligned}$$

giving that

$$\begin{aligned}&h_\xi(\alpha t_1+\beta t_2)\\&\leq(|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|)^{-3}\\&=(\alpha[|\xi|-\|(1-t_1)x+t_1y\|]+\beta[|\xi|-\|(1-t_2)x+t_2y\|])^{-3}.\end{aligned}$$

By using the convexity of the function $(\cdot)^{-3}$ we have

$$\begin{aligned}&(\alpha[|\xi|-\|(1-t_1)x+t_1y\|]+\beta[|\xi|-\|(1-t_2)x+t_2y\|])^{-3}\\&\leq\alpha[|\xi|-\|(1-t_1)x+t_1y\|]^{-3}+\beta[|\xi|-\|(1-t_2)x+t_2y\|]^{-3}\\&=\alpha h_\xi(t_1)+\beta h_\xi(t_2).\end{aligned}$$

Therefore

$$h_\xi(\alpha t_1+\beta t_2)\leq\alpha h_\xi(t_1)+\beta h_\xi(t_2),$$

which proves the convexity of h_ξ on $[0,1]$.

By using the Hermite-Hadamard type inequality, see for instance [12, p. 11], for h_ξ on $[0,1]$ we get

$$\int_0^1 h_\xi(t)\,dt\leq\frac{1}{2}\left\{\frac{1}{2}[h_\xi(1)+h_\xi(0)]+h_\xi\left(\frac{1}{2}\right)\right\}\leq\frac{1}{2}[h_\xi(1)+h_\xi(0)]$$

namely

$$\int_0^1 \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt$$
$$\leq \frac{1}{2}\left\{\frac{1}{2}\left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] + \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\}$$
$$\leq \frac{1}{2}\left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right]$$

for $\xi \in \gamma$.

By making use of this inequality, we have

$$\int_\gamma |f(\xi)| \left(\int_0^1 \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt\right) |d\xi|$$
$$\leq \frac{1}{4}\int_\gamma |f(\xi)| \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] |d\xi|$$
$$+ \frac{1}{2}\int_\gamma |f(\xi)| \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3} |d\xi|$$
$$\leq \frac{1}{2}\int_\gamma |f(\xi)| \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] |d\xi| .$$

By making use of Theorem 2 we then derive the desired result (2.14). □

Corollary 4. *With the assumptions of Theorem 2 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty,$$

then

$$\left\|f(y) - f(x) - \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1} d\xi\right\| \tag{2.15}$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 (1-t)\left\|(\xi - (1-t)x - ty)^{-1}\right\|^3 dt\right) |d\xi|$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left(\int_0^1 (1-t)\left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt\right) |d\xi|$$
$$\leq \frac{1}{2\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \left\{\frac{1}{2}\int_\gamma \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] |d\xi|\right.$$
$$\left.+ \int_\gamma \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3} |d\xi|\right\}$$
$$\leq \frac{1}{2\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] |d\xi| .$$

Remark 2. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open* *disk centered in* 0 *and of radius* R*. Then by (2.14),*

$$\left\| f(y) - f(x) - R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it}-x\right)^{-1}(y-x)\left(Re^{2\pi it}-x\right)^{-1}dt\right\| \tag{2.16}$$
$$\leq 2R\left\|y-x\right\|^2\left\|f\right\|_{R,\infty} \times \int_0^1\left(\int_0^1 (1-s)\left\|\left(Re^{2\pi it}-(1-s)x-sy\right)^{-1}\right\|^3 ds\right)dt$$
$$\leq 2R\left\|y-x\right\|^2\left\|f\right\|_{R,\infty}\int_0^1 (1-s)\left(R-\left\|(1-s)x+sy\right\|\right)^{-3}ds$$
$$\leq R\left\|y-x\right\|^2\left\|f\right\|_{R,\infty}\left\{\frac{1}{2}\left[\left(R-\|y\|\right)^{-3}+\left(R-\|x\|\right)^{-3}\right] + \left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\}$$
$$\leq R\left\|y-x\right\|^2\left\|f\right\|_{R,\infty}\left[\left(R-\|y\|\right)^{-3}+\left(R-\|x\|\right)^{-3}\right],$$

where $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$.

Remark 3. *We observe that from (2.13) and (2.16) we have the following upper bounds*

$$B_1(x,y,R,f) := \frac{R\|f\|_{R,\infty}\|y-x\|^2}{\left(R-\|x\|\right)^2\left(R-\|y\|\right)}$$

and

$$B_2(x,y,R,f) := R\left\|y-x\right\|^2\left\|f\right\|_{R,\infty}\left\{\frac{1}{2}\left[\left(R-\|y\|\right)^{-3}+\left(R-\|x\|\right)^{-3}\right] + \left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\}$$

for the quantity

$$\left\| f(y) - f(x) - R\int_0^1 e^{2\pi it} f\left(Re^{2\pi it}\right)\left(Re^{2\pi it}-x\right)^{-1}(y-x)\left(Re^{2\pi it}-x\right)^{-1}dt\right\|,$$

where $f : D \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* D *and* $x, y \in \mathcal{B}$ *with* $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ *and* $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$.

A 3d-plot for the difference of the quantities

$$K_1(x,y) := \frac{1}{\left(1-\|x\|\right)^2\left(1-\|y\|\right)}$$

and

$$K_1(x,y) := \frac{1}{2}\left[\left(1-\|y\|\right)^{-3}+\left(1-\|x\|\right)^{-3}\right] + \left(1-\left\|\frac{x+y}{2}\right\|\right)^{-3}$$

for $(x,y) \in (0.8, 0.9) \times (-0.9, -0.8)$ *and* $\|\cdot\| = |\cdot|$ *shows that it takes both positive and negative values, meaning that some time the bound* B_1 *is better than* B_2 *and other time is worse, depending on the values of* $(x,y) \in (0.8, 0.9) \times (-0.9, -0.8)$.

3. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1 &\exp\left[R\cos(2\pi t)\right]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).\end{aligned} \tag{3.1}$$

Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By (2.12) for the exponential function we get

$$\Bigg\|\exp y - \exp x - R\int_0^1 e^{2\pi it}\exp\left(Re^{2\pi it}\right)\left(Re^{2\pi it} - x\right)^{-1}(y-x)\left(Re^{2\pi it} - x\right)^{-1}dt\Bigg\| \tag{3.2}$$

$$
\begin{aligned}
&\leq 2R\left\|y-x\right\|^{2} \\
&\times\int_{0}^{1}\exp\left[R\cos\left(2\pi t\right)\right]\left(\int_{0}^{1}\left(1-s\right)\left\|\left(Re^{2\pi it}-\left(1-s\right)x-sy\right)^{-1}\right\|^{3}ds\right)dt \\
&\leq 2R\left\|y-x\right\|^{2}I_{0}(R)\int_{0}^{1}\left(1-s\right)\left(R-\left\|\left(1-s\right)x+sy\right\|\right)^{-3}ds \\
&\leq\frac{R\left\|y-x\right\|^{2}I_{0}(R)}{\left(R-\left\|x\right\|\right)^{2}\left(R-\left\|y\right\|\right)}
\end{aligned}
$$

and from (2.14)

$$
\begin{aligned}
&\Big\|\exp y-\exp x \\
&-R\int_{0}^{1}e^{2\pi it}\exp\left(Re^{2\pi it}\right)\left(Re^{2\pi it}-x\right)^{-1}\left(y-x\right)\left(Re^{2\pi it}-x\right)^{-1}dt\Big\| \\
&\leq 2R\left\|y-x\right\|^{2}\int_{0}^{1}\exp\left[R\cos\left(2\pi t\right)\right] \\
&\times\left(\int_{0}^{1}\left(1-s\right)\left\|\left(Re^{2\pi it}-\left(1-s\right)x-sy\right)^{-1}\right\|^{3}ds\right)dt \\
&\leq 2R\left\|y-x\right\|^{2}I_{0}(R)\int_{0}^{1}\left(1-s\right)\left(R-\left\|\left(1-s\right)x+sy\right\|\right)^{-3}ds \\
&\leq R\left\|y-x\right\|^{2}I_{0}(R) \\
&\times\left\{\frac{1}{2}\left[\left(R-\left\|y\right\|\right)^{-3}+\left(R-\left\|x\right\|\right)^{-3}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \\
&\leq R\left\|y-x\right\|^{2}I_{0}(R)\left[\left(R-\left\|y\right\|\right)^{-3}+\left(R-\left\|x\right\|\right)^{-3}\right].
\end{aligned}
\tag{3.3}
$$

Let f be an analytic functions on the open disk $D\left(0,\rho\right)$ given by the *power series* $f\left(\lambda\right):=\sum_{j=0}^{\infty}\alpha_{j}\lambda^{j}\ \left(\left|\lambda\right|<\rho\right)$. If $\nu\left(a\right)<\rho$, then the series $\sum_{j=0}^{\infty}\alpha_{j}a^{j}$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty}\left|\alpha_{j}\right|\left\|a^{j}\right\|<\infty$, and we can define $f\left(a\right)$ to be its sum. Clearly $f\left(a\right)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$
\lambda\mapsto f_{A}(\lambda):D(0,\rho)\to\mathbb{C},f_{A}(\lambda):=\sum_{n=0}^{\infty}\left|\alpha_{n}\right|\lambda^{n}.
$$

As some natural examples that are useful for applications, we can point out that, if

$$
\begin{aligned}
f\left(\lambda\right)&=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n}\lambda^{n}=\ln\frac{1}{1+\lambda},\ \ \lambda\in D\left(0,1\right); \\
g\left(\lambda\right)&=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n\right)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};
\end{aligned}
\tag{3.4}
$$

$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$
$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{3.5}$$
$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$
$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$
$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.6}$$
$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\, n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 3. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \le f_A(|\lambda|) \ \textit{for}\ \lambda \in D(0,\rho). \tag{3.7}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty} \left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \le \lim_{n\to\infty} \sum_{j=0}^{n} \left|\alpha_j \lambda^j\right| \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} |\alpha_j| |\lambda|^j = \sum_{j=0}^{\infty} |\alpha_j| |\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned}
&\left\| f(y) - f(x) \right.\\
&\left. -R\int_0^1 e^{2\pi i t} f\left(Re^{2\pi i t}\right)\left(Re^{2\pi i t} - x\right)^{-1}(y-x)\left(Re^{2\pi i t} - x\right)^{-1} dt \right\| \\
&\leq 2R\|y-x\|^2 f_A(R) \\
&\times \int_0^1 \left(\int_0^1 (1-s) \left\| \left(Re^{2\pi i t} - (1-s)x - sy\right)^{-1} \right\|^3 ds \right) dt \\
&\leq 2R\|y-x\|^2 f_A(R) \int_0^1 (1-s)\left(R - \|(1-s)x + sy\|\right)^{-3} ds \\
&\leq \frac{R\|y-x\|^2 f_A(R)}{(R-\|x\|)^2 (R - \|y\|)}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\left\| f(y) - f(x) \right.\\
&\left. -R\int_0^1 e^{2\pi i t} f\left(Re^{2\pi i t}\right)\left(Re^{2\pi i t} - x\right)^{-1}(y-x)\left(Re^{2\pi i t} - x\right)^{-1} dt \right\| \\
&\leq 2R\|y-x\|^2 f_A(R) \\
&\times \int_0^1 \left(\int_0^1 (1-s) \left\| \left(Re^{2\pi i t} - (1-s)x - sy\right)^{-1} \right\|^3 ds \right) dt \\
&\leq 2R\|y-x\|^2 f_A(R) \int_0^1 (1-s)\left(R - \|(1-s)x + sy\|\right)^{-3} ds \\
&\leq R\|y-x\|^2 f_A(R) \left\{ \frac{1}{2}\left[(R-\|y\|)^{-3} + (R-\|x\|)^{-3} \right] \right. \\
&\left. + \left(R - \left\| \frac{x+y}{2} \right\| \right)^{-3} \right\} \\
&\leq R\|y-x\|^2 f_A(R) \left[(R-\|y\|)^{-3} + (R-\|x\|)^{-3} \right].
\end{aligned} \tag{3.9}$$

The proof follows by Remarks 1, 2 and Lemma 3. As examples, one can consider the functions f and f_A listed above.

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