OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. In this paper we show among others that

$$\max\left\{\left\|f((1-\lambda)x+\lambda y)-\int_0^1 f((1-t)x+ty)\,dt\right\|,\right.$$
$$\left.\left\|(1-\lambda)f(y)+\lambda f(x)-\int_0^1 f((1-t)x+ty)\,dt\right\|\right\}$$
$$\leq \frac{1}{2\pi}\|y-x\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}|d\xi|$$

for $\lambda \in [0,1]$. Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

In 1938, A. Ostrowski [17], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [17])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $u \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The corresponding trapezoid inequality was obtained in 2000 by Cerone and Dragomir, [3].

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Theorem 2. *With the assumptions of Theorem 1 we have*

$$\left|\frac{(u-a)\,f(a)+(b-u)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.2}$$

for all $u\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

For a recent survey on Ostrowski's inequality for scalar functions and Lebesgue integral see [7].

In order to extend Ostrowski's and Trapezoid inequalities for functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\,\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)\,R_a(z)\,R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [14] and [18].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[12].

2. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y - x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y - x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{x,y}(t) = \frac{df_{x,y}(t)}{dt} &= \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y - x), \end{aligned}$$

which proves (2.5).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f:G\to\mathbb{C}$ is analytic on the domain G and $x\in\mathcal{B}$, with $\sigma(x)\subset G$, then for $v\in\mathcal{B}$ we have*

$$D(f)(x)(v)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-x)^{-1}v(\xi-x)^{-1}d\xi, \tag{2.4}$$

where $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)\subset\operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v\in\mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv)\subset\operatorname{ins}(\delta)\subset G$. Then

$$\begin{aligned}&f(x+hv)-f(x)\\&=\frac{1}{2\pi i}\left(\int_{\gamma}f(\xi)(\xi-x-hv)^{-1}d\xi-\int_{\gamma}f(\xi)(\xi-x)^{-1}d\xi\right)\\&=\frac{1}{2\pi i}\int_{\gamma}f(\xi)\left[(\xi-x-hv)^{-1}-(\xi-x)^{-1}\right]d\xi\\&=\frac{1}{2\pi i}h\int_{\gamma}f(\xi)\left[(\xi-x-hv)^{-1}v(\xi-x)^{-1}\right]d\xi,\end{aligned}$$

which gives for $h\neq 0$ that

$$\frac{f(x+hv)-f(x)}{h}=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-x-hv)^{-1}v(\xi-x)^{-1}d\xi.$$

By taking the limit over $h\to 0$ and using the properties of the integral, we get (2.4). □

Lemma 3. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y)\subset\operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}d\xi \tag{2.5}$$

for all $t\in(0,1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1}d\xi, \tag{2.6}$$

and

$$f'_{x,y}(1-)=\frac{1}{2\pi i}\int_{\gamma}f(\xi)(\xi-y)^{-1}(y-x)(\xi-y)^{-1}d\xi. \tag{2.7}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *With the assumptions of Lemma 3 we have the bounds*

$$\begin{aligned}
(2.8)\qquad & \left\| f_{x,y}'(t) \right\| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| (|\xi| - \|(1-t)x + ty\|)^{-2} |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| [(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2} |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2} \right] |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|
\end{aligned}$$

for all $t \in [0,1]$.

Proof. By taking the norm in (2.5) we get

$$\begin{aligned}
(2.9)\qquad & \left\| f_{x,y}'(t) \right\| \\
& \le \frac{1}{2\pi} \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} (y-x) (\xi - (1-t)x - ty)^{-1} \right\| |d\xi| \\
& \le \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 |d\xi| \\
& = \frac{1}{2\pi} \|y-x\| \int_\gamma |f(\xi)| |\xi|^{-2} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^2 |d\xi|
\end{aligned}$$

for all $t \in [0,1]$, which proves the first inequality in (2.8).

Since

$$\left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \le (1-t)\left\| \frac{x}{\xi} \right\| + t\left\| \frac{y}{\xi} \right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} = \sum_{k=0}^{\infty} \left[(1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right]^k.$$

Therefore

$$\begin{aligned}
\left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\| & \le \sum_{k=0}^{\infty} \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\|^k \\
& = \left(1 - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
& = \left(\frac{|\xi|}{|\xi|} - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
& = |\xi| (|\xi| - \|(1-t)x + ty\|)^{-1}
\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi} \right)^{-1} \right\|^{2} \leq |\xi|^{2} \left(|\xi| - \|(1-t)x+ty\| \right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned} &\int_{\gamma} |f(\xi)| \, |\xi|^{-2} \left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi} \right)^{-1} \right\|^{2} |d\xi| \\ &\leq \int_{\gamma} |f(\xi)| \left(|\xi| - \|(1-t)x+ty\| \right)^{-2} |d\xi| \end{aligned}$$

and we derive the second inequality in (2.8).

By the triangle inequality we have

$$\begin{aligned} |\xi| - \|(1-t)x+ty\| &\geq |\xi| - (1-t)\|x\| - t\|y\| \\ &= (1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) > 0 \end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi| - \|(1-t)x+ty\| \right)^{-1} \leq \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-1}$$

and thus

$$\left(|\xi| - \|(1-t)x+ty\| \right)^{-2} \leq \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$. This proves the third inequality in (2.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned} &\left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2} \\ &\leq (1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2} \end{aligned}$$

for $t \in [0,1]$, which proves the fourth inequality in (2.8).

Finally, observe that

$$(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2} \leq \max\left\{ (|\xi| - \|x\|)^{-2}, (|\xi| - \|y\|)^{-2} \right\}$$

and the last part of (2.8) is thus proved. □

Corollary 1. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open* *disk centered in* 0 *and of radius* R, *then*

$$\begin{aligned}(2.10)\quad & \left\|f_{x,y}^{'}(t)\right\| \\ & \leq R\|y-x\| \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| \left\|\left(Re^{2\pi is}-(1-t)x-ty\right)^{-1}\right\|^2 ds \\ & \leq R\|y-x\|\left(R-\|(1-t)x+ty\|\right)^{-2} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\ & \leq R\|y-x\|\left[(1-t)(R-\|x\|)+t(R-\|y\|)\right]^{-2} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\ & \leq R\|y-x\|\left[(1-t)(R-\|x\|)^{-2}+t(R-\|y\|)^{-2}\right] \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\ & \leq \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds,\end{aligned}$$

for all $t \in [0,1]$.

It follows by taking γ parametrized by $\xi(s) = Re^{2\pi is}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi iRe^{2\pi is}ds$, $|d\xi(s)| = 2\pi Rds$, $|\xi| = R$ and by (2.8) we get (2.10).

Remark 1. *If* $\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma}|f(\xi)| < 0$, *then from (2.8) we derive*

$$\begin{aligned}(2.11)\quad & \left\|f_{x,y}^{'}(t)\right\| \\ & \leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty} \int_\gamma \left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 |d\xi| \\ & \leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty} \int_\gamma (|\xi|-\|(1-t)x+ty\|)^{-2} |d\xi| \\ & \leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2} |d\xi| \\ & \leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right] |d\xi| \\ & \leq \frac{1}{2\pi}\|y-x\|\|f\|_{\gamma,\infty} \int_\gamma \frac{1}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}} |d\xi|\end{aligned}$$

for all $t \in [0,1]$.

Also, if $\|f\|_{R,\infty} := \sup_{s\in[0,1]}\left|f\left(Re^{2\pi is}\right)\right| < \infty$, *then from (2.10) we derive*

$$\begin{aligned}(2.12)\quad \left\|f_{x,y}^{'}(t)\right\| & \leq R\|y-x\|\|f\|_{R,\infty} \int_0^1 \left\|\left(Re^{2\pi is}-(1-t)x-ty\right)^{-1}\right\|^2 ds \\ & \leq R\|y-x\|\|f\|_{R,\infty}\left(R-\|(1-t)x+ty\|\right)^{-2} \\ & \leq R\|y-x\|\|f\|_{R,\infty}\left[(1-t)(R-\|x\|)+t(R-\|y\|)\right]^{-2} \\ & \leq R\|y-x\|\|f\|_{R,\infty}\left[(1-t)(R-\|x\|)^{-2}+t(R-\|y\|)^{-2}\right] \\ & \leq \frac{R\|y-x\|\|f\|_{R,\infty}}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}},\end{aligned}$$

for all $t \in [0,1]$.

3. Main Results

We have the following Ostrowski and midpoint type inequalities:

Theorem 3. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x)$, $\sigma(y) \subset G$. *Then we have the norm inequality*

$$\left\| f\left((1-\lambda)x+\lambda y\right) - \int_0^1 f\left((1-t)x+ty\right)dt \right\| \leq \frac{1}{2}\left[\lambda^2 \left\|f'_{x,y}\right\|_{[0,\lambda],\infty} + (1-\lambda)^2 \left\|f'_{x,y}\right\|_{[\lambda,1],\infty}\right] \leq \left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|f'_{x,y}\right\|_{[0,1],\infty} \tag{3.1}$$

for all $\lambda \in [0,1]$.

In particular, we have the midpoint inequality

$$\left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f\left((1-t)x+ty\right)dt \right\| \leq \frac{1}{8}\left[\left\|f'_{x,y}\right\|_{[0,1/2],\infty} + \left\|f'_{x,y}\right\|_{[1/2,1],\infty}\right] \leq \frac{1}{4}\left\|f'_{x,y}\right\|_{[0,1],\infty}. \tag{3.2}$$

Proof. If we use the integration by parts formula for the Bochner's integral, we have

$$\int_0^\lambda t f'_{x,y}(t)\,dt = \lambda f_{x,y}(\lambda) - \int_0^\lambda f_{x,y}(t)\,dt$$

and

$$\int_\lambda^1 (t-1) f'_{x,y}(t)\,dt = (1-\lambda) f_{x,y}(\lambda) - \int_\lambda^1 f_{x,y}(t)\,dt,$$

for all $\lambda \in [0,1]$.

Adding these two equalities, we obtain the Montgomery type identity (see for example [16, p. 565]):

$$f_{x,y}(\lambda) - \int_0^1 f_{x,y}(t)\,dt = \int_0^\lambda t f'_{x,y}(t)\,dt + \int_\lambda^1 (t-1) f'_{x,y}(t)\,dt, \tag{3.3}$$

namely, by (2.2), the equality of interest in terms of Fréchet derivative

$$f\left((1-\lambda)x+\lambda y\right) - \int_0^1 f\left((1-t)x+ty\right)dt = \int_a^\lambda tD(f)\left((1-t)x+ty\right)(y-x)\,dt + \int_\lambda^1 (t-1)D(f)\left((1-t)x+ty\right)(y-x)\,dt \tag{3.4}$$

for all $\lambda \in [0,1]$.

If we take the norm in (3.3) and use the integral's properties, we have

$$\begin{aligned}
(3.5)\qquad & \left\| f_{x,y}(\lambda) - \int_0^1 f_{x,y}(t)\,dt \right\| \\
& \leq \left\| \int_0^\lambda t f'_{x,y}(t)\,dt \right\| + \left\| \int_\lambda^1 (t-1) f'_{x,y}(t)\,dt \right\| \\
& \leq \int_0^\lambda t \left\| f'_{x,y}(t) \right\| dt + \int_\lambda^1 (1-t) \left\| f'_{x,y}(t) \right\| dt \\
& \leq \frac{1}{2}\lambda^2 \left\| f'_{x,y} \right\|_{[0,\lambda],\infty} + \frac{1}{2}(1-\lambda)^2 \left\| f'_{x,y} \right\|_{[\lambda,1],\infty} \\
& = \frac{1}{2}\left[\lambda^2 + (1-\lambda)^2\right] \max\left\{ \left\| f'_{x,y} \right\|_{[0,\lambda],\infty}, \left\| f'_{x,y} \right\|_{[\lambda,1],\infty} \right\} \\
& = \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \left\| f'_{x,y} \right\|_{[0,1],\infty},
\end{aligned}$$

which proves (3.1). □

Corollary 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}
(3.6)\qquad & \left\| f((1-\lambda)x + \lambda y) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\
& \leq \frac{1}{2\pi} \|y - x\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \\
& \times \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(3.7)\qquad & \left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\
& \leq \frac{1}{8\pi} \|y - x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|.
\end{aligned}$$

The proof follows by Lemma 4 and the last inequality in (3.1).

Remark 2. *If $\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < 0$, then from (3.6) and (3.7) we derive*

$$\begin{aligned}
(3.8)\qquad & \left\| f((1-\lambda)x + \lambda y) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\
& \leq \frac{1}{2\pi} \|y - x\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|f\|_{\gamma,\infty} \\
& \times \int_\gamma \frac{1}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|.
\end{aligned}$$

In particular, we have

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{1}{8\pi}\|y-x\|\,\|f\|_{\gamma,\infty}\int_\gamma \frac{1}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}|d\xi|. \tag{3.9}$$

Corollary 3. *Assume that $f: D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and $x, y\in\mathcal{B}$ with $\sigma(x), \sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\left\|f((1-\lambda)x+\lambda y)-\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \times\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds. \tag{3.10}$$

In particular,

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{R\|y-x\|}{4\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds. \tag{3.11}$$

The proof follows by Corollary 1 and the last inequality in (3.1).

Remark 3. *If $\|f\|_{R,\infty}:=\sup_{s\in[0,1]}\left|f\left(Re^{2\pi is}\right)\right|<\infty$, then from*

$$\left\|f((1-\lambda)x+\lambda y)-\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{R\|y-x\|\,\|f\|_{R,\infty}}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]. \tag{3.12}$$

In particular,

$$\left\|f\left(\frac{x+y}{2}\right)-\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{R\|y-x\|\,\|f\|_{R,\infty}}{4\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}. \tag{3.13}$$

The interested reader may state sharper inequalities than these by using Lemma 4 and Corollary 1, but they are more complicated and we do not state them here.

Theorem 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$. Then we have the norm inequality*

$$\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) dt\right\| \leq \frac{1}{2}\left[\lambda^2\left\|f_{x,y}'\right\|_{[0,\lambda],\infty}+(1-\lambda)^2\left\|f_{x,y}'\right\|_{[\lambda,1],\infty}\right] \leq\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|f_{x,y}'\right\|_{[0,1],\infty} \tag{3.14}$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid inequality

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) dt\right\| \leq \frac{1}{8}\left[\left\|f_{x,y}'\right\|_{[0,1/2],\infty}+\left\|f_{x,y}'\right\|_{[1/2,1],\infty}\right] \leq \frac{1}{4}\left\|f_{x,y}'\right\|_{[0,1],\infty}. \tag{3.15}$$

Proof. If we use the integration by parts formula for the Bochner's integral, we have

$$\int_0^1(t-\lambda) f_{x,y}'(t) dt=(1-\lambda) f_{x,y}(1)+\lambda f_{x,y}(0)-\int_0^1 f_{x,y}(t) dt,$$

for $\lambda \in [0,1]$, namely we have the following equality of interest in terms of Fréchet derivative

$$(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) dt =\int_0^1(t-\lambda) f_{x,y}'(t) dt=\int_0^1(t-\lambda) D(f)((1-t) x+t y)(y-x) dt \tag{3.16}$$

for $\lambda \in [0,1]$.

If we take the norm in (3.16), then we get

$$\begin{aligned}
&\left\|(1-\lambda) f(y)+\lambda f(x)-\int_0^1 f((1-t) x+t y) dt\right\| \\
&\leq \int_0^1|t-\lambda|\left\|f_{x,y}'(t)\right\| dt \\
&=\int_0^\lambda(\lambda-t)\left\|f_{x,y}'(t)\right\| dt+\int_\lambda^1(t-\lambda)\left\|f_{x,y}'(t)\right\| dt \\
&\leq\left\|f_{x,y}'\right\|_{[0,\lambda]} \int_0^\lambda(\lambda-t) dt+\left\|f_{x,y}'\right\|_{[\lambda,1]} \int_\lambda^1(t-\lambda) dt \\
&=\frac{1}{2}\left[\lambda^2\left\|f_{x,y}'\right\|_{[0,\lambda],\infty}+(1-\lambda)^2\left\|f_{x,y}'\right\|_{[\lambda,1],\infty}\right] \\
&=\frac{1}{2}\left[\lambda^2+(1-\lambda)^2\right] \max\left\{\left\|f_{x,y}'\right\|_{[0,\lambda],\infty},\left\|f_{x,y}'\right\|_{[\lambda,1],\infty}\right\} \\
&=\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|f_{x,y}'\right\|_{[0,1],\infty},
\end{aligned}$$

which proves the inequality (3.14). □

Corollary 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\left\| (1-\lambda) f(y) + \lambda f(x) - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{1}{2\pi} \|y-x\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \times \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi| . \tag{3.17}$$

In particular, we have

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{1}{8\pi} \|y-x\| \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi| . \tag{3.18}$$

The proof follows by Lemma 4 and the last inequality in (3.14).

Remark 4. *If $\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < 0$, then from (3.17) and (3.18) we derive*

$$\left\| (1-\lambda) f(y) + \lambda f(x) - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{1}{2\pi} \|y-x\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f\|_{\gamma,\infty} \times \int_\gamma \frac{1}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi| . \tag{3.19}$$

In particular, we have

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{1}{8\pi} \|y-x\| \|f\|_{\gamma,\infty} \int_\gamma \frac{1}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi| . \tag{3.20}$$

Corollary 5. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\left\| (1-\lambda) f(y) + \lambda f(x) - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{R\|y-x\|}{\min\left\{ (R - \|x\|)^2, (R - \|y\|)^2 \right\}} \times \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \int_0^1 \left| f\left(Re^{2\pi i s} \right) \right| ds. \tag{3.21}$$

In particular,

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \leq \frac{R\|y-x\|}{4\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds. \tag{3.22}$$

The proof follows by Corollary 1 and the last inequality in (3.14).

Remark 5. *If* $\|f\|_{R,\infty} := \sup_{s\in[0,1]} \left|f\left(Re^{2\pi is}\right)\right| < \infty$, *then from (3.19) and (3.20)*

$$\left\| (1-\lambda)f(y)+\lambda f(x) - \int_0^1 f((1-t)x+ty)\,dt \right\| \leq \frac{R\|y-x\|\,\|f\|_{R,\infty}}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]. \tag{3.23}$$

In particular,

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \leq \frac{R\|y-x\|\,\|f\|_{R,\infty}}{4\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}}. \tag{3.24}$$

4. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a\in\mathcal{B}$. Assume that $x, y\in\mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R>0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t)+i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned} (4.1) \qquad & \int_0^1 \exp\left[R\cos\left(2\pi t\right)\right] dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R). \end{aligned}$$

Assume that $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By (3.6) and (3.7) for the exponential function we get

$$\begin{aligned} (4.2) \qquad & \left\|\exp\left((1-\lambda)x + \lambda y\right) - \int_0^1 \exp\left((1-t)x + ty\right) dt\right\| \\ &\leq \frac{R I_0(R)\left\|y - x\right\|}{\min\left\{\left(R - \left\|x\right\|\right)^2, \left(R - \left\|y\right\|\right)^2\right\}}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \end{aligned}$$

In particular, we have the midpoint inequality

$$\begin{aligned} (4.3) \qquad & \left\|\exp\left(\frac{x+y}{2}\right) - \int_0^1 \exp\left((1-t)x + ty\right) dt\right\| \\ &\leq \frac{R I_0(R)\left\|y - x\right\|}{4\min\left\{\left(R - \left\|x\right\|\right)^2, \left(R - \left\|y\right\|\right)^2\right\}}. \end{aligned}$$

By (3.21) and (3.22) we get

$$\begin{aligned} (4.4) \qquad & \left\|(1-\lambda)\exp y + \lambda \exp x - \int_0^1 \exp\left((1-t)x + ty\right) dt\right\| \\ &\leq \frac{R I_0(R)\left\|y - x\right\|}{\min\left\{\left(R - \left\|x\right\|\right)^2, \left(R - \left\|y\right\|\right)^2\right\}}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]. \end{aligned}$$

In particular, we have the trapezoid inequality

$$\begin{aligned} (4.5) \qquad & \left\|\frac{f(x) + f(y)}{2} - \int_0^1 f\left((1-t)x + ty\right) dt\right\| \\ &\leq \frac{R I_0(R)\left\|y - x\right\|}{4\min\left\{\left(R - \left\|x\right\|\right)^2, \left(R - \left\|y\right\|\right)^2\right\}}. \end{aligned}$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(a) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \tag{4.6}$$

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C};$$

$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$

$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{4.7}$$

$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$

$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{4.8}$$

$$\frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 5. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \leq f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{4.9}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty} \left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \leq \lim_{n\to\infty} \sum_{j=0}^{n} \left|\alpha_j \lambda^j\right| \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} |\alpha_j| |\lambda|^j = \sum_{j=0}^{\infty} |\alpha_j| |\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned} &\left\| f((1-\lambda)x + \lambda y) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ &\leq \frac{R\|y-x\| f_A(R)}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \end{aligned} \tag{4.10}$$

for all $\lambda \in [0,1]$ and

$$\begin{aligned} &\left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ &\leq \frac{R\|y-x\| f_A(R)}{4\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}}. \end{aligned} \tag{4.11}$$

We also have

$$\begin{aligned} &\left\| (1-\lambda) f(y) + \lambda f(x) - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ &\leq \frac{R\|y-x\| f_A(R)}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \end{aligned} \tag{4.12}$$

for all $\lambda \in [0,1]$ and

$$\begin{aligned} &\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x + ty)\,dt \right\| \\ &\leq \frac{R\|y-x\| f_A(R)}{4\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}}. \end{aligned} \tag{4.13}$$

The proof follows by Corollaries 3, 4 and Lemma 5. As examples, one can consider the functions f and f_A listed above.

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